MATH 162B WINTER 2012
INTRODUCTION TO p-ADIC GALOIS REPRESENTATIONS
OVERVIEW

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Introduction

A main goal of algebraic number theory is to understand continuous Galois representations of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}(n, R)$$

where R is a topological ring. Often R is one of the rings $\mathbb{C}, \overline{\mathbb{Q}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{F}}_\ell$ (or subextensions of these). Here continuity is with respect to the profinite topology on $G_{\mathbb{Q}}$ and the topology of R; note that the topology on R matters: as fields, $\mathbb{C} \cong \overline{\mathbb{Q}}_p$, but not as topological fields.

The study of these "global" Galois representations is a fundamentally hard problem, and an often simpler approach is to first study continuous "local" Galois representations of $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}(n, R)$ where $G_{\mathbb{Q}_p} \subset G_{\mathbb{Q}}$ is the decomposition group at p. One desires to understand/classify local and global Galois representations.

1. Examples

For simplicity we'll only talk about $\mathbb{Q}$ and $\mathbb{Q}_p$.

Global Galois representations.

Artin representations. These are continuous Galois representations $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(n, \mathbb{C})$, which necessarily have finite image (if $n = 2$ this means that the image is cyclic, dihedral, tetrahedral, octahedral or icosahedral).

The main conjecture related to these representations is that the Artin L-function

$$L(\rho, s) = \prod_p \det(1 - \mathrm{Frob}_p\, p^{-s} | \rho^{I_p})^{-1}$$

has analytic continuation to $\mathbb{C}$ if ρ does not contain the trivial representation. For function fields proven by Weil; meromorphic continuation follows from Tate's thesis and Brauer's theorem on induced characters (implies conjecture for cyclic and dihedral); Langlands proved the tetrahedral case; Tunnell proved the octahedral case; the icosahedral case is in progress. Should follow in general from the Langlands program.

In terms of classification, two dimensional Artin representations should correspond to weight 1 modular forms.

Mod p representations. These are continuous Galois representations $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(n, \overline{\mathbb{F}}_p)$.

The main conjecture is Serre's conjecture, which says that such ρ which are irreducible and "odd" come from modular forms with predictable weight and level. Known for $\mathbb{Q}$ by Khare and Wintenberger.

ℓ-adic representations. These are continuous Galois representations $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$. If they come in compatible systems, meaning one is given a Galois representation for each prime ℓ such that Frob_p acts compatibly in ρ_ℓ as ℓ varies, then one can attach a meaningful L-function.

The main conjecture is the global Langlands conjecture. This conjecture has several components, but the first direction is that attached to "algebraic" automorphic representations one can always attach compatible systems of Galois representations. See the first homework for $n = 1$; the $n = 2$ case the first part is Eichler-Shimura (weight 2), Deligne (weight ≥ 3), Deligne-Serre (weight 1). The second direction is that one can describe the compatible systems that come from automorphic representations (this description requires p-adic Hodge theory). For $n = 2$ the first such result was Wiles' proof of Fermat's Last Theorem, later the Taniyama-Shimura conjecture; finally the Fontaine-Mazur conjecture for $n = 2$ mostly known by Kisin.

Local Galois representations.

Complex representations. These are continuous Galois representations $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}(n, \mathbb{C})$. The local Langlands conjecture states that these are in bijection with smooth representations of $\mathrm{GL}(n, \mathbb{Q}_p)$. Proven by Harris-Taylor and Henniart (90's), a different proof given by Scholze ('10).

ℓ-adic representations ($\ell \neq p$). These are continuous Galois representations $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$ for $\ell \neq p$.

Grothendieck's ℓ-adic monodromy theorem implies that these are in bijection with certain Weil-Deligne representations, which are pairs (r, N) of a continuous (here this means open stabilizers) Galois representation r and a nilpotent matrix N such that $r(g)N = p^d N r(g)$ where d is the exponent of Frob_p in g. The correspondence is

$$\rho(\mathrm{Frob}^n \sigma) = r(\mathrm{Frob}^n \sigma) \exp(t_\ell(\sigma) N)$$

where $t_\ell : I_p \to I_p/P_p \cong \prod_{q \neq p} \mathbb{Z}_q \to \mathbb{Z}_\ell$.

Remark 1. A better way to describe N is that $N : \rho \to \rho(-1)$ is an intertwining operator.

p-adic Galois representations. These are continuous Galois representations $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}(n, \overline{\mathbb{Q}}_p)$.

Although Grothendieck's ℓ-adic monodromy theorem no longer applies since $\ell = p$, one can still attach Weil-Deligne representations to such ρ via Fontaine's p-adic Hodge theory. This is complicated, and one of the focuses of this course.

