Complex Representations of $\mathrm{GL}(2, \mathbb{F}_q)$

Andrei Jorza

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The complex representation theory of GL(2) over finite fields is explained well in many places, and it an excellent toy setting for graduate students who want to study Jacquet-Langlands. For other sources see for instance Piatetski-Shapiro's book and Paul Garrett's notes.

These exercises are intended for graduate students who want to study the representation theory of $\mathrm{GL}(2, \mathbb{F}_q)$ hands-on, as a guide to the various results.

Contents

1 Representation theory of finite groups

1.1 Basic properties and constructions

Exercise 1.1.1 (Representations as modules over the group ring) Let G be a group.

1. Suppose (V, ρ) is a representation of G on the K-vector space V. Show that V is a $K[G]$-module under $\sum a_g[g] \cdot v = \sum a_g \rho(g)(v)$.

2. Suppose V is a $K[G]$-module. Show that $\rho(g)v := g \cdot v$ defines a representation of G.

Exercise 1.1.2 (Semisimplicity of representations: Maschke's theorem) Let K be a field and G a finite group such that either $\operatorname{char} K = 0$ or $\operatorname{char} K \nmid |G|$. Let V be a representation of G and $W \subset V$ a G-subrepresentation.

1. Let $\pi : V \to W$ be any vector space projection, i.e., a linear map such that $\pi|_W$ is the identity map. Show that $\sigma : V \to W$ given by $\sigma(v) = \frac{1}{|G|} \sum_{g \in G} \pi(\rho(g)(v))$ is a well-defined G-equivariant linear map.

2. Show that $U = \ker \sigma \subset V$ is a G-subrepresentation and $V = W \oplus U$ is a G-representation decomposition.

3. Conclude that all finite dimensional G-representations are semisimple.

4. Show that the representation of $\mathbb{Z}/2\mathbb{Z}$ acting on $\mathbb{F}_2^2$ such that $1 \in \mathbb{Z}/2\mathbb{Z}$ sends (x, y) to $(x + y, y)$ is a non-semisimple representation.

Exercise 1.1.3 (Dual representations) Let (V, ρ) be a representation of G over a field K. Let $V^\vee = \mathrm{Hom}_K(V, K)$ and $(\rho^\vee(g)f)(v) = f(\rho(g)^{-1}(v))$.

1. Show that $(V^\vee, \rho^\vee)$ is a representation of G.

2. Show that $V \otimes V^\vee \cong \mathrm{End}_K(V)$ where G acts on $f \in \mathrm{End}_K(V)$ by $(gf)(v) = \rho(g)(f(\rho(g)^{-1}(v)))$.

3. Show that the scalar matrices in $\mathrm{End}_K(V)$ form a one-dimensional irreducible subrepresentation of $V \otimes V^\vee$ isomorphic to the trivial character.

4. Show that V is irreducible if and only if $\dim \mathrm{Hom}_K(1, V \otimes V^\vee) = 1$. [Hint: Use Schur's lemma.]

Exercise 1.1.4 (Induced representations) Let G be a profinite group and H a finite index subgroup.

1. Let (V, ρ) be a representation of H over a field K. Define $(\mathrm{Ind}_H^G V, \mathrm{Ind}_H^G \rho)$ as follows: $\mathrm{Ind}_H^G V$ is the vector space $\{f : G \to V | f(hg) = \rho(h)(f(g)), \forall h \in G, g \in G\}$ and if $g \in G$ and $f \in \mathrm{Ind}_H^G V$ then the action is $((\mathrm{Ind}_H^G \rho)(g)f)(x) = f(xg)$. Show that $\mathrm{Ind}_H^G V$ is a representation of G.

2. Let M be an H-module, by which we mean an abelian group with an action of G. Define $\mathrm{Ind}_H^G M$ analogously. Show that $\mathrm{Ind}_H^G M$ is a G-module.

1.2 Induced representations

Exercise 1.2.1 (Induction as an operation on modules over the group ring) Let H, G and (ρ, V) be as in Exercise 1.1.4. Fix representatives $G/H = \cup g_j H$ and a basis v_i of V.

1. Show that $\mathrm{Ind}_H^G V$ has as basis functions $f_{v_i, g_j} : G \to V$ taking g_j to v_i and g_k to 0 for $k \neq j$.

2. Show that the map $V \otimes_{K[H]} K[G] \to \mathrm{Ind}_H^G V$

$$\sum a_{v_i, g_j} v_i \otimes [g_j] \mapsto \sum a_{v_i, g_j} f_{v_i, g_j}$$

is a G-equivariant isomorphism of K-vector spaces.

