Exercises 1
Monday 6/12/17

I encourage you to look particularly at the problems marked with *.

1. For a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})$ and $z \in \mathbb{C}$ define

$$g \cdot z = \frac{az+b}{cz+d}$$

   (a) Show that for any two matrices $g, h \in \mathrm{GL}(2,\mathbb{R})$, $(gh)\cdot z = g \cdot (h \cdot z)$.

   (b) * Show that $\mathrm{Im}(g \cdot z) = \dfrac{\det(g)\,\mathrm{Im}(z)}{|cz+d|^2}$.

   (c) * Deduce that $g \cdot z$ defines an action of the group $\mathrm{GL}(2,\mathbb{R})^+$ of positive determinant matrices on the upper half plane $\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$.

   (d) * Using calculus we may define $g \cdot \infty = \dfrac{a}{c}$. Show that the orbit of ∞ under $\mathrm{SL}(2,\mathbb{Z})$ is all of $\mathbb{Q} \cup \{\infty\} = \mathbb{P}^1\mathbb{Q}$.

2. Show that the group $\mathrm{SL}(2,\mathbb{Z}) = \{g \in M_{2\times 2}(\mathbb{Z}) \mid \det g = 1\}$ is generated by the matrices $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. [Hint: Suppose $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$. Compute S^2, Sg, $T^{-q}g$ and argue by induction.]

3. Let $\mathcal{F} \subset \mathcal{H}$ be the region described by $-1/2 < \mathrm{Re}\, z < 1/2$ and $|z| > 1$ together with its boundary in the first quadrant.

   (a) Show that $\mathcal{F}$ is a fundamental domain for the action of $\mathrm{SL}(2,\mathbb{Z})$ acting on $\mathcal{H}$. [Hint: If $z \in \mathcal{H}$ show that as $g \in \mathrm{SL}(2,\mathbb{Z})$, the set $\{\mathrm{Im}(g \cdot z)\}$ contains a maximum. A translate of this will work.]

   (b) * Compute the volume of $\mathcal{F}$ under the hyperbolic measure, i.e., compute

$$\mathrm{vol}(\mathcal{F}) = \int_{\mathcal{F}} \frac{dx dy}{y^2}$$

4. * Let $u \in \mathcal{H}$ and define $f(z) = \dfrac{z-u}{z-\overline{u}}$.

   (a) Show that $|f(z)| < 1$ if and only if $z \in \mathcal{H}$. [Hint: Use basic geometry.]

   (b) Show that $f(z)$ is a biholomorphic function $f : H \to D$ where D is the open unit disc. Bihomolomorphic means bijective and holomorphic with holomorphic inverse.