# Exercises 2
# Monday 6/13/17

5. Define the Bernoulli number $B_{2n}$ using the Taylor expansion[1]

$$z\cot\text{an}(z) = 1 + \sum_{n=1}^{\infty} (-4)^n B_{2n} \frac{z^{2n}}{(2n)!}$$

From calculus we also know that

$$z\cot\text{an}(z) = 1 + 2\sum_{n=1}^{\infty} \frac{z^2}{z^2 - n^2\pi^2}$$

Expand each geometric series $\dfrac{z^2}{z^2 - n^2\pi^2}$ to show that

$$\boxed{\zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!}}$$

6. Let $q = e^{2\pi i z}$. The second formula from the previous exercise can be rewritten as

$$\pi\cot\text{an}(\pi z) = \frac{1}{z} + \sum_{n\geq 1}\left(\frac{1}{z+n} + \frac{1}{z-n}\right) = \sum_{n\in\mathbb{Z}} \frac{1}{z+n}$$

(a) Show directly from the definition of cotan that

$$\pi\cot\text{an}(\pi z) = \pi i \frac{q-1}{q+1} = \pi i - 2\pi i \sum_{n=0}^{\infty} q^n$$

(b) Differentiate $k-1$ times to show that

$$\boxed{\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} q^n}$$

7. Let $k$ be an integer. Show that if $f(z+1) = f(z)$ and $f(-1/z) = z^k f(z)$ then $f(g\cdot z) = (cz+d)^k f(z)$ for all $g \in \mathrm{SL}(2,\mathbb{Z})$. Here $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. [Hint: The group $\mathrm{SL}(2,\mathbb{Z})$ is generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.]

8. Let $S$ be the set of pairs of integers $(m,n)$ not both zero. Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$. Show that

$$\{(ma+nc, mb+nd) \mid (m,n) \in S\} = S$$

[1] The more standard definition is that $\dfrac{x}{e^x - 1} = \sum B_n \dfrac{x^n}{n!}$. Plug in $x = 2iz$ to deduce the cotan Taylor series.