Exercise 1.2.2 (Frobenius reciprocity) Suppose H is a finite index subgroup of a profinite group G, M is an H-module and N is a G-module. Show that

$$\mathrm{Hom}_G(N, \mathrm{Ind}_H^G M) \cong \mathrm{Hom}_H(N, M)$$

sending a G-equivariant map $f : N \to \mathrm{Ind}_H^G M$ to the H-equivariant map $n \mapsto f(n)(1)$ in $\mathrm{Hom}_H(M, N)$ and the H-equivariant map g to the G-equivariant map $n \mapsto (g \mapsto f(g(n)))$ in $\mathrm{Hom}_G(N, \mathrm{Ind}_H^G M)$.

Exercise 1.2.3 (Restriction of induced representations) Let H and N be finite index subgroups of a profinite group G and let (ρ, V) be a representation of H over a field K.

1. For $g \in G$ define $V^g = V$ and $\rho^g(x) = \rho(gxg^{-1})$. Show that (ρ^g, V^g) is a representation of $g^{-1}Hg$.

2. Show that $f \mapsto f^g$ (defined as $f^g(x) = f(gxg^{-1})$ gives an isomorphism $(\mathrm{Ind}_H^G V)^g \cong \mathrm{Ind}_{g^{-1}Hg}^G (V^g)$.

3. Show that the map $\phi \mapsto \oplus_{g \in H\backslash G/N}(n \mapsto \phi(ng))$ gives an isomorphism of N-representations

$$(\mathrm{Ind}_H^G V)|_N \cong \oplus_{g \in H\backslash G/N} (\mathrm{Ind}_{H \cap gNg^{-1}}^{gNg^{-1}} W)^g \cong \oplus_{g \in H\backslash G/N} \mathrm{Ind}_{g^{-1}Hg \cap N}^N (W^g)$$

4. In the special case when N is a normal subgroup of G show that

$$(\mathrm{Ind}_H^G V)|_N \cong \oplus_{g \in H\backslash G/N} (\mathrm{Ind}_{H \cap N}^N V)^g$$

Exercise 1.2.4 (Irreducibility of induced representations) Let H be a finite index subgroup of a profinite group G and V a G-representation.

1. Suppose $V|_H = W_1 \oplus \cdots \oplus W_n$ where W_i are non-isomorphic irreducible H-representations. Suppose that for any $i \neq j$ there exists $g \in G$ and $w \in W_i$ such that the projection of $g(w)$ to W_j is nonzero. Show that V is irreducible.

2. Suppose $H \lhd G$ and W is an H-representation such that $W^g \not\cong W$ for any $g \in G - H$. Show that $\operatorname{Ind}_H^G W$ is an irreducible G-representation.

Exercise 1.2.5 (Induction and tensor product) Let H be an open subgroup of the profinite group G.

1. Suppose V is an H-representation and W is a G-representation. Show that

$$\operatorname{Ind}_H^G(V) \otimes W \cong \operatorname{Ind}_H^G(V \otimes W|_H)$$

2. Suppose V, W are representations of H. Show that

$$\operatorname{Ind}_H^G V \otimes \operatorname{Ind}_H^G W \cong \oplus_{g \in H\backslash G/H} \operatorname{Ind}_H^G((\operatorname{Ind}_{H \cap gHg^{-1}}^{gHg^{-1}} V)^g \otimes W)$$

Exercise 1.2.6 (Induced characters) Let $H \subset G$ be finite groups and V a representation of H. Show that

$$\operatorname{Tr} \operatorname{Ind}_H^G(\rho)(g) = \sum_{k \in H\backslash G} \operatorname{Tr}(\rho(kgk^{-1}))$$

1.3 Explicit examples of representations of finite groups

Exercise 1.3.1 (The standard representation of S_3) Consider the character $\tau : A_3 \cong \mathbb{Z}/3\mathbb{Z} \to \mathbb{C}^\times$ sending (123) to ζ_3. Show that $\operatorname{Ind}_{A_3}^{S_3} \tau$ is isomorphic to the permutation representation of S_3 on $\{(x, y, z) \in \mathbb{C} | x + y + z = 0\}$.

Exercise 1.3.2 (The standard representation of S_n) Let (V, ρ) be the permutation representation of S_n on $\mathbb{C}^n$.

1. Show that $U = \{(x, \ldots, x) \in \mathbb{C}^n | x \in \mathbb{C}\}$ and $W = \{(x_1, \ldots, x_n) | \sum x_i = 0\}$ are S_n-subrepresentations of V and $V = U \oplus W$.

2. Show that if $\sigma \in S_n$ then $\operatorname{Tr} \rho(\sigma)$ is the number of fixed points of the permutation σ.

