GALOIS REPRESENTATIONS FOR HOLOMORPHIC SIEGEL MODULAR FORMS

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Abstract. We prove local global compatibility (up to a quadratic twist) of Galois representations associated to holomorphic Hilbert-Siegel modular forms in many cases (induced from Borel or Klingen parabolic). For Siegel modular forms, when the local representation is an irreducible principal series we get local global compatibility without a twist. We achieve this by proving a version of rigidity (strong multiplicity one) for GSp(4) using, on the one hand the doubling method to compute the standard L-function, and on the other hand the explicit classification of the irreducible local representations of GSp(4); then we refer to [Sor10] for local global compatibility in the case of globally generic Hilbert-Siegel modular forms.

1. Introduction

To a holomorphic modular form f of weight $k \geq 2$ and level $\Gamma_1(N)$ Deligne associates an ℓ-adic Galois representation $\rho_f : G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(2, \overline{\mathbb{Q}}_\ell)$ which at primes $p \nmid N\ell$ is unramified such that the characteristic polynomial of Frob_p is the same as the Hecke polynomial at p of the modular form f. By completely different methods, to the local representation at p of the automorphic representation associated to f Jacquet and Langlands associate a local Galois representation $\rho_p : G_{\mathbb{Q}_p} \to \mathrm{GL}(2, \mathbb{C})$. One of the basic premises of the Langlands program over number fields is that the local representation ρ_p should, in some sense, be the restriction to $G_{\mathbb{Q}_p}$ of the global representation ρ_f. This phenomenon is known as local-global compatibility and for modular forms it was proven for $p \neq \ell$ by Carayol and for $p = \ell$ by Saito.

More generally, one desires to attach to each regular algebraic cuspidal representation of a reductive group a global Galois representation compatible with local Galois representations to be attached to the local components of the automorphic representation. Both these statements are still conjectural in general, but such results have been obtained for Hilbert modular forms (the case of GL(2) over totally real fields) and to GL(n) over totally real or CM fields with a restriction on the allowed algebraic cuspidal representation (essentially self-dual for totally real fields and essentially conjugate self-dual for CM fields).

The focus of this article is the study of the above described constructions in the context of the reductive group GSp(4) over a totally real field F, in which case the classically defined forms are known as Hilbert-Siegel modular forms. The expected result for regular algebraic cuspidal representations is the following (for a description of Hilbert-Siegel modular forms and the source of the following conjecture see the introduction to [Sor10])

Conjecture 1.1. *Let F be a totally real field and let π be a cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A}_F)$ and assume there is a cuspidal weak lift Π of π to $\mathrm{GL}(4, \mathbb{A}_F)$. If λ is the*

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multiplier character on $\mathrm{GSp}(4)$*, assume there exists an integer* w *such that* $\pi^\circ = \pi \otimes |\lambda|^{w/2}$ *is unitary.*

For each archimedean place v *assume that* π_v *is an essentially discrete series representation with Harish-Chandra parameter* $\mu(v) = (\mu_1(v), \mu_2(v))$ *with* $\mu_1(v)+\mu_2(v) \equiv w \pmod 2$*. Then for each* $\iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ *there is a unique continuous irreducible representation* $\rho_{\pi,\iota} : G_F \to \mathrm{GSp}(4, \overline{\mathbb{Q}}_\ell)$ *such that*

- *(Local-global compatibility) For each nonarchimedean place:*

$$\iota\,\mathrm{WD}(\rho_{\pi,\iota}|_{G_{F_v}})^{\text{Fr-ss}} \cong \mathrm{rec}_{\mathrm{GT}}(\pi_v \otimes |\lambda|^{-3/2})$$

  (here WD *is the Weil-Deligne representation associated to the local Galois representation* $\rho_{\pi,\iota}|_{G_{F_v}}$*,* Fr-ss *refers to Frobenius semisimplification, and* $\mathrm{rec}_{\mathrm{GT}}$ *is the local Langlands reciprocity map constructed in* [GT10a, Main Theorem, p. 1]*)*
- *(Ramanujan-Petersson conjecture)* π° *is tempered everywhere.*
- $\rho_{\pi,\iota}^\vee \cong \rho_{\pi,\iota} \otimes \chi^{-1}$ *where* $\chi = \omega_{\pi^\circ}\chi_{\mathrm{cycl}}^{-w-3}$ *is totally odd.*

