MA 1A (SECTION 1) HW1 SOLUTIONS

Problem 1. (Apostol I 3.5 9)
There is no real number a such that $x \le a$ for all real x.

Solution. Assume contrary, let $a \in \mathbb{R}$ be such that for all $x \in \mathbb{R}$, $x \le a$. As $\mathbb{R}$ is closed under addition, $a, 1 \in \mathbb{R} \Rightarrow a + 1 \in \mathbb{R}$. Hence, by assumption, $a + 1 \le a$. By Thm I.21, $0 < 1$. By Axiom 4, $a = a + 0$. By Thm I.18, $a = a + 0 < a + 1$, contradiction.
(At most half credit if only applied Archimedean property (Thm I.30).)

Problem 2. (Apostol I 3.12 6)
If x and y are arbitrary real numbers, $x < y$, prove that there exists at least one rational number r satisfying $x < r < y$, and hence infinitely many. This property is often described by saying that the rational numbers are *dense* in the real-number system.

Solution. Assume contrary, there does not exists $r \in \mathbb{Q}$ such that $x < r < y$.

As $y - x > 0$, there exists $M \in \mathbb{N}$ such that $M > \frac{1}{y-x}$ (Archimedean property (Thm I.30)). There also exists $a, b \in \mathbb{Z}$ such that $a < xM$ and $b > yM$, i.e. $\frac{a}{M} < x < y < \frac{b}{M}$.

Consider $\frac{a}{M}, \frac{a+1}{M}, \frac{a+2}{M}, \ldots, \frac{b-1}{M}, \frac{b}{M}$. As they are all in $\mathbb{Q}$, none of them is in the interval (x, y) by assumption. Then there exists $i \in \mathbb{Z}$, $a \le i \le b$, such that $\frac{a}{M} \le \frac{i}{M} \le x < y \le \frac{i+1}{M} \le \frac{b}{M}$. This implies that $\frac{1}{M} = \frac{i+1}{M} - \frac{i}{M} \ge y - x$, and hence $M \le \frac{1}{y-x}$, contradicting with the definition of M.

Problem 3. (Apostol I 4.4 1(c))
Prove by induction that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2.$$

Solution. When $n = 1$, L.H.S. $= 1^3 = 1$, and R.H.S. $= 1^2 = 1 =$ L.H.S.

Assume that the equality holds for some $k \in \mathbb{N}$, i.e. $1^3 + 2^3 + 3^3 + \cdots + k^3 = (1 + 2 + 3 + \cdots + k)^2$. When $n = k + 1$,

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$$
\begin{aligned}
\text{R.H.S.} &= (1+2+\cdots+k+(k+1))^2 \\
&= (1+2+\cdots+k)^2 + 2(1+2+\cdots+k)(k+1) + (k+1)^2 \\
&= 1^3+2^3+\cdots+k^3 + 2(1+2+\cdots+k)(k+1) + (k+1)^2 \text{ (induction assumption)} \\
&= 1^3+2^3+\cdots+k^3 + 2\frac{k(k+1)}{2}(k+1) + (k+1)^2 \text{ (proved in recitation)} \\
&= 1^3+2^3+\cdots+k^3 + (k+1)^3 = \text{L.H.S.}
\end{aligned}
$$

Hence, by induction, $1^3+2^3+3^3+\cdots+n^3 = (1+2+3+\cdots+n)^2$ for all $n \in \mathbb{N}$.

Problem 4. (Apostol I 4.4 9)

Prove the following statement by induction: If a line of unit length is given, then a line of length $\sqrt{n}$ can be constructed with straightedge and compass for each positive integer n.

Solution. When $n = 1$, a line of length $\sqrt{1} = 1$ can be constructed since it is given.

Assume that for some $k \in \mathbb{N}$, a line of length $\sqrt{k}$ can be constructed with straightedge and compass. Note that by Pythagoras' Theorem, if the two vertical lines of a right-angled triangle are of lengths 1 and $\sqrt{k}$, then the hypothenuse is of length $\sqrt{k+1}$. Since we can construct perpendicular lines at any given point on a straight line with straightedge and compass (proved in recitation), a line of length $\sqrt{k+1}$ can be constructed.

Hence, by induction, a line of length $\sqrt{n}$ can be constructed for all $n \in \mathbb{N}$.

Problem 5. (Apostol I 4.7 12)

Guess and prove a general rule which simplifies the sum

$$\sum_{k=1}^{n} \frac{1}{k(k+1)}.$$

Solution. Claim: $\sum\limits_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ for all $n \in \mathbb{N}$.

When $n = 1$, L.H.S. $= \frac{1}{1\times 2} = \frac{1}{2}$, and R.H.S. $= \frac{1}{1+1} = \frac{1}{2}$ =L.H.S.

Assume that the equality holds for some $r \in \mathbb{N}$, i.e. $\sum\limits_{k=1}^{r} \frac{1}{k(k+1)} = \frac{r}{r+1}$. When $n = r+1$, L.H.S. $= \sum\limits_{k=1}^{r+1} \frac{1}{k(k+1)} = \sum\limits_{k=1}^{r} \frac{1}{k(k+1)} + \frac{1}{(r+1)(r+2)} = \frac{r}{r+1} + \frac{1}{(r+1)(r+2)}$ (induction assumption) $= \frac{r(r+2)+1}{(r+1)(r+2)} = \frac{r+1}{r+2} =$ R.H.S. Hence, by induction, $\sum\limits_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ for all $n \in \mathbb{N}$.