MIDTERM

MA1A SECTION 1 FALL 2011

Time: 2 hours

This exam should be completed in a **standard blue book** (or as many as necessary) within **two** hours. You may do the exam in **multiple sittings**. If you go over the time limit, indicate clearly where the work for the two hour limit ends: you will get half credit for what you write in an additional half an hour (also marked clearly). Anything written outside the two and a half hours will not be graded.

The blue book(s) with your solutions must be deposited, by **noon** on **October 31, 2011** (Monday), into the **Ma1a Section 1** mailbox. Do **NOT** turn the midterm exam to anyone else or anywhere else, and do **NOT** slip it under the door anywhere. Loose sheets of paper will **NOT** be accepted.

You may use the notes you yourself took in class and in recitation. You may use the sections from the textbook that were covered in class, Dinakar Ramakrishnan's online notes for Ma1a, as well as the posted HW solutions. But nothing else, and no one else.

Before starting to work on this midterm I encourage you to go over your course notes, paying special attention to all the examples covered in class.

Each of the following problems is worth **40** points, but your final grade will be out of **100** points, so you need not solve all the problems to get a maximum score. In the case of problems with multiple parts, you may use any part in the solution of any other part even if you have not solved it. Give details in your solutions, don't simply write the answer.

(1) (40 points) Let $x \neq 1$. For an integer $n \geq 1$ show that

$$\prod_{k=1}^{n}(1 + x^{2^{k-1}}) = \frac{1 - x^{2^n}}{1 - x}$$

(2) (a) (20 points) Show that the limit of a convergent sequence of integers is an integer.
(b) (10 points) Let $(x_n)_{n \geq 1}$ be a convergent sequence of rational numbers. Let q_n be the denominator of x_n (when written in lowest terms). If the sequence $(q_n)_{n \geq 1}$ is a bounded sequence of integers, show that there exists an integer N such that $N \cdot x_n$ is an integer for all $n \geq 1$.
(c) (10 points) Under the assumptions of part (b) deduce that $\lim\limits_{n\to\infty} x_n$ is a rational number.

(3) Let $x \in (0, \pi/2)$. Consider the sequence

$$x_n = \underbrace{\sin(\sin(\cdots \sin}_{n \text{ times}} x))$$

(a) (20 points) Is the sequence (x_n) monotonic?
(b) (20 points) Does it converge? If yes, find its limit, if not, show why that is the case.

(4) (a) (20 points) Show that if $x > 0$ is a real number then $\lim\limits_{n\to\infty} \dfrac{\lfloor xn \rfloor}{n} = x$, where $\lfloor a \rfloor$ represents the largest integer $\leq a$.
(b) (20 points) Let $f : (0, \infty) \to \mathbb{R}$ be an increasing function, i.e., such that $f(x) < f(y)$ whenever $x < y$. Show that the limit

$$\lim_{n\to\infty} f\left(\frac{\lfloor xn \rfloor}{n}\right)$$

exists.