The first result towards the conjecture was made in the context of globally generic automorphic representations and is contained in the main theorem of [Sor10, p. 627], updated to include recent results ([Car10, Theorems 1.1 and 1.2]) for automorphic representations of $\mathrm{GL}(n)$:

Theorem 1.2 (Sorensen)**.** *Let* π *be as in the conjecture such that* π *is globally generic, i.e., it has a global Whittaker model. Then local-global compatibility holds for* $v \nmid \ell$*. When* $v \mid \ell$ *then* $\rho_{\pi,\iota}|_{G_{F_v}}$ *is de Rham of Hodge-Tate weights* $(w - \mu_1(v') - \mu_2(v'))/2 + \{0, \mu_2(v') + 1, \mu_1(v') + 2, \mu_1(v') + \mu_2(v') + 3\}$ *where* v' *is the infinite place such that* $\iota(v') = v$*; the Galois representation is crystalline if* π_v *is unramified.*

It is now worth explaining what is left towards the conjecture given the methods of [Sor10] and the impending publication of the work of Arthur on functorial transfer between semisimple matrix groups. Indeed now that the fundamental lemma is a theorem Arthur's expected work gives a functorial transfer of, the the language of the conjecture, the representation π on $\mathrm{GSp}(4)$ to the representation Π on $\mathrm{GL}(4)$. However, the compatibility of L-parameters is checked at unramified places, while at ramified places what is being provided are character identities; deducing the compatibility of L-parameters from this is a formidable task and this work can be thought of as bypasses of this problem.

A brief remark about the condition of genericity in the above theorem: the local representation π_v is *generic* if it has a Whittaker model, while π is globally generic if it has a global Whittaker model. If v is an archimedean place and π_v is a discrete series, then π_v is not generic if it is holomorphic; if it is generic it appears in the $H^{2,1}$ cohomology of a local system on a Shimura variety associated to $\mathrm{GSp}(4)$. If the global representation π is generic then each local representation π_v is generic. The converse is not known to hold ([JS07, p. 383]).

We will study the above conjecture in the case when π_v is holomorphic for each archimedean place v, hence, π is not globally generic. We keep the assumption that there is a weak lift to $\mathrm{GL}(4)$ as this, while expected, is known only in the case of $F = \mathbb{Q}$ by [Wei07, Theorem 1]. The strategy is to use the strong theta lift of Π from $\mathrm{GL}(4)$ to a globally generic π^G cuspidal automorphic representation on $\mathrm{GSp}(4)$ ([GT10a, Theorem 12.1]), whose local representations at the archimedean places are in the same L-packets as the ones for π. Then

π and π^G will be weakly equivalent and we will prove a rigidity statement for GSp(4) and then apply the Conjecture to the case of the globally generic π^G: by the Chebotarëv density theorem $\rho_{\pi,\iota} \cong \rho_{\pi^G,\iota}$ (this can be taken either as definition or as a theorem in the case when $F = \mathbb{Q}$ and $\rho_{\pi,\iota}$ is constructed by Taylor et al.).

Throughout this article we will assume that the local place v does not divide 2. The reason is that when $v \mid p > 2$ all supercuspidal L-parameters are dihedral, and the list of L-parameters in [Vig86] is complete.

The main results of this paper are sometimes too technical to summarize, but we include the following example result:

Theorem 1.3. *Let π be as in the conjecture such that π_v is a holomorphic discrete series for every infinite place v. Then for every finite place $v \nmid 2$ such that π_v has Iwahori invariant vectors and is generic local-global compatibility is satisfied up to a quadratic twist and the Ramanujan-Petersson conjecture is satisfied. If π_v is not assumed to be generic, then local-global compatibility is satisfied up to semisimplification and a quadratic twist.*

The main results of this paper are Corollary 6.3 on local-global compatibility of monodromy, Corollary 6.4 on the Skinner-Urban conjecture for para-sphericals, Proposition 6.5 on rigidity for GSp(4) and Theorem 7.1 on local-global compatibility and temperedness of generic local components of cuspidal automorphic representations.