3. Let $s_{n,k}$ be the number of $\sigma \in S_n$ with exactly k fixed points. Show that

$$||\chi_V||^2 = \frac{1}{n!} \sum_{k=0}^{n} k^2 s_{n,k}$$

4. Show that

$$s_{n,0} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! \approx \frac{n!}{e}$$

and that

$$s_{n,k} = \binom{n}{k} s_{n-k,0}$$

5. Conclude that

$$||\chi_V||^2 \approx 2$$

and use the fact that $||\chi_v||^2$ is an integer to show that it is 2.

6. Deduce that W is irreducible. This is the standard representation of S_n.

Exercise 1.3.3 (The irreducible representations of S_4)

1. Show that the dimensions of the irreducible complex representations of S_4 are $1, 1, 2, 3, 3$. [Hint: You already know two characters (the trivial and the sign) and the three dimensional standard.]

2. If ε is the sign character show that the two 3-dimensional representations are std and std $\otimes \varepsilon$.

3. Show that $[A_4, A_4] = V = \{1, (12)(34), (13)(24), (14)(23)\} \cong (\mathbb{Z}/2\mathbb{Z})^2$ and thus the abelizanization of A_4 is $\mathbb{Z}/3\mathbb{Z} \cong \{1, (123), (132)\}$. Let τ be the character of A_4 sending (123) to ζ_3. Show that $\operatorname{Ind}_{A_4}^{S_4} \tau$ is the irreducible 2-dimensional representation.

4. An alternate construction of the irreducible 2-dimensional representation of S_4. Show that $S_4/V \cong S_3$ and thus that the irreducible 2 dimensional representation of S_4 is the standard representation of S_3.

Exercise 1.3.4 (Representations of finite Heisenberg groups) Let p be a prime, $q = p^m$ for some $m \geq 1$ and G be the set of matrices $(n+2) \times (n+2)$ of the form

$$m(a_i, b_i, c) = \begin{pmatrix} 1 & a_1 & \dots & a_n & c \\ & 1 & 0 & \dots & b_1 \\ & & \ddots & & \vdots \\ & & & 1 & b_n \\ & & & & 1 \end{pmatrix}$$

where $a_i, b_i, c \in \mathbb{F}_q$.

1. Show that under matrix multiplication G is a group of size q^{2n+1}. (It is called the *Heisenberg group*.)

2. Let $H \subset G$ be the subset of matrices $m(a_i, b_i, c)$ with $a_1 = \ldots = a_n = 0$. Show that H is an abelian subgroup of G, isomorphic to $\mathbb{F}_q^{n+1}$.

3. Show that $Z(G)$ consists of those $m(a_i, b_i, c)$ with $a_i = 0$ and $b_i = 0$ for all i.

4. Show that $[G, G] = Z(G)$. (A p-group whose commutant equals its center and is isomorphic to a cyclic group of size p is said to be *extraspecial*. When $q = p$ this shows that G is an extraspecial group.)

5. Choose a collection of characters $\chi_i, \eta_j : \mathbb{F}_q \to \mathbb{C}^\times$ for $1 \leq i, j \leq n$. Show that

$$\chi(m(a_i, b_i, c)) = \prod \chi_i(a_i) \prod \eta_j(b_j)$$

   gives a character $\chi : G \to \mathbb{C}^\times$.

6. Suppose $\eta, \chi_1, \ldots, \chi_n : \mathbb{F}_q \to \mathbb{C}^\times$ and define $\chi(m(0, b_i, c)) = \eta(c) \prod \chi_i(b_i)$.

   (a) Show that χ is a character of H.

   (b) If η is not the trivial character show that $\operatorname{Ind}_H^G \chi$ is an irreducible representation of dimension q^n.

   (c) If η is as above show that

$$\operatorname{Tr} \operatorname{Ind}_H^G \chi(m(0,0,c)) = q^n \eta(c)$$

   and conclude that different characters η give nonisomorphic induced representations.

7. Show that the irreducible complex representations of G are the q^{2n} characters and the $q-1$ irreducible induced representations above. [Hint: Look at dimensions.]

1.4 Complex representations of $\mathrm{GL}(2)$ over finite fields

The exercises of this section are consecutive and the notation is common.

Exercise 1.4.1 (Structure of GL(2) over finite fields) Let p be a prime and $G = \mathrm{GL}(2, \mathbb{F}_q)$ where $q = p^r$. Let $B \subset G$ be the upper triangular matrices (the Borel subgroup) and let $T \subset B$ be the diagonal matrices.

1. Let $w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. Show that $G = B \sqcup BwB$ and conclude that $B \backslash G / B = \{1, w\}$.

