MA 1A (SECTION 1) MID-TERM SOLUTIONS

Problem 1. (40 points)

Let $x \neq 1$. For an integer $n \geq 1$ show that $\prod_{k=1}^{n}(1+x^{2^{k-1}}) = \frac{1-x^{2^n}}{1-x}$.

Solution. When $n = 1$, L.H.S. $= 1 + x^{2^{1-1}} = 1 + x$, and R.H.S. $= \frac{1-x^{2^1}}{1-x} = \frac{1-x^2}{1-x} = 1 + x =$ L.H.S. since $x \neq 1$.

Assume that for some $r \in \mathbb{N}$, $\prod_{k=1}^{r}(1+x^{2^{k-1}}) = \frac{1-x^{2^r}}{1-x}$. When $n = r+1$, L.H.S. $= \prod_{k=1}^{r+1}(1+x^{2^{k-1}}) = (1 + x^{2^{r+1-1}})\prod_{k=1}^{r}(1 + x^{2^{k-1}}) = (1 + x^{2^r})\frac{1-x^{2^r}}{1-x} = \frac{1^2-(x^{2^r})^2}{1-x} = \frac{1-x^{2^{r+1}}}{1-x} =$ R.H.S. Therefore, by induction, $\prod_{k=1}^{n}(1 + x^{2^{k-1}}) = \frac{1-x^{2^n}}{1-x}$ for all $n \in \mathbb{N}$.

Problem 2.

(a) (20 points) Show that the limit of a convergent sequence of integers is an integer.

(b) (10 points) Let $(x_n)_{n\geq 1}$ be a convergent sequence of rational numbers. Let q_n be the denominator of x_n (when written in lowest terms). If the sequence $(q_n)_{n\geq 1}$ is a bounded sequence of integers, show that there exists an integer N such that $N \cdot x_n$ is an integer for all $n \geq 1$.

(c) (10 points) Under the assumptions of part (b) deduce that $\lim_{n\to\infty} x_n$ is a rational number.

Solution. (a) Let $(a_n)_{n\geq 1}$ be a convergent sequence of integers. Assume contrary, let the limit be $L \in \mathbb{R}\backslash\mathbb{Z}$. Then for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all n with $n \geq N$, $|a_n - L| < \epsilon$. In particular, we can pick $\epsilon = \min\{\lceil L\rceil - L, L - \lfloor L\rfloor\} > 0$. However, as $a_n \in \mathbb{Z}$, by the definition of ceiling and floor function, $|a_n - L| \geq \lceil L\rceil - L$ or $|a_n - L| \geq L - \lfloor L\rfloor$, i.e. $|a_n - L| \geq \epsilon$ for all $n \in \mathbb{N}$, contradiction. Therefore, $L \in \mathbb{Z}$.

(b) Let a bound of $(q_n)_{n\geq 1}$ be $M \in \mathbb{N}$, i.e. $-M \leq q_n \leq M$ for all $n \in \mathbb{N}$. If $N = M!$, $q_n \mid N$ for all $n \in \mathbb{N}$, and hence, $N \cdot x_n$ is an integer for all $n \in \mathbb{N}$.

(c) Let $L = \lim_{n\to\infty} x_n$. Then $NL = \lim_{n\to\infty} N \cdot x_n$, where N is the same as part (b). As $(N \cdot x_n)_{n\geq 1}$ is a convergent sequence of integers, by part (a), NL is an integer. Therefore, $L = \frac{NL}{N}$ is a fraction of integers with $N \neq 0$, i.e. rational.

Problem 3.

Let $x \in (0, \frac{\pi}{2})$. Consider the sequence $x_n = \underbrace{\sin(\sin(\cdots\sin x))}_{n \text{ times}}$.

(a) (20 points) Is the sequence (x_n) monotonic?

(b) (20 points) Does it converge? If yes, find its limit; if not, show why that is the case.

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Solution. (a) For all $x \in (0, \frac{\pi}{2})$, $0 < \sin x < x < \frac{\pi}{2}$.
When $n = 1$, $x_1 = \sin x \in (0, \frac{\pi}{2})$. Assume that for some $k \in \mathbb{N}$, $x_k \in (0, \frac{\pi}{2})$. When $n = k + 1$, $x_{k+1} = \sin x_k \in (0, \frac{\pi}{2})$. By induction, $x_n \in (0, \frac{\pi}{2})$ for all $n \in \mathbb{N}$. As a result, $x_{n+1} = \sin x_n < x_n$ for all $n \in \mathbb{N}$. Therefore, the sequence is monotonic (decreasing).
(b) As the sequence is monotonic and bounded, it is convergent. Let $L = \lim_{n\to\infty} x_n$. Then $\sin L = \sin\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} \sin x_n$ (since sin is continuous) $= \lim_{n\to\infty} x_{n+1} = L$, implying $L = 0$.

Problem 4.
(a) (20 points) Show that if $x > 0$ is an irrational number, then $\lim_{n\to\infty} \frac{\lfloor xn \rfloor}{n} = x$, where $\lfloor a \rfloor$ represents the largest integer $\leq a$.
(b) (20 points) Let $f : (0, \infty) \to \mathbb{R}$ be an increasing function, i.e. $f(x) < f(y)$ whenever $x < y$. Show that the limit $\lim_{n\to\infty} f(\frac{\lfloor xn \rfloor}{n})$ exists.

Solution. (a) For all $\epsilon > 0$, by Archimedean principle, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. For all n with $n \geq N$, $\left|\frac{\lfloor xn \rfloor}{n} - x\right| = \frac{xn - \lfloor xn \rfloor}{n} < \frac{1}{n} \leq \frac{1}{N} < \epsilon$.
(b) $\frac{\lfloor xn \rfloor}{n} \leq \frac{xn}{n} = x$, but the equality case cannot hold since x is irrational, so $\frac{\lfloor xn \rfloor}{n} < x$ for all $n \in \mathbb{N}$. Hence, $\left(\frac{\lfloor xn \rfloor}{n}\right)_{n\geq 1}$ is converging to x from the left. As f is an increasing function, the left limit $\lim_{y\to y_0^-} f(y)$ exists for all $y_0 \in \mathbb{R}$. Taking $y_0 = x$, we have $\lim_{n\to\infty} f(\frac{\lfloor xn \rfloor}{n})$ exists.