This work was motivated by the author's thesis, written under the supervision of Andrew Wiles, and we are grateful to him and to Chris Skinner for suggesting the problem and for his invaluable help. The intended application of this work is the study of Galois representations associated to regular algebraic cuspidal automorphic representations of GL(2) over quadratic imaginary fields.

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2. Rigidity and γ-factors

(2.1) The classical rigidity statement for GL(2) is the following: if two cuspidal automorphic representations with the same central character are weakly equivalent, then they are isomorphic. This statement is usually combined with multiplicity one and known as strong multiplicity one. We note that multiplicity one is known for globally generic representations for GSp(4) by [JS07].

There are several proofs in the case of GL(2) and we recall [Cas73, Theorem 2]: if $\pi_{1,v} \cong \pi_{2,v}$ for $v \notin S$ then for all global characters η, $\pi_{1,v} \otimes \eta_v \cong \pi_{2,v} \otimes \eta_v$. Considering the Jacquet-Langlands γ-factors

$$\gamma(s, \pi_v, \eta_v, \psi_v) = \varepsilon(s, \pi_v \otimes \eta_v, \psi_v) \frac{L(1-s, \widetilde{\pi}_v \otimes \eta_v^{-1})}{L(s, \pi_v \otimes \eta_v)}$$

(where ψ is an additive character) it follows by the global functional equation that

$$\prod_v \gamma(s, \pi_{1,v}, \eta_v, \psi_v) = 1 = \prod_v \gamma(s, \pi_{2,v}, \eta_v, \psi_v)$$

Let $v \in S$. For $w \notin S$, $\gamma(s, \pi_{1,w}, \eta_w, \psi_w) = \gamma(s, \pi_{2,w}, \eta_w, \psi_w)$ by assumption. For $w \in S - \{v\}$ we may choose η_w very ramified in which case again

$$\gamma(s, \pi_{1,w}, \eta_w, \psi_w) = \varepsilon(s, \pi_{1,w} \otimes \eta_w) = \varepsilon(s, \pi_{2,w} \otimes \eta_w) = \gamma(s, \pi_{2,w}, \eta_w, \psi_w)$$

by [JL70, Proposition 3.8] (this is known as "stability" of γ-factors). Therefore we may cancel out the equal terms and deduce that $\gamma(s, \pi_{1,v}, \eta_v, \psi_v) = \gamma(s, \pi_{2,v}, \eta_v, \psi_v)$ for all η_v. Since the central characters of π_1 and π_2 are the same (rigidity for global characters) it follows by [JL70, Corollary 2.19] that $\pi_{1,v} \cong \pi_{2,v}$.

(2.2) For more general groups, it is expected (by local-global compatibility and the Chebotarëv density theorem) that if π_1 and π_2 are weakly equivalent then $\pi_{1,v}$ and $\pi_{2,v}$ are in a same L-packet. In the cases when γ-factors can be defined with sufficiently good properties, the above proof can be adapted. Indeed, in the case of $\mathrm{GL}(n)$ there is a theory of γ-factors due to [GJ72] giving $\gamma(s, \pi, \sigma, \psi)$ where π is an irreducible smooth representation of $\mathrm{GL}(n)$ and σ is an irreducible smooth representation of $\mathrm{GL}(m)$. As above, using a global functional equation and stability for γ-factors, one deduces that $\gamma(s, \pi_{1,v}, \sigma_v, \psi_v) = \gamma(s, \pi_{2,v}, \sigma_v, \psi_v)$ for all σ_v irreducible smooth representations of $\mathrm{GL}(m)$ for $m = 1, \ldots, n-1$. One then deduces that $\pi_{1,v} \cong \pi_{2,v}$. It is expected that one only needs $m = 1, \ldots, \lfloor n/2 \rfloor$, but this is a theorem only in the case of $m = 1, \ldots, n-2$ (cf. [Che06]).