2. Show that $B \backslash G \cong \mathbb{P}^1(\mathbb{F}_q)$.

3. Let $g \in \mathrm{GL}(2, \mathbb{F}_q)$ with characteristic polynomial $P_g(X)$ with roots $\lambda_1, \lambda_2 \in \overline{\mathbb{F}}_q$.
   (a) If $\lambda_1 = \lambda_2 = \lambda$ show that $\lambda \in \mathbb{F}_q$ and that g is conjugate in $\mathrm{GL}(2, \mathbb{F}_q)$ to
   $$\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} \text{ or } \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}$$
   and there are $q-1$ conjugacy classes of the first type (called semisimple singular) and $q-1$ of the second type (called nonsemisimple singular). [Remark: the characteristic 2 case requires special care.]
   (b) If $\lambda_1, \lambda_2 \in \mathbb{F}_q$ show that g is conjugate in $\mathrm{GL}(2, \mathbb{F}_q)$ to
   $$\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$$
   This is the split regular case and show that there are $(q-1)(q-2)/2$ split regular conjugacy classes.
   (c) If $\lambda_1, \lambda_2 \notin \mathbb{F}_q$ then $\lambda_1, \lambda_2 \in \mathbb{F}_{q^2}$ and $\lambda_1 = \tau(\lambda_2)$ where τ is Frobenius on $\mathbb{F}_q$. Writing $\lambda = \lambda_2$ show that g is conjugate in $\mathrm{GL}(2, \mathbb{F}_q)$ to
   $$\begin{pmatrix} & -N_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\lambda) \\ 1 & \mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\lambda) \end{pmatrix}$$
   This is the nonsplit regular case and show that there are $(q^2-q)/2$ nonsplit regular conjugacy classes.

Exercise 1.4.2 (The principal series representations) Suppose $\chi_1, \chi_2 : \mathbb{F}_q^\times \to \mathbb{C}^\times$ are two characters.

1. Show that
   $$\eta\left(\begin{pmatrix} a & b \\ & d \end{pmatrix}\right) := \chi_1(a)\chi_2(d)$$
   defines a character of B.
2. We denote $I(\eta) := \mathrm{Ind}_B^G \eta$. Show that $I(\eta)^\vee \cong I(\eta^{-1})$. [Hint: Show that $(\mathrm{Ind}_H^G V)^\vee \cong \mathrm{Ind}_H^G(V^\vee)$.]
3. Show that $I(\eta)|_B \cong \oplus_{x \in B\backslash G/B} \mathrm{Ind}_{B \cap xBx^{-1}}^B(\eta^x)$ where $\eta^x : B \cap xBx^{-1} \to \mathbb{C}$ is defined by $\eta^x(b) = \eta(x^{-1}bx)$.
4. Deduce that
   $$I(\eta) \otimes I(\eta)^\vee \cong I(1) \oplus \mathrm{Ind}_T^G(\eta^w/\eta)$$
   [Hint: Recall that if $K \subset H \subset G$ and V is a representation of K then $\mathrm{Ind}_K^G V \cong \mathrm{Ind}_H^G \mathrm{Ind}_K^H V$.]
5. Show that
   $$\dim \mathrm{Hom}(1, I(\eta) \otimes I(\eta)) = 1 + \dim \mathrm{Hom}_T(1, \eta^x/\eta)$$
   and conclude that $I(\eta)$ is irreducible if and only if $\chi_1 \neq \chi_2$ in which case $I(\eta)$ is called the principal series representation of (χ_1, χ_2). If $\mathbb{F}_q^\times = \langle a \rangle$ show that the map attaching to $I(\chi_1, \chi_2)$ the matrix $\begin{pmatrix} \chi_1(a) & \\ & \chi_2(a) \end{pmatrix}$ gives a bijection between the principal series representations and the split regular conjugacy classes.
6. If $\chi_1 = \chi_2 = \chi$ show that $G \to \mathbb{C}^\times$ given by $g \mapsto \chi(\det(g))$ is a 1-dimensional subrepresentation of $I(\eta)$. Show that the map $\chi \circ \det \mapsto \begin{pmatrix} \chi(a) & \\ & \chi(a) \end{pmatrix}$ gives a bijection between the 1-dimensional irreducible representations of G and the semisimple singular conjugacy classes.

7. When $\chi_1 = \chi_2 = \chi$ decompose $I(\eta) = \chi \circ \det \oplus \mathrm{St}_\chi$ where St_χ is called the Steinberg representation. Show that St_χ is irreducible and that $\mathrm{St}_\chi \cong \mathrm{St}_1 \otimes \chi$. Show that the map $\mathrm{St}_\chi \mapsto \begin{pmatrix} \chi(a) & 1 \\ & \chi(a) \end{pmatrix}$ gives a bijection between the Steinberg representations and the nonsemisimple singular conjugacy classes.

