MA 1A (SECTION 1) HW2 SOLUTIONS

Problem 1. (Apostol I 4.10 3)

Prove that $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$. This is called the *law of Pascal's triangle.*

Solution. R.H.S. $= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} = \frac{n!}{k!(n-k+1)!}(k + (n-k+1)) = \frac{(n+1)!}{k!(n+1-k)!} =$ L.H.S.

Problem 2. (Apostol I 4.10 4)

Use induction to prove the binomial theorem $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$. Then use the theorem to derive the formulae $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ and $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$, if $n > 0$.

Solution. When $n = 1$, L.H.S. $= a + b$, and R.H.S. $= \binom{1}{0} a^0 b^{1-0} + \binom{1}{1} a^1 b^{1-1} = a + b =$ L.H.S.

Assume that for some $r \in \mathbb{N}$, $(a+b)^r = \sum_{k=0}^{r} \binom{r}{k} a^k b^{r-k}$. When $n = r + 1$,

$$\begin{aligned}
\text{L.H.S.} = (a+b)^r(a+b) &= \left(\sum_{k=0}^{r} \binom{r}{k} a^k b^{r-k} \right) (a+b) \text{ (by induction assumption)} \\
&= \sum_{k=0}^{r} \binom{r}{k} a^{k+1} b^{r-k} + \sum_{k=0}^{r} \binom{r}{k} a^k b^{r-k+1} \\
&= \sum_{k=1}^{r+1} \binom{r}{k-1} a^k b^{r-k+1} + \sum_{k=0}^{r} \binom{r}{k} a^k b^{r-k+1} \\
&= a^{r+1} + \sum_{k=1}^{r} \binom{r}{k-1} a^k b^{r-k+1} + \sum_{k=1}^{r} \binom{r}{k} a^k b^{r-k+1} + b^{r+1} \\
&= a^{r+1} + \sum_{k=1}^{r} \left(\binom{r}{k-1} + \binom{r}{k} \right) a^k b^{r-k+1} + b^{r+1} \\
&= a^{r+1} + \sum_{k=1}^{r} \binom{r+1}{k} a^k b^{r-k+1} + b^{r+1} \text{ (by Problem 1)} \\
&= \text{R.H.S.}
\end{aligned}$$

By induction, $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ for all $n \in \mathbb{N}$.

In particular, if $a = b = 1$, $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$ and $0 = ((-1)+1)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$ for all $n \in \mathbb{N}$.

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Problem 3. (Apostol 9.6 12)

Let $w = \frac{az+b}{cz+d}$, where a, b, c, and d are real. Prove that $w - \bar{w} = \frac{(ad-bc)(z-\bar{z})}{|cz+d|^2}$. If $ad - bc > 0$, prove that the imaginary parts of z and w have the same sign.

Solution.

$$\begin{aligned} \text{L.H.S.} &= \frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d} \\ &= \frac{(az+b)(c\bar{z}+d) - (a\bar{z}+b)(cz+d)}{(cz+d)\overline{(cz+d)}} \\ &= \frac{adz + bc\bar{z} - ad\bar{z} - bcz}{|cz+d|^2} = \text{R.H.S.} \end{aligned}$$

Since $2 \cdot \text{Im}(w) = w - \bar{w} = \frac{ad-bc}{|cz+d|^2}(z - \bar{z}) = 2\frac{ad-bc}{|cz+d|^2} \cdot \text{Im}(z)$, if $ad - bc > 0$, then $\text{Im}(w)$ and $\text{Im}(z)$ will have the same sign.

Problem 4. (Apostol 9.10 4)

(a) If θ is real, show that $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$.

(b) Use the formula in (a) to deduce the identities $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$.

Solution. (a) $\frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{\cos\theta+i\sin\theta+\cos(-\theta)+i\sin(-\theta)}{2} = \frac{\cos\theta+i\sin\theta+\cos\theta-i\sin\theta}{2} = \frac{2\cos\theta}{2} = \cos\theta$, and $\frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{\cos\theta+i\sin\theta-\cos(-\theta)-i\sin(-\theta)}{2i} = \frac{\cos\theta+i\sin\theta-\cos\theta+i\sin\theta}{2i} = \frac{2i\sin\theta}{2i} = \sin\theta$.

(b) $\cos^2\theta = \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^2 = \frac{e^{i2\theta}+e^{-i2\theta}+2}{4} = \frac{\cos 2\theta+i\sin 2\theta+\cos(-2\theta)+i\sin(-2\theta)+2}{4} = \frac{2\cos 2\theta+2}{4} = \frac{\cos 2\theta+1}{2}$, and $\sin^2\theta = \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^2 = \frac{e^{i2\theta}+e^{-i2\theta}-2}{-4} = \frac{\cos 2\theta+i\sin 2\theta+\cos(-2\theta)+i\sin(-2\theta)-2}{-4} = \frac{2\cos 2\theta-2}{-4} = \frac{1-\cos 2\theta}{2}$.

Problem 5.

Let $x = 1 + y > 1$ be a real number. Show that $x^n > 1 + ny$ and deduce another proof of the fact that the sequence $(x_n)_{n\geq 1}$ where $x_n = x^n$ is divergent, i.e., it does not have a limit.

Solution. By Problem 2, $x^n = (y+1)^n = \sum\limits_{k=0}^{n} \binom{n}{k} y^k = 1 + ny + \sum\limits_{k=2}^{n} \binom{n}{k} y^k > 1 + ny$ for $n > 1$ since $y > 0$.

Assume contrary, let $L \in \mathbb{R}$ be such that $\lim\limits_{n\to\infty} x^n = L$. Let $\epsilon = 1$. As $y > 0$, by Achimedean Principle, there exists $N' \in \mathbb{N}$ such that $N'y > L$. For all $N \in \mathbb{N}$, let $n = \max\{N, N'\}$. Then $n \geq N$, and $|x^n - L| \geq x^n - L > 1 + ny - L \geq 1 + N'y - L > 1 = \epsilon$, so $\lim\limits_{n\to\infty} x^n \neq L$, contradiction. Hence, $(x_n)_{n\geq 1}$ does not have a limit.