MA 1A (SECTION 1) HW6 SOLUTIONS

Problem 1.
For $r > 0$ compute $\lim_{n\to\infty} n(r^{\frac{1}{n}} - 1)$.

Solution. $\lim_{n\to\infty} n(r^{\frac{1}{n}} - 1) = \lim_{n\to\infty} \frac{r^{\frac{1}{n}} - r^0}{\frac{1}{n} - 0} = \frac{d}{dx} r^x\Big|_{x=0} = \log r$.

Problem 2.
Let $0 < a < b$ and T_n the following partition:

$$T_n = \left[a, a\left(\tfrac{b}{a}\right)^{\frac{1}{n}}\right] \cup \left[a\left(\tfrac{b}{a}\right)^{\frac{1}{n}}, a\left(\tfrac{b}{a}\right)^{\frac{2}{n}}\right] \cup \cdots \cup \left[a\left(\tfrac{b}{a}\right)^{\frac{j-1}{n}}, a\left(\tfrac{b}{a}\right)^{\frac{j}{n}}\right] \cup \cdots \cup \left[a\left(\tfrac{b}{a}\right)^{\frac{n-1}{n}}, b\right].$$

(a) Show that $f : [a, b] \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ is integrable.
(b) Compute $\lim_{n\to\infty} L(f, T_n)$ and deduce that

$$\int_a^b \frac{dx}{x} = \log b - \log a.$$

Solution. (a) Let $s_n, t_n : [a, b] \to \mathbb{R}$ be such that for all $x \in \left[a\left(\frac{b}{a}\right)^{\frac{j-1}{n}}, a\left(\frac{b}{a}\right)^{\frac{j}{n}}\right]$, $s_n(x) = \frac{1}{a}\left(\frac{a}{b}\right)^{\frac{j}{n}}$ and $t_n(x) = \frac{1}{a}\left(\frac{a}{b}\right)^{\frac{j-1}{n}}$, $j = 1, 2, \ldots, n$.

$\int_a^b s_n(x)dx = \sum_{j=1}^{n} \frac{1}{a}\left(\frac{a}{b}\right)^{\frac{j}{n}} \left[a\left(\frac{b}{a}\right)^{\frac{j}{n}} - a\left(\frac{b}{a}\right)^{\frac{j-1}{n}}\right] = n\left(1 - \left(\frac{a}{b}\right)^{\frac{1}{n}}\right) \xrightarrow{n\to\infty} -\log\frac{a}{b}$ by Problem 1, and $\int_a^b t_n(x)dx = \sum_{j=1}^{n} \frac{1}{a}\left(\frac{a}{b}\right)^{\frac{j-1}{n}} \left[a\left(\frac{b}{a}\right)^{\frac{j}{n}} - a\left(\frac{b}{a}\right)^{\frac{j-1}{n}}\right] = n\left(\left(\frac{b}{a}\right)^{\frac{1}{n}} - 1\right) \xrightarrow{n\to\infty} \log\frac{b}{a}$ by Problem 1. As $-\log\frac{a}{b} = \log\frac{b}{a} = \log b - \log a$, the upper and lower integrals of f are equal. By Theorem 1.9, f is integrable over $[a, b]$.
(b) By part (a), $\lim_{n\to\infty} L(f, T_n) = \log\frac{b}{a} = \log b - \log a$. By Theorem 1.9, $\lim_{n\to\infty} L(f, T_n) = \log b - \log a$.

Problem 3. (Apostol 2.4.1)
Let $f(x) = 4 - x^2$, $g(x) = 0$, $a = -2$, $b = 2$. Compute the area of the region S between the graphs of f and g over the interval $[a, b]$. Make a sketch of the two graphs and indicate S by shading.

Solution. For all $x \in [-2, 2]$, $f(x) = 4 - x^2 \geq 4 - 4 = 0 = g(x)$. Hence, the area of S $= \int_{-2}^{2} f(x) - g(x) = \int_{-2}^{2} 4 - x^2 = 4x - \frac{x^3}{3}\Big|_{-2}^{2} = 16 - \frac{16}{3} = \frac{32}{3}$.

Problem 4. (Apostol 2.4.15)
The graphs of $f(x) = x^2$ and $g(x) = cx^3$, where $c > 0$, intersect at the points $(0, 0)$ and $(\frac{1}{c}, \frac{1}{c^2})$. Find c so that the region which lies between these graphs and over the interval $[0, \frac{1}{c}]$

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has area $\frac{2}{3}$.

Solution. For all $x \in [0, \frac{1}{c}]$, $f(x) = x^2 \geq x^2 \frac{x}{\frac{1}{c}} = cx^3 = g(x)$. Hence, the area of the region $= \int_0^{\frac{1}{c}} f(x) - g(x) dx = \int_0^{\frac{1}{c}} x^2 - cx^3 = \frac{x^3}{3} - c\frac{x^4}{4}\Big|_0^{\frac{1}{c}} = \frac{1}{3c^3} - \frac{1}{4c^3} = \frac{1}{12c^3} = \frac{2}{3} \Rightarrow 8c^3 = 1 \Rightarrow c = \frac{1}{2}$.

Problem 5. (Apostol 1.15.7)

Let $[x]$ denote the greatest integer $\leq x$.

(a) Compute $\int_0^9 [\sqrt{t}] dt$.

(b) If n is a positive integer, prove that $\int_0^{n^2} [\sqrt{t}] dt = \frac{n(n-1)(4n+1)}{6}$.

Solution. (a) $[\sqrt{t}] = \begin{cases} 0 & \text{if } 0 \leq t < 1; \\ 1 & \text{if } 1 \leq t < 4; \\ 2 & \text{if } 4 \leq t < 9. \end{cases}$. So $\int_0^9 [\sqrt{t}] dt = 0(1-0) + 1(4-1) + 2(9-4) = 13$.

(b) When $n = 1$, L.H.S. $= \int_0^{1^2} [\sqrt{t}] dt = 0$, and R.H.S. $= \frac{1(1-1)(4\times 1+1)}{6} = 0 =$ L.H.S.

Assume that $\int_0^{k^2} [\sqrt{t}] dt = \frac{k(k-1)(4k+1)}{6}$ for some $k \in \mathbb{N}$. When $n = k+1$, L.H.S. $= \int_0^{(k+1)^2} [\sqrt{t}] dt = \int_0^{k^2} [\sqrt{t}] dt + \int_{k^2}^{(k+1)^2} [\sqrt{t}] dt = \frac{k(k-1)(4k+1)}{6} + k((k+1)^2 - k^2) = k\big(\frac{4k^2-3k-1}{6} + 2k+1\big) = \frac{k(k+1)(4k+1)}{6} =$ R.H.S. Hence, by induction, $\int_0^{n^2} [\sqrt{t}] dt = \frac{n(n-1)(4n+1)}{6}$ for all $n \in \mathbb{N}$.