Exercise 1.4.3 (Jacquet modules) Let $U = \{\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{F}_q)\}$, the unipotent radical of B and $T = \{\begin{pmatrix} x & \\ & y \end{pmatrix}$ the diagonal torus in $\mathrm{GL}(2, \mathbb{F}_q)\}$. If (V, ρ) is a representation then $J_U(\rho)$ is the quotient representation V/V_U where $V_U \subset V$ is the subspace generated by the vectors $\{\rho(u)v - v | u \in U, v \in V\}$.

1. Show that V_U is the set of $v \in V$ such that

$$\sum_{u \in U} \rho(u)v = 0$$

2. Show that if $t \in T$ then $\rho(t)$ is an automorphism of $J_U(\rho)$ and thus $J_U(\rho)$ is a T-representation. [Hint: Use the previous criterion.]

3. Show that a G-equivariant morphism $f : V \to W$ where V, W are two G-representations gives a T-equivariant morphism $J_U(V) \to J_U(W)$ and thus J_U is a functor from G-representations to T-representations.

4. Show that J_U is an exact functor, i.e., if $0 \to A \to B \to C \to 0$ is an exact sequence of G-representations then

$$0 \to J_U(A) \to J_U(B) \to J_U(C) \to 0$$

is an exact sequence of T-representations.

5. Suppose $\chi : B \to \mathbb{C}^\times$ is a character. Show that $\mathrm{Hom}_B(V|_B, \chi) \cong \mathrm{Hom}_T(J_U(V), \chi)$ and conclude that

$$\mathrm{Hom}_G(V, \mathrm{Ind}_B^G \chi) \cong \mathrm{Hom}_T(J_U(V), \chi)$$

6. Show that every finite dimensional representation of T is abelian and thus a representation V of G can be realized inside some $\mathrm{Ind}_B^G \chi$ if and only if $J_U(V) \neq 0$.

Exercise 1.4.4 (The cuspidal representations, the ones corresponding to the nonsplit regular conjugacy classes) Let $M = \{\begin{pmatrix} a & b \\ & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{F}_q)\}$, called the mirabolic subgroup. A representation V of G is said to be cuspidal if $J_U(V) = 0$, which, by the previous part, is equivalent to $V \not\subset \mathrm{Ind}_B^G \chi$ for any χ. We already classified the non-cuspidal representations and would like to classify and construct the cuspidal ones.

1. Show that there are $q(q-1)/2$ nonisomorphic irreducible cuspidal representations of G.

2. Suppose V is an irreducible cuspidal representation of G. Show that $\mathrm{Hom}_U(1, V) = 0$ and conclude that $V|_U$, which is a finite dimensional representation of the abelian group U, is a direct sum of nontrivial characters of U.

3. Let $\psi : U \cong \mathbb{F}_q \to \mathbb{C}^\times$ be such a nontrivial character contained in $V|_U$. Show that $\mathrm{Ind}_U^M \psi$ is irreducible of dimension $q - 1$.

4. Deduce that $V|_M$ contains $\mathrm{Ind}_U^M \psi$ and thus that cuspidal representations have dimension $\geq q - 1$.

5. Use the dimension formula $\sum(\dim V_i)^2 = |G|$ to show that every cuspidal representation has dimension exactly $q - 1$.

Exercise 1.4.5 (Whittaker models for noncuspidal representations) A Whittaker model for a G-representation V is any nonzero map $V \to W(\psi) = \mathrm{Ind}_U^G \psi$. Suppose $V = \mathrm{Ind}_B^G \eta$ where $\eta = (\chi_1, \chi_2)$ is a character on T extended to B by acting trivially on U.

1. Show that for nontrivial $\psi : U \to \mathbb{C}^\times$

$$\mathrm{Hom}_U(\mathrm{Ind}_B^G \eta, \psi) = \oplus_{g \in B\backslash G/U} \mathrm{Hom}_{U \cap g^{-1}Ug}(\eta^g, \psi)$$

2. Show that $B\backslash G/U = \{1, w\}$ and conclude that $\dim \mathrm{Hom}_U(\mathrm{Ind}_B^G \eta, \psi) = 1$ and thus that $\dim \mathrm{Hom}_G(\mathrm{Ind}_B^G \eta, \mathrm{Ind}_U^G(\psi)) = 1$, which means that noncuspidal representations have Whittaker models.

Exercise 1.4.6 (L-parameters) Let $\Gamma_q = \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$.

1. Show that as profinite topological groups $\Gamma_q \cong \widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}/n\mathbb{Z}$ generated topologically by Frob_q taking x to x^q.

2. Show that a homomorphism $\rho : \Gamma_q \to \mathrm{GL}(2, \mathbb{C})$ is continuous if and only if $\ker \rho \subset \Gamma_q$ is an open subgroup.