(2.3) In the case of the group $\mathrm{GSp}(4)$ one expects that if π_1 and π_2 are two weakly equivalent cuspidal automorphic representations then $\mathrm{rec}_{\mathrm{GT}}(\pi_{1,v}) \cong \mathrm{rec}_{\mathrm{GT}}(\pi_{2,v})$ for all nonarchimedean places v. However, the adaptation of the proof from the case of $\mathrm{GL}(n)$ is not apparent because the definition of γ-factors for $\mathrm{GSp}(4)$ in [Sha90, Theorem 3.5] only applies to generic representations. In the case of nongeneric representations Shahidi's result gives information about the Plancherel measure, which does not suffice for our purposes.

3. Standard γ-factors and the doubling method

(3.1) Let us recall the standard definition of γ-factors for (generic) local representations of $\mathrm{GL}(n)$. For the representation π with (unique) Whittaker model $\mathcal{W}$ one defines two zeta integrals, each of which gives a Whittaker functional. By uniqueness of the Whittaker model, the two zeta integrals differ by a constant $\gamma(s, \pi, \psi)$, which is a rational function in q^{-s}. For tempered representations π, the L-function $L(s, \pi)$ is the inverse of the monic polynomial in q^{-s} which is the numerator of $\gamma(s, \pi, \psi)$, while the ε-factor is defined such that
$$\gamma(s, \pi_v, \eta_v, \psi_v) = \varepsilon(s, \pi_v \otimes \eta_v, \psi_v) \frac{L(1-s, \widetilde{\pi}_v \otimes \eta_v^{-1})}{L(s, \pi_v \otimes \eta_v)}$$

(3.2) There are two ways to define γ-factors for local nongeneric representations of $\mathrm{GSp}(4)$. One, specific to $\mathrm{GSp}(4)$, uses Bessel models (generalized Whittaker models) and is achieved in [PS97, §3]. However, the computation of the local L-function was done only in certain cases. The other method, developed in a series of articles starting with [PSR86] and ending with [LR], is known as the doubling method, which gives the standard γ-factors for $\mathrm{Sp}(4)$. These have very good properties (known as the Ten Commandments) [LR, Theorem 4], but are only defined for twists by characters. As previously mentioned, to show that the local representations $\pi_{1,v}$ and $\pi_{2,v}$ on $\mathrm{GL}(4)$ are isomorphic, one needs to check equality of γ-factors for twists by characters and representations of $\mathrm{GL}(2)$. We have to make do with less.

(3.3) Let us briefly review the doubling method for Sp(4). Let F be a local field and $\mathcal{V} = (V, h)$ be a symplectic 4-dimensional vector space over F. Let $\mathcal{V} \times \mathcal{V} = (V \times V, h^{\Box})$ where $h^{\Box} = h \oplus (-h)$. Let $G = \mathrm{Isom}(\mathcal{V}) \cong \mathrm{Sp}(4)$ and $G^{\Box} = \mathrm{Isom}(\mathcal{V} \times \mathcal{V}) \cong \mathrm{Sp}(8)$. We identify $G \times G$ with the subgroup of $G^{\Box}$ stabilizing $V \times 0$ and $0 \times V$. Let $V^{\diamond} = \{(v, v) : v \in V\} \subset V \times V$ and let $S^{\diamond}$ be the Siegel parabolic stabilizing $V^{\diamond}$ and let $G^{\diamond} = G \times G \cap S^{\diamond}$. For a character ω of $F^{\times}$, let

$$I^{\mathcal{V}}(\omega, s) = \mathrm{Ind}_{S^{\diamond}}^{G^{\Box}}(\omega||^{s}) \circ \det$$

be the normalized induction.