2. This course

Goals. The main goals of this course are the following:

(1) Enumerate all p-adic Galois representations.
(2) Enumerate the p-adic Galois representations arising from geometry.

To motivate the second goal, let's look at the case of Fermat's Last Theory. The theorem follows from a modularity lifting theorem which is proven by showing that a deformation space of Galois representations (the ring R) is isomorphic to a deformation space of modular forms (the ring T); if one allows too many Galois representations as deformations one gets a too large ring R, so one needs to specify constraints which force these deformations to correspond to modular forms and algebraic geometry).

How to study? How to study p-adic Galois representations? Seek inspiration from algebraic geometry. Local Galois representations arise naturally in the etale cohomology of varieties over $\mathbb{Q}_p$; perhaps one may find a different cohomology group which is simpler to describe.

The proto-example is Torelli's theorem for compact Riemann surfaces. If X is a compact connected Riemann surface (equivalently a smooth projective connected curve over $\mathbb{C}$) then $H^0(X, \mathbb{Z}) = \mathbb{Z}$ since X is connected, $H^2(X, \mathbb{Z}) = \mathbb{Z}$ since X is proper, while $H^1(X, \mathbb{Z})$ is a $\mathbb{Z}$-module (of rank twice the genus of the curve).

The Hodge decomposition theorem states that

$$H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^{1,0}(X) \oplus H^{0,1}(X)$$

where $H^{1,0}(X) = H^0(X, \Omega^1_X)$ and $H^{0,1} = H^1(X, \Omega^0_X)$ are differential cohomology groups (the left hand side is the de Rham cohomology group $H^1_{\mathrm{dR}}(X/\mathbb{C})$ while the right hand side is the Hodge cohomology group $H^1_{\mathrm{Hodge}}(X)$). So from the category of smooth projective connected curves over $\mathbb{C}$ we obtained an object in a different category whose objects are "Hodge structures", i.e., $\mathbb{Z}$-modules M together with a decomposition $V = M \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$. The amazing thing is that the curve X is determined by its associated Hodge structure.

So we replaced a complicated object (the curve X) with a linear algebra datum (the Hodge structure). This is the defining theme of p-adic Hodge theory, where one seeks linear algebra data in the hope of finding an equivalence of categories; this will not be possible always.

The algebraic geometry story. The general example is that of a proper smooth scheme of finite type over $\mathbb{Q}_p$. The p-adic etale cohomology group $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ gives rise to p-adic Galois representations. One seeks linear algebra data in the other cohomology groups attached to X.

Hodge-Tate. In general the de Rham cohomology group $H^n_{\text{dR}}(X/\mathbb{Q}_p)$ is not a direct sum of Hodge cohomology groups; instead, $H^n_{\text{dR}}(X/\mathbb{Q}_p)$ is a $\mathbb{Q}_p$ vector space endowed with a filtration (which comes from a degenerating spectral sequence whose first sheet contains Hodge cohomology groups). The first linear algebra datum is $\text{gr}^\bullet H^n_{\text{dR}}(X/\mathbb{Q}_p)$ which is a graded $\mathbb{Q}_p$ vector space.

While in the case of smooth projective complex curves the comparison between the two cohomology groups occured over $\mathbb{C}$, the story is more complicated over $\mathbb{Q}_p$ since it needs to take into account the Galois action. Tate conjectured in the 60's and Faltings proved in the 80's that

$$\underbrace{H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)}_{\text{Galois action}} \otimes_{\mathbb{Q}_p} \underbrace{B_{\text{HT}}}_{\substack{\text{graded vector space} \\ \text{with Galois action}}} \cong \underbrace{\text{gr}^\bullet H^n_{\text{dR}}(X/\mathbb{Q}_p)}_{\text{graded vector space}} \otimes_{\mathbb{Q}_p} B_{\text{HT}}$$

where B_{HT} is a graded vector space with Galois action over the p-adic completion $\mathbb{C}_p$ of $\overline{\mathbb{Q}}_p$ (implicit in this is that the Galois group acts on each graded piece).

de Rham. The Hodge-Tate linear algebra datum is convenient, since it is only a graded vector space with the Galois acting on each graded piece, but it is much too coarse to describe the Galois representation on the etale cohomology group. One could instead not base change to $\mathbb{C}_p$ and work directly with the de Rham cohomology group $H^n_{\text{dR}}(X/\mathbb{Q}_p)$ which is a filtered vector space with Galois action (meaning that the Galois group acts on each filtered piece, but not necessarily semisimply, so not necessarily on each graded piece).