3. The Weil group of $\mathbb{F}_q$ is $W_q = \mathrm{Frob}_q^{\mathbb{Z}} \subset \Gamma_q$ and denote by $W_{q,n}$ its projection to $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$. Show that the natural action of $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ on $\mathbb{F}_{q^n}$ gives an extension

$$1 \to \mathbb{F}_{q^n}^\times \to \widetilde{\Gamma}_{q,n} \to \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \to 1$$

such that $(\Gamma_{q,n})_n$ forms an inverse system.

4. Let $\widetilde{\Gamma}_q = \varprojlim \widetilde{\Gamma}_{q,n}$ and let $\widetilde{W}_q \subset \widetilde{\Gamma}_q$ be the preimage of $W_q \subset \Gamma_q$ under the natural projection map $\widetilde{\Gamma}_q \to \Gamma_q$. Show that every continuous homomorphism $\rho : \widetilde{W}_q \to \mathrm{GL}(2, \mathbb{C})$ factors through $\widetilde{\Gamma}_{q,n}$.

5. An L-parameter for $\mathrm{GL}(2, \mathbb{F}_q)$ is a finite dimensional continuous complex representation ϕ of $\widetilde{W}_q \times \mathrm{SL}(2, \mathbb{C})$. Show that if ϕ is irreducible then $\phi = \rho \otimes \tau$ where ρ is an irreducible representation of $\widetilde{W}_q$ and τ is a finite dimensional representation of $\mathrm{SL}(2, \mathbb{C})$. Moreover, $\dim \phi = \dim \rho \dim \tau$.

6. Every character $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$ can be thought of as a one dimensional representation $\widetilde{W}_q \to \widetilde{\Gamma}_{q,1} \cong \mathbb{F}_q^\times \to \mathbb{C}^\times$. To the irreducible principal series $I(\chi_1, \chi_2)$ attach the L-parameter $\phi_{I(\chi_1,\chi_2)} = (\chi_1 \oplus \chi_2) \otimes 1$; to the characters $\chi \circ \det$ attach the L-parameter $\phi_{\chi\circ\det} = (\chi \oplus \chi) \otimes 1$; to the Steinberg representation St_χ attach the L-parameter $\phi_{\mathrm{St}_\chi} = \chi \otimes \mathrm{std}$ where std is the standard representation of $\mathrm{SL}(2, \mathbb{C})$ on $\mathbb{C}^2$ given by usual matrix multiplication. Show that this gives a bijection between the noncuspidal and non-Steinberg representations of $\mathrm{GL}(2, \mathbb{F}_q)$ and the reducible L-parameters of dimension 2. [Hint: The complex continuous representations of $\mathrm{SL}(2, \mathbb{C})$ are semisimple and the irreducible ones are all of the form $\mathrm{Sym}^n \mathrm{std}$, of dimension $n + 1$, where $\mathrm{Sym}^0 \mathrm{std} = 1$ and $\mathrm{Sym}^1 \mathrm{std} = \mathrm{std}$.]

7. Show that every irreducible two-dimensional L-parameter is either $\chi \otimes \mathrm{std}$, in bijection with the Steinberg representations, or of the form $\rho \otimes 1$ where ρ is an irreducible two-dimensional representation of $\widetilde{\Gamma}_{q,2} = \mathbb{F}_{q^2}^\times \rtimes \mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q) \cong \mathbb{F}_{q^2} \rtimes \mathbb{Z}/2\mathbb{Z}$.

8. Irreducible two-dimensional L-parameters.

   (a) Show that if $x \in \mathbb{F}_{q^2}^\times$ is such that $N_{\mathbb{F}_{q^2}/\mathbb{F}_q}(x) = 1$ then $x = y^{q-1}$ for some $y \in \mathbb{F}_{q^2}$.

   (b) Show that $\chi : \mathbb{F}_{q^2}^\times \to \mathbb{C}^\times$ satisfies $\chi = \chi^q$ (where $\chi^q(x) = \chi(x^q)$) if and only if $\chi = \nu \circ N_{\mathbb{F}_{q^2}/\mathbb{F}_q}$ for some character $\nu : \mathbb{F}_q^\times \to \mathbb{C}^\times$. In that case χ is said to be the base change from $\mathbb{F}_q$ of ν.

   (c) Conclude that there are $q(q-1)/2$ equivalence classes of characters $\chi : \mathbb{F}_{q^2}^\times \to \mathbb{C}^\times$ which are not base changes from $\mathbb{F}_q$ where $\chi \sim \chi^q$ are the equivalences.

(d) Suppose χ is as above, not a base change from $\mathbb{F}_q$. Show that $\rho_\chi := \mathrm{Ind}_{\mathbb{F}_{q^2}^\times}^{\widetilde{\Gamma}_{q,2}} \chi$ is an irreducible two dimensional representation. Show that $\rho_\chi \cong \rho_{\chi'}$ if and only if $\chi' \sim \chi$.

