MA 1A (SECTION 1) HW5 SOLUTIONS

Problem 1. (Apostol 4.15.4)

Let $f(x) = 1 - x^{\frac{2}{3}}$. Show that $f(1) = f(-1) = 0$, but that $f'(x)$ is never zero in the interval $[-1, 1]$. Explain how this is possible, in view of Rolle's theorem.

Solution. $f(1) = 1 - 1^{\frac{2}{3}} = 0$, and $f(-1) = 1 - [(-1)^2]^{\frac{1}{3}} = 1 - 1 = 0$.
$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}} \neq 0$ for all $x \in \mathbb{R}\backslash\{0\}$, and $f'(x)$ does not exist at $x = 0$, so $f'(x)$ is never zero in the interval $[-1, 1]$.
Rolle's theorem works only if $f'(x)$ exists for all x in the interval $(-1, 1)$. As $f'(x)$ does not exist at $x = 0$, we cannot apply the theorem.

Problem 2. (Apostol 4.15.8)

Use the mean-value theorem to deduce the following inequalities:
(a) $|\sin x - \sin y| \leq |x - y|$.
(b) $ny^{n-1}(x - y) \leq x^n - y^n \leq nx^{n-1}(x - y)$ if $0 < y \leq x$, $n \in \mathbb{N}$.

Solution. (a). Without loss of generality, let $x \geq y$. $f(t) = \sin t$ is continuous on $[y, x]$ and is differentiable on (y, x), with $f'(t) = \cos t$. By mean-value theorem, there exists $z \in (x, y)$ such that $\sin x - \sin y = \cos z(x - y)$, so $|\sin x - \sin y| = |\cos z||x - y| \leq |x - y|$.
(b). $f(t) = t^n$ is continuous on $[y, x]$ and is differentiable on (y, x), with $f'(t) = nt^{n-1}$. By mean-value theorem, there exists $z \in (x, y)$ such that $x^n - y^n = nt^{n-1}(x - y)$. Note that $f(t) = t^n$ is increasing on $[0, +\infty)$, so $ny^{n-1}(x - y) \leq x^n - y^n \leq nx^{n-1}(x - y)$.

Problem 3. (Apostol 4.19.4)

$f(x) = x^3 - 6x^2 + 9x + 5$.
(a) Find all points x such that $f'(x) = 0$.
(b) Examine the sign of f' and determine those intervals in which f is monotonic.
(c) Examine the sign of f'' and determine those intervals in which f' is monotonic.
(d) Make a sketch of the graph f.

Solution. (a) $f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$, so $f'(x) = 0$ if and only if $x = 1$ or 3.
(b) $f'(x) = 3(x - 1)(x - 3)$, so $f'(x) > 0$ if $x < 1$ or $x > 3$, and $f'(x) < 0$ if $1 < x < 3$. Hence, $f(x)$ is increasing on $(-\infty, 1] \cup [3, \infty)$ and decreasing on $[1, 3]$.
(c) $f''(x) = 6x - 12 = 0$ if $x = 2$, $f''(x) > 0$ if $x > 2$, and $f''(x) < 0$ if $x < 2$. Hence, $f'(x)$ is increasing on $[2, \infty)$ and decreasing on $(-\infty, 2]$.
(d) Local maximum at $x = 1$ with $f(x) = 9$ and local minimum at $x = 3$ with $f(x) = 5$. y-intercept is 5.

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Problem 4. (Apostol 4.21.4)

Given $S > 0$. Prove that among all positive numbers x and y with $x + y = S$, the sum $x^2 + y^2$ is smallest when $x = y$.

Solution. $y = S - x$, so $x^2 + y^2 = x^2 + (S - x)^2 = 2x^2 - 2xS + S^2$. Taking the derivative with respect to x, we get $4x - 2S$, which is zero if and only if $x = \frac{S}{2}$. As the second derivative is $4 > 0$, $x^2 + y^2$ has a local minimum at $x = \frac{S}{2}$. As $f(x) \to \infty$ when $x \to \pm\infty$, $x^2 + y^2$ has the global minimum at $x = \frac{S}{2}$, and this is exactly when $x = y$.

Problem 5. (Apostol 4.21.20)

A cylinder is obtained by revolving a rectangle about the x-axis, the base of the rectangle lying on the x-axis and the entire rectangle lying in the region between the curve $y = \frac{x}{x^2+1}$ and the x-axis. Find the maximum possible volume of the cylinder.

Solution. $y = \frac{x}{x^2+1}$ is an odd function. By symmetry, we take the branch $x, y \geq 0$. $y = \frac{x}{x^2+1} \Rightarrow x = \frac{1 \pm \sqrt{1-4y^2}}{2y}$, so for each given $0 \leq y \leq \frac{1}{2}$, the length of the base on the x-axis is $\frac{1+\sqrt{1-4y^2}}{2y} - \frac{1-\sqrt{1-4y^2}}{2y} = \frac{\sqrt{1-4y^2}}{y}$, and the corresponding volume of the cylinder is $\pi y^2 \frac{\sqrt{1-4y^2}}{y} = \pi y \sqrt{1-4y^2}$. Taking the derivative with respect to y, we get $\sqrt{1-4y^2} + y \frac{1}{2} \frac{-8y}{\sqrt{1-4y^2}} = \frac{1-8y^2}{\sqrt{1-4y^2}}$, which is zero if and only if $y = \frac{\sqrt{2}}{4}$. As the second derivative at $y = \frac{\sqrt{2}}{4}$ is $\left.\frac{-12y+32y^3}{(1-4y^2)^{\frac{3}{2}}}\right|_{y=\frac{\sqrt{2}}{4}} = -8 < 0$, $\pi y \sqrt{1-4y^2}$ has local maximum at $y = \frac{\sqrt{2}}{4}$. $\left.\pi y \sqrt{1-4y^2}\right|_{y=0} = 0$ and $\left.\pi y \sqrt{1-4y^2}\right|_{y=\frac{1}{2}} = 0$, so $\pi y \sqrt{1-4y^2}$ has the global maximum at $y = \frac{\sqrt{2}}{4}$, and $\left.\pi y \sqrt{1-4y^2}\right|_{y=\frac{\sqrt{2}}{4}} = \frac{\pi}{4}$.