The corresponding comparison theorem was conjectured by Fontaine and proven by Faltings (using almost math), Nizioł (using K-theory), Beilinson (using derived geometry) and Scholze (using perfectoid spaces)

$$\underbrace{H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)}_{\text{Galois action}} \otimes_{\mathbb{Q}_p} \underbrace{B_{\text{dR}}}_{\substack{\text{filtered vector space} \\ \text{with Galois action}}} \cong \underbrace{H^n_{\text{dR}}(X/\mathbb{Q}_p)}_{\text{filtered vector space}} \otimes_{\mathbb{Q}_p} B_{\text{dR}}$$

Crystalline. The de Rham linear algebra datum of a filtered $\mathbb{Q}_p$ vector space with Galois action is still convenient and works generally, but the functor attaching the linear algebra datum to the Galois representation is not faithful, meaning it cannot detect maps between Galois representations. However, if the proper smooth scheme $X/\mathbb{Q}_p$ has a proper smooth model over $\mathbb{Z}_p$, in other words it has good reduction at p, then one has an additional cohomology group: the crystalline cohomology group $H^n_{\text{cris}}(X/\mathbb{Z}_p)$ which is a $\mathbb{Q}_p$ filtered vector space with an action not of the whole Galois group but just of Frobenius.

The relevant comparison theorem is that if X has proper smooth reduction at p then

$$\underbrace{H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)}_{\text{Galois action}} \otimes_{\mathbb{Q}_p} \underbrace{B_{\text{cris}}}_{\substack{\text{filtered vector space} \\ \text{with Frobenius}}} \cong \underbrace{H^n_{\text{dR}}(X/\mathbb{Q}_p)}_{\text{filtered vector space}} \otimes_{\mathbb{Q}_p} B_{\text{cris}}$$

which goes via the cristalline cohomology group $H^n_{\text{cris}}(X/\mathbb{Z}_p)$.

The remarkable fact about this functor is that it is faithful on the subcategory of crystalline representations, and that one can characterize all filtered $\mathbb{Q}_p$ vector spaces with Frobenius action that arise from p-adic Galois representations (these are the admissible modules).

Semistable. One may object that X having proper smooth reduction at p is too strong a condition. One may weaken this and only require that X has a "semistable" reduction at p, which means that singularities behave like the intersection of two lines. Such singularities are still nice enough because if one leaves the category of schemes and goes into the category of "log schemes", such semistable singularities become "log smooth". In that case the log crystalline cohomology group $H^n_{\text{log-cris}}(X/\mathbb{Z}_p)$ is a $\mathbb{Q}_p$ filtered vector space with an action of Frobenius and a monodromy operator N with the commutation relation $N\phi = p\phi N$. (The monodromy N is supposed to encode the monodromy operator acting on the generic cohomology around the singular special fiber.)

The relevant comparison theorem, proven by Tsuji in the 90's, is

$$\underbrace{H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)}_{\text{Galois action}} \otimes_{\mathbb{Q}_p} \underbrace{B_{\text{st}}}_{\substack{\text{filtered vector space} \\ \text{with Frobenius and } N}} \cong \underbrace{H^n_{\text{dR}}(X/\mathbb{Q}_p)}_{\text{filtered vector space}} \otimes_{\mathbb{Q}_p} B_{\text{st}}$$

which goes via the log cristalline cohomology group $H^n_{\text{log-cris}}(X/\mathbb{Z}_p)$.

Again, remarkably, the functor from semistable representations to $\mathbb{Q}_p$ filtered vector spaces with Frobenius and monodromy is faithful and the image can be described.

Fields of norms. What about the other p-adic Galois representations, not coming from algebraic geometry? Describing these representations proved to be crucial in the p-adic local Langlands program. The approach is via fields of norms of Fontaine and Wintenberger, which in effect says that there exists a characteristic p field E such that the Galois theory of a very ramified extension of $\mathbb{Q}_p$ is the same as the Galois theory of E. Then one can prove comparison theorems in the characteristic p setting and automatically transport results to characteristic 0. This theory relies on ramification estimates due to Tate.

The plan. Time permitting we will look at:

(1) Review local class field theory
(2) Hodge-Tate(-Sen) theory and B_{HT}
(3) Admissible representations
(4) De Rham representations and B_{dR}
(5) Crystalline representations and B_{cris}
(6) Semistable representations and B_{st}
(7) Admissibility and weak admissibility
(8) Fields of norms
(9) (ϕ, Γ)-modules
(10) Kisin's $\mathfrak{S}$-modules and integral p-adic Hodge theory
(11) Families of p-adic Galois representations