MA 1A (SECTION 1) HW3 SOLUTIONS

Problem 1. (Apostol 10.4.10)

Consider the sequence $f(n) = n^{(-1)^n}$.
(a) Determine whether the sequence converges or diverges;
(b) find the limit if it is convergent.

Solution. Assume that $\lim_{n\to\infty} f(n) = L$ for some $L \in \mathbb{R}$. Pick $\epsilon = 1$. For all $N \in \mathbb{N}$, pick $n = 2\max\{N, \lceil \frac{L+1}{2} \rceil\}$. Then $n \geq 2N \geq N$, and $|f(n) - L| = |n^{(-1)^n} - L| = |n^1 - L|$ (as n is even) $\geq n - L \geq 2(\frac{L+1}{2}) - L = 1 = \epsilon$, contradicting that $\lim_{n\to\infty} f(n) = L$. Therefore, $(f(n))_{n\geq 1}$ diverges.

Problem 2.

Let $(z_n)_{n\geq 1}$ be a sequence of complex numbers. Show that $(z_n)_{n\geq 1}$ converges with $\lim_{n\to\infty} z_n = 0$ if and only if the sequence $(|z_n|)_{n\geq 1}$ converges with $\lim_{n\to\infty} |z_n| = 0$.

Solution. For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all n with $n \geq N$, $|z_n - 0| < \epsilon$ if and only if $|z_n| < \epsilon$, which is equivalent to $||z_n| - 0| < \epsilon$. Therefore, $\lim_{n\to\infty} z_n = 0$ is equivalent to $\lim_{n\to\infty} |z_n| = 0$.

Problem 3.

Use the previous exercise and what you learned in class to show that if z is a complex number with $|z| < 1$ then the sequence $(a_n)_{n\geq 1}$ where $a_n = 1 + z + z^2 + \cdots + z^n$ converges, and compute its limit. [Hint: can you recall what we proved in class about the expression a_n?]

Solution. By the formula in class, $a_n = \frac{1-z^{n+1}}{1-z}$. As $|z| < 1$, $\lim_{n\to\infty} |z|^{n+1} = 0$. By Problem 2, we have $\lim_{n\to\infty} z^{n+1} = 0$. Hence, $\lim_{n\to\infty} a_n = \frac{1 - \lim_{n\to\infty} z^{n+1}}{1-z} = \frac{1}{1-z}$ as $z \neq 1$ by $|z| < 1$.

Problem 4.

Find the domain of definition of the function $f(x) = \sqrt{x^2 - 2} + \frac{1}{x^2-4}$, taking the real variable x to the set of real numbers.

Solution. The domain where $\sqrt{x^2 - 2}$ is real is $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$ since $x^2 \geq 2$. The domain where $\frac{1}{x^2-4}$ is real is $\mathbb{R} \backslash \{-2, 2\}$ since $x^2 \neq 4$. Hence, the domain where $f(x)$ is real is $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty) \backslash \{-2, 2\}$.

Date: Fall 2011.

Problem 5.

Show that the function $f : [2, \infty) \to [4, \infty)$ defined by $f(x) = x^2 - 4x + 8$ is bijective and compute its inverse $f^{-1} : [4, \infty) \to [2, \infty)$.

Solution. Let $g(y) = 2 + \sqrt{y - 4}$ with the domain $[4, \infty)$. Then for all $y \geq 4$, $g(y) \geq 2 + \sqrt{4 - 4} = 2$, so $g : [4, \infty) \to [2, \infty)$.

For all $x \in [2, \infty)$, $g(f(x)) = 2 + \sqrt{x^2 - 4x + 8 - 4} = 2 + |x - 2| = x$ since $x \geq 2$. On the other hand, for all $y \in [4, \infty)$, $f(g(y)) = (2 + \sqrt{y - 4})^2 - 4(2 + \sqrt{y - 4}) + 8 = 4 + y - 4 + 4\sqrt{y - 4} - 8 - 4\sqrt{y - 4} + 8 = y$. Therefore, we have $f^{-1}(y) = g(y) = 2 + \sqrt{y - 4}$ and f is bijective.