Let π be an irreducible smooth representation of $\mathrm{Sp}(4, F)$, and let $\widetilde{\pi}$ be the contragredient representation. For elements $f \in I^{\mathcal{V}}(\omega, s)$ and $\alpha \otimes \widetilde{\alpha} \in \pi \otimes \widetilde{\pi}$ there is a zeta integral $Z^{\mathcal{V}}(f, \alpha \otimes \widetilde{\alpha})$ and an intertwining operator $M^{\mathcal{V}}(\omega, s) : I^{\mathcal{V}}(\omega, s) \to I^{\mathcal{V}}(\omega^{-1}, -s)$. Then, as in the classical definition due to Tate, there is a Γ-factor such that

$$Z^{\mathcal{V}}(M^{\mathcal{V}}(\omega, s)f, \alpha \otimes \widetilde{\alpha}) = \Gamma^{\mathcal{V}}(s, \pi, \omega) Z(f, \alpha \otimes \widetilde{\alpha})$$

(cf. [LR, p. 315])

(3.4) Finally, in [LR, §9], from $\Gamma^{\mathcal{V}}(s, \pi, \omega)$ one obtains $\gamma^{\mathcal{V}}(s, \pi, \omega, \psi)$ where ψ is an additive character of F. It has the property that if $\mathrm{BC}(\pi)$ is the expected functorial transfer of π from Sp(4) to GL(5) then $\gamma^{\mathcal{V}}(s, \pi, \omega, \psi) = \gamma^{\mathrm{GJ}}(s, \mathrm{BC}(\pi) \otimes \omega, \psi)$ where γ^{GJ} are the Godement-Jacquet γ-factors.

The $\gamma^{\mathcal{V}}$ factors satisfy the following properties (we omit ψ from the formulae; here B is the Borel, P is the Siegel parabolic and Q is the Klingen parabolic), which are part of [LR, Theorem 4]:

Theorem 3.5. *Let π be an irreducible smooth representation of $\mathrm{GSp}(4, K)$ where K is a finite extension of $\mathbb{Q}_p$.*

- *If π is a subrepresentation of $\mathrm{Ind}_B^G \chi_1 \times \chi_2$ then*

$$\gamma^{\mathcal{V}}(s, \pi, \omega) = \gamma(s, \chi_1\omega)\gamma(s, \chi_1^{-1}\omega)\gamma(s, \chi_2\omega)\gamma(s, \chi_2^{-1}\omega)\gamma(s, 1)$$

- *If π is a subrepresentation of $\mathrm{Ind}_P^G \sigma$ then $\gamma^{\mathcal{V}}(s, \pi, \omega) = \gamma(s, \sigma \otimes \omega)\gamma(s, \widetilde{\sigma} \otimes \omega)\gamma(s, 1)$. Here π is a representation of* GL(2).
- *If π is a subrepresentation of $\mathrm{Ind}_Q^G \chi \rtimes \sigma$ then $\gamma^{\mathcal{V}}(s, \pi, \omega) = \gamma(s, \chi\omega)\gamma(s, \chi^{-1}\omega)\gamma(s, \sigma \otimes \omega, \mathrm{Ad} \otimes 1)$. Here σ is a representation on $\mathrm{Sp}(2) \cong \mathrm{SL}(2)$.*
- *If F is a number field and π is a cuspidal automorphic representation of $\mathrm{Sp}(4, \mathbb{A}_F)$ then there is a global functional equation*

$$\prod_v \gamma^{\mathcal{V}}(s, \pi_v, \omega_v, \psi_v) = 1$$

We would like to mention the following stability property of the γ-factors obtained from the doubling method, which is the content of [RS05, Theorem 1]

Theorem 3.6. *If π_1 and π_2 are admissible representations of $\mathrm{Sp}(4, F)$ where F is a nonarchimedean local field and ω is a sufficiently ramified character, then $\gamma^{\mathcal{V}}(s, \pi_1, \omega, \psi) = \gamma^{\mathcal{V}}(s, \pi_2, \omega, \psi)$.*

(3.7) To recover the L-functions in the case of tempered representations, we have already mentioned that one uses the numerator of the γ-factor. In the case of nontempered representations, one writes the representation uniquely as a Langlands quotient of the induction of a tempered representation on a Levi subgroup, and one defines the L-function to be the L-function of this tempered representation.

