MA 1A (SECTION 1) HW4 SOLUTIONS

Problem 1. (Apostol 3.8.18)
Calculate the limit $\lim_{x\to 0} \frac{1-\cos 2x}{x^2}$.

Solution. $\lim_{x\to 0} \frac{1-\cos 2x}{x^2} = \lim_{x\to 0} \frac{1-(1-2\sin^2 x)}{x^2} = 2\lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sin x}{x}$ (by Thm 3.1(iii) since $\lim_{x\to 0} 2$ and $\lim_{x\to 0} \frac{\sin x}{x}$ exist) $= 2 \cdot 1 \cdot 1 = 2$.

Problem 2. (Apostol 4.6.38)
Given the formula $1 + x + x^2 + \cdots + x^n = \frac{x^{n+1}-1}{x-1}$ (valid if $x \neq 1$), determine, by differentiation, formulae for the following sums:
(a) $1 + 2x + 3x^2 + \cdots + nx^{n-1}$,
(b) $1^2x + 2^2x^2 + 3^2x^3 + \cdots + n^2x^n$.

Solution. (a) $1 + 2x + 3x^2 + \cdots + nx^{n-1} = \frac{d}{dx}(1 + x + x^2 + \cdots + x^n) = \frac{d}{dx}\frac{x^{n+1}-1}{x-1} = \frac{(x-1)(n+1)x^n-(x^{n+1}-1)1}{(x-1)^2} = \frac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2}$.
(b) $1^2x + 2^2x^2 + 3^2x^3 + \cdots + n^2x^n = x(1^2 + 2^2x + 3^2x^2 + \cdots + n^2x^{n-1}) = x\frac{d}{dx}(x + 2x^2 + 3x^3 + \cdots + nx^n) = x\frac{d}{dx}(x(1 + 2x + 3x^2 + \cdots + nx^{n-1})) = x\frac{d}{dx}\frac{nx^{n+2}-(n+1)x^{n+1}+x}{(x-1)^2}$ (by part (a)) $= x\frac{(x-1)^2(n(n+2)x^{n+1}-(n+1)^2x^n+1)-2(x-1)(nx^{n+2}-(n+1)x^{n+1}+x)}{(x-1)^4} = \frac{n^2x^{n+3}-(2n^2+2n-1)x^{n+2}+(n+1)x^{n+1}-x^2-x}{(x-1)^3}$.

Problem 3. (Apostol 4.9.15)
Given that the derivative $f'(a)$ exists. State which of the following statements are true and which are false. Give a reason for your decision in each case.
(a) $f'(a) = \lim_{h\to a} \frac{f(h)-f(a)}{h-a}$.
(b) $f'(a) = \lim_{t\to 0} \frac{f(a+2t)-f(a)}{t}$.
(c) $f'(a) = \lim_{h\to 0} \frac{f(a)-f(a-h)}{h}$.
(d) $f'(a) = \lim_{t\to 0} \frac{f(a+2t)-f(a+t)}{2t}$.

Solution. (a) True. $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$. Let $\ell = a + h$, so $h = \ell - a$, and $h \to 0$ is equivalent to $\ell - a \to 0$, or $\ell \to a$. Then $f'(a) = \lim_{\ell\to a} \frac{f(\ell)-f(a)}{\ell-a} = \lim_{h\to a} \frac{f(h)-f(a)}{h-a}$ since h and ℓ are dummy variables.
(b) False. $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$. Let $h = 2t$, so $h \to 0$ is equivalent to $2t \to 0$, or $t \to 0$. Then $f'(a) = \lim_{t\to 0} \frac{f(a+2t)-f(a)}{2t} = \frac{1}{2}\lim_{t\to 0} \frac{f(a+2t)-f(a)}{t} \neq \lim_{t\to 0} \frac{f(a+2t)-f(a)}{t}$ unless $f'(a) = 0$.
(c) True. $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$. Let $h = -\ell$, so $h \to 0$ is equivalent to $-\ell \to 0$, or $\ell \to 0$.

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Then $f'(a) = \lim_{\ell\to 0} \frac{f(a-\ell)-f(a)}{-\ell} = \lim_{\ell\to 0} \frac{f(a)-f(a-\ell)}{\ell} = \lim_{h\to 0} \frac{f(a)-f(a-h)}{h}$ since h and ℓ are dummy variables.
(d) False $f'(a) = 2f'(a) - f'(a) = 2 \cdot \frac{1}{2}\lim_{t\to 0} \frac{f(a+2t)-f(a)}{t} - \lim_{t\to 0} \frac{f(a+t)-f(a)}{t}$ (by part (b)) $= \lim_{t\to 0} \frac{f(a+2t)-f(a+t)}{t} \neq \lim_{t\to 0} \frac{f(a+2t)-f(a+t)}{2t}$ unless $f'(a) = 0$.

Problem 4. (Apostol 4.12.14)
Determine the derivative $f'(x)$, where x is restricted to those values for which $f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}$ is meaningful.

Solution. $f'(x) = \frac{\frac{d}{dx}(x+\sqrt{x+\sqrt{x}})}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} = \frac{1+\frac{\frac{d}{dx}(x+\sqrt{x})}{2\sqrt{x+\sqrt{x}}}}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} = \frac{1+\frac{1+\frac{1}{2\sqrt{x}}}{2\sqrt{x+\sqrt{x}}}}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} = \frac{2\sqrt{x+\sqrt{x}}+1+\frac{1}{2\sqrt{x}}}{4\sqrt{x+\sqrt{x}}\sqrt{x+\sqrt{x+\sqrt{x}}}} = \frac{4\sqrt{x}\sqrt{x+\sqrt{x}}+2\sqrt{x}+1}{8\sqrt{x}\sqrt{x+\sqrt{x}}\sqrt{x+\sqrt{x+\sqrt{x}}}}$.

Problem 5. (Apostol 4.12.30)
The equation $x^3 + y^3 = 1$ defines y as one or more functions of x.
(a) Assuming the derivative y' exists, and without attempting to solve for y, show that y' satisfies the equation $x^2 + y^2 y' = 0$.
(b) Assuming the second derivative y'' exists, show that $y'' = -2xy^{-5}$ whenever $y \neq 0$.

Solution. (a) $0 = \frac{d}{dx}1 = \frac{d}{dx}(x^3 + y^3) = 3x^2 + 3y^2\frac{dy}{dx}$, so $x^2 + y^2 y' = 0$ since $y' = \frac{dy}{dx}$ by definition.
(b) Note that $y' = -\frac{x^2}{y^2}$ if $y \neq 0$, so $y'' = -\frac{d}{dx}\frac{x^2}{y^2} = -\frac{y^2\cdot 2x - x^2\cdot 2yy'}{y^4} = -\frac{2xy^2-2x^2y(-\frac{x^2}{y^2})}{y^4} = -\frac{2xy^3+2x^4}{y^5} = -2xy^{-5}(x^3+y^3) = -2xy^{-5}$.