(e) Suppose ρ is an irreducible two-dimensional representation of $\widetilde{\Gamma}_{q,2} = \mathbb{F}_{q^2}^\times \rtimes \mathbb{Z}/2\mathbb{Z}$. Show that $\rho|_{\mathbb{F}_{q^2}^\times} \cong \chi \oplus \chi^q$ for some χ a character of $\mathbb{F}_{q^2}^\times$ which is not a base change from $\mathbb{F}_q$.

(f) Deduce that there is a bijection between the cuspidal representations of G and the irreducible two-dimensional L-parameters which, in turn, are in bijection with equivalence classes of characters of $\mathbb{F}_{q^2}^\times$ which are not base changes from $\mathbb{F}_q$. The next subpart will make this bijection explicit.

(g) In the notation of Exercise 1.4.7 show that $\rho_\psi^\chi(a(u)) = \mathrm{id}$ for $u \in L^1$ and conclude that ρ_ψ^χ factors through $\mathcal{G}/\{a(u)|u \in L^1\} \cong \mathrm{GL}(2, \mathbb{F}_q)$.

Exercise 1.4.7 (Constructing the cuspidal representations using the Weil representation) Let $K = \mathbb{F}_q$ and $L = \mathbb{F}_{q^2}$, $L^1 = \{x \in L | N_{L/K}x = 1\}$ and $\mathcal{G} = \ker\left(\det \cdot N_{L/K} : \mathrm{GL}(2, K) \times L^\times \to \mathbb{C}^\times\right)$. For simplicity of notation for $z \in L$ write $\overline{z} = z^q$ for the nontrivial automorphism of $\mathrm{Gal}(L/K)$.

1. Show that $N_{L/K}$ is surjective.

2. Let $n(t) = \left(\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}, 1\right)$, $a(u) = \left(\begin{pmatrix} N_{L/K}u & \\ & 1 \end{pmatrix}, \overline{u}^{-1}\right)$, $z(v) = \left(\begin{pmatrix} v & \\ & v \end{pmatrix}, v^{-1}\right)$ and $w = \left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, 1\right)$. Write $\mathcal{N} = \{n(t)|t \in K\}$, $\mathcal{A} = \{a(u)|u \in L^\times\}$ and $\mathcal{Z} = \{z(v)|v \in K^\times\}$. Show that

$$\mathcal{ZAN}\backslash\mathcal{G}/\mathcal{N} = \{1, w\}$$

3. Deduce that a presentation of $\mathcal{G}$ is given by $n(t) \in \mathcal{N}, a(u) \in \mathcal{A}, z(v) \in \mathcal{Z}$ and w subject to the relations

$$\begin{aligned} n(t)z(v) &= z(v)n(t) \\ a(u)z(v) &= z(v)a(u) \\ a(u)n(t)a(u)^{-1} &= n(tN_{L/K}(u)) \\ w^2 &= z(-1)a(-1) \\ wz(v) &= z(v)w \\ wa(u) &= z(N_{L/K}(u))a(\overline{u}^{-1})w \\ wn(t)w &- z(-t)a(-t^{-1})n(-t)wn(-t^{-1}) \end{aligned}$$

4. Fix a nontrivial character $\psi : K \to \mathbb{C}^\times$. Show that there exists a representation ρ_ψ of $\mathcal{G}$, called the Weil representation, on the set of maps $\mathcal{C}(L) = \{f : L \to \mathbb{C}\} \cong \mathbb{C}^{q^2}$ such that if $f : L \to \mathbb{C}$ then

$$\begin{aligned} (\rho_\psi(n(t))f)(x) &= \psi(tN_{L/K}(x))f(x) \\ (\rho_\psi(a(u))f)(x) &= f(ux) \\ (\rho_\psi(z(v))f)(x) &= f(x) \\ (\rho_\psi(w)f)(x) &= \widehat{f}(\overline{x}) \end{aligned}$$

where $\widehat{f}$ is the Fourier transform

$$\widehat{f}(y) = \frac{1}{q}\sum_{x \in L} f(x)\psi(\mathrm{Tr}_{L/K}(xy))$$

[Hint: You need to check that these formulae give a homomorphism $\rho_\psi : \mathcal{G} \to \mathrm{Aut}(L^2(L, \mathbb{C}))$. E.g., you'll need to check that $\widehat{\widehat{f}(\overline{x})}(\overline{x}) = f(-x)$.]

5. Suppose $\chi : L^\times \to \mathbb{C}^\times$ is a character which is not a base change from K. Let $\mathcal{C}(L,\chi) \subset \mathcal{C}(L)$ be the set of functions $f : L \to \mathbb{C}$ such that $f(xy) = \chi(x)f(y)$ for $x \in L^1$ and $y \in L$. Show that if $f \in \mathcal{C}(L,\chi)$ then $f(0) = 0$ and then show that $\mathcal{C}(L,\chi)$ is an irreducible $\mathcal{G}$-subrepresentation ρ_ψ^χ of $\mathcal{C}(L)$ of dimension $q-1$.