4. Rigidity for GSp(4)

Conjecture 4.1 (Rigidity). *Let π_1 and π_2 be two weakly equivalent cuspidal automorphic representations of $\mathrm{GSp}(4, \mathbb{A}_F)$. Then $\mathrm{rec}_{\mathrm{GT}}(\pi_{1,v}) \cong \mathrm{rec}_{\mathrm{GT}}(\pi_{2,v})$ for all places v.*

(4.2) Let π_1 and π_2 be two weakly equivalent cuspidal automorphic representations of $\mathrm{GSp}(4, \mathbb{A}_F)$. Let S be a finite set of places such that $\pi_{1,v} \cong \pi_{2,v}$ for $v \notin S$. For $i = 1, 2$ let π'_i be an irreducible component of the restriction of π_i to $\mathrm{Sp}(4, \mathbb{A}_F)$. Then $\pi'_{i,v}$ is an irreducible component of the restriction of $\pi_{i,v}$ to $\mathrm{Sp}(4, F_v)$. By [GT10b, Main Theorem, (v)] it follows that $\mathrm{rec}_{\mathrm{GT}}(\pi'_{i,v}) = \mathrm{std} \circ \mathrm{rec}_{\mathrm{GT}}(\pi_{i,v})$. By [GT10b, Main Theorem, (i)] and [GT10a, Theorem 8.3]we have

$$\begin{aligned}\gamma^{\mathcal{V}}(s, \pi'_{i,v}, \omega_v, \psi_v) &= \gamma(s, \mathrm{rec}_{\mathrm{GT}}(\pi'_{i,v}) \otimes \omega_v, \mathrm{std} \otimes \mathrm{std}, \psi) \\ &= \gamma(s, \mathrm{rec}_{\mathrm{GT}}(\pi_{i,v}) \otimes \omega_v, \mathrm{std} \otimes \mathrm{std}, \psi) \\ &= \gamma(s, \pi_{i,v} \otimes \omega_v, \mathrm{std} \otimes \mathrm{std}, \psi)\end{aligned}$$

where the last equality makes sense for generic representations $\pi_{i,v}$. For $w \notin S$, (we may increase S to contain all representations which are not unramified principal series) we have $\gamma(s, \pi_{1,w} \otimes \omega_w, \mathrm{std} \otimes \mathrm{std}, \psi_w) = \gamma(s, \pi_{2,w} \otimes \omega_w, \mathrm{std} \otimes \mathrm{std}, \psi_w)$, and therefore $\gamma^{\mathcal{V}}(s, \pi'_{1,w}, \omega_w, \psi_w) = \gamma^{\mathcal{V}}(s, \pi'_{2,w}, \omega_w, \psi_w)$.

(4.3) Fix $v \in S$ and let $w \in S - \{v\}$. Making ω_w sufficiently ramified, Theorem 3.6 implies that

$$\gamma^{\mathcal{V}}(s, \pi'_{1,w}, \omega_w, \psi_w) = \gamma^{\mathcal{V}}(s, \pi'_{2,w}, \omega_w, \psi_w)$$

Finally, by the global functional equation in Theorem 3.5 and the above equalities of γ-factors, we deduce that

$$\gamma^{\mathcal{V}}(s, \pi'_{1,v}, \omega_v, \psi_v) = \gamma^{\mathcal{V}}(s, \pi'_{2,v}, \omega_v, \psi_v)$$

for all characters ω_v, or, equivalently,

$$\gamma(s, \mathrm{rec}_{\mathrm{GT}}(\pi_{1,v}) \otimes \omega_v, \mathrm{std} \otimes \mathrm{std}, \psi_v) = \gamma(s, \mathrm{rec}_{\mathrm{GT}}(\pi_{2,v}) \otimes \omega_v, \mathrm{std} \otimes \mathrm{std}, \psi_v)$$

(4.4) Since we only have equality of the γ-factors for twists by characters, we cannot appeal to general converse theorems and we must compute the standard γ-factors for all the irreducible smooth representations of GSp(4) over local fields and analyze when two such factors can be equal.