6. Show that $1 \to L^1 \to \mathcal{G} \to \mathrm{GL}(2,K) \to 1$ given by $u \mapsto a(u)$ and $(g,x) \mapsto g$ is an exact sequence.

7. Let χ as above and write $\pi_{\psi,\chi}(g,x) = \rho_\psi^\chi(g,x) \otimes \chi^{-1}(x)$. Show that $\pi_{\psi,\chi}$ is trivial on the image of L^1 in $\mathcal{G}$ and deduce that it gives a representation of $\mathrm{GL}(2,\mathbb{F}_q)$ of dimension $q-1$.

8. Show that $\sum_{t\in K} \pi_{\psi,\chi}(n(t))f = 0$ for all $f \in \mathcal{C}(L,\chi)$ and deduce that $\pi_{\psi,\chi}$ is cuspidal.

9. Suppose χ and χ' are two characters of $L^\times$ which are not base changes from K and suppose that $\pi_{\psi,\chi} \cong \pi_{\psi,\chi'}$. Show that $\chi \sim \chi'$.

10. Conclude that $\chi \mapsto \pi_{\psi,\chi}$ gives a bijection between the two dimensional L-parameters with irreducible representation of $\widehat{W}_q$ and the set of cuspidal representations of G.

Exercise 1.4.8 (Hecke theory) Let $K = \mathbb{F}_q$ and $\psi : K \to \mathbb{C}^\times$ a nontrivial character. For a function $\phi : \mathcal{M}_n(K) \to \mathbb{C}$ define the Fourier transform $\widehat{\phi} : \mathcal{M}_n(K) \to \mathbb{C}$ by

$$\widehat{\phi}(X) = q^{-n^2/2} \sum_{Y\in\mathcal{M}_n(K)} \phi(Y)\psi(\mathrm{Tr}(XY))$$

Here $\mathcal{M}_n(K)$ are $n \times n$ matrices and Tr is usual matrix trace.

1. Show that $\widehat{\widehat{\phi}}(X) = \phi(-X)$.

2. (Zeta functions) Let (V,π) be a finite dimensional representation of $\mathrm{GL}(n,K)$ and $\phi : \mathcal{M}_n(K) \to \mathbb{C}$. Define the following two endomorphisms in $\mathrm{End}(V)$:

$$Z(\Phi,\pi) = \sum_{g\in\mathrm{GL}(n,K)} \phi(g)\pi(g)$$
$$W_\pi(\psi,X) = q^{-n^2/2} \sum_{g\in\mathrm{GL}(n,K)} \psi(\mathrm{Tr}(gX))\pi(g)$$

for $X \in \mathcal{M}_n(K)$. Show that

$$Z(\phi,\pi) = \sum_{X\in\mathcal{M}_n(K)} \widehat{\phi}(-X)W_\pi(\psi,X)$$

3. For $X \in \mathcal{M}_n(K)$ and $g,h \in \mathrm{GL}(n,K)$ show that $W_\pi(\psi,gXh) = \pi(h)^{-1}W_\pi(\psi,X)\pi(g)^{-1}$.

4. For an irreducible representation π of $\mathrm{GL}(n,K)$ show that $W_\pi(\psi,I_n) \in \mathrm{End}(\pi)$ commutes with $\pi(g)$ for all g and conclude that it is a scalar. Let $\varepsilon(\pi,\psi)$ be the scalar $W_{\pi^\vee}(\psi,I_n)$ attached to the irreducible dual representation $\pi^\vee$.

5. Let $\mathcal{S}$ be the set of functions $f : \mathcal{M}_n(K) \to \mathbb{C}$ such that $f(X) = 0$ for all matrices X with $\det(X) = 0$. Show that if $\phi \in \mathcal{S}$ then $\widehat{\phi} \in \mathcal{S}$.

6. (The functional equation) Show that for π irreducible and $\phi \in \mathcal{S}$:

$$Z(\widehat{\phi},\pi^\vee)^t = \varepsilon(\pi,\psi)Z(\phi,\pi)$$

where superscript t means dual endomorphism (i.e., matrix transposition).

7. Show that if $\pi = \mathrm{Ind}_B^G \chi_1 \otimes \chi_2$ is an irreducible principal series representation of $\mathrm{GL}(2, \mathbb{F}_q)$ then

$$\varepsilon(\pi, \psi) = q^{-2} \sum_{a,b \in \mathbb{F}_q^\times} \psi(a+b) \chi_1^{-1}(a) \chi_2^{-1}(b)$$