5. Representations of GSp(4) over p-adic fields

(5.1) Let F be a finite extension of $\mathbb{Q}_p$, let ϖ_F be a uniformizer and $\mathcal{O}_F$ be the ring of integers. The irreducible smooth nonsupercuspidal representations of $\mathrm{GSp}(4, F)$ have been classified by [ST93] and the classification appears concisely in [RS07, Table A.1, p.270]. Moreover, in [RS07, Table A.7, p.281] one can find a list of L-parameters associated to the nonsupercuspidal representations of $\mathrm{GSp}(4, F)$. The computation of $\gamma(s, \mathrm{rec}_{\mathrm{GT}}(\pi) \otimes \omega_v, \mathrm{std} \otimes \mathrm{std}, \psi_v)$ for nonsupercuspidal π becomes an exercise in computing the standard factors associated to these L-parameters. If π is a supercuspidal representation of GSp(4), we appeal to [GT10b, Theorem 6.5] to aid in the computation of the standard γ-factor of π.

(5.2) The Sally-Tadic classification of nonsupercuspidal representations has six classes of representations I-VI which are constituents of inductions from the Borel, three classes VII-IX which are inductions from the Klingen parabolic and two X-XI which are inductions from the Siegel parabolic. By the multiplicativity principle of Theorem 3.5, it follows that the γ-factors are the same for all the constituents of an induction; the numerator of this γ-factor is also the L-function associated to the (unique) generic constituent in each induction. (This is because for nonsupercuspidals, temperedness implies that there is a generic representation in the local L-packet.)

(5.3) Therefore, we will assume that π is a nonsupercuspidal generic representation, so it must be of the form I, IIa, IIIa, IVa, Va, VIa, VII, VIIIa, IXa, X or XIa. (For the actual representations, in the Sally-Tadic notation, see [RS07, Table A.1].) For convenience we consolidate our computations in a series of tables.

TABLE 1. Nonsupercuspidal generic representations of $\mathrm{GSp}(4,F)$

Class	Constituent of	$\mathrm{rec}_{\mathrm{GT}}(\pi)$	
I	$\chi_1 \times \chi_2 \rtimes \sigma$	$\chi_1\chi_2\sigma, \chi_1\sigma, \chi_2\sigma, \sigma$	0
IIa	$\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$	$\chi^2\sigma, \nu^{1/2}\chi\sigma, \nu^{-1/2}\chi\sigma, \sigma$	N_1
IIIa	$\chi \times \nu \rtimes \nu^{-1/2}\sigma$	$\nu^{1/2}\chi\sigma, \nu^{-1/2}\chi\sigma, \nu^{1/2}\sigma, \nu^{-1/2}\sigma$	N_4
IVa	$\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$	$\nu^{3/2}\sigma, \nu^{1/2}\sigma, \nu^{-1/2}\sigma, \nu^{-3/2}\sigma$	N_5
Va	$\nu\xi \times \xi \rtimes \nu^{-1/2}\sigma$	$\nu^{1/2}\sigma, \nu^{1/2}\xi\sigma, \nu^{-1/2}\sigma, \nu^{-1/2}\xi\sigma$	N_3
VIa	$\nu \times 1 \rtimes \nu^{-1/2}\sigma$	$\nu^{1/2}\sigma, \nu^{1/2}\sigma, \nu^{-1/2}\sigma, \nu^{-1/2}\sigma$	N_3
VII	$\chi \rtimes \pi$	$\chi\omega_\pi\phi'_\pi, \phi_\pi$	0
VIIIa	$1 \rtimes \pi$	$\omega_\pi\phi'_\pi, \phi_\pi$	0
IXa	$\nu\xi \rtimes \nu^{-1/2}\pi$	$\xi\nu^{1/2}\omega_\pi\phi'_\pi, \nu^{-1/2}\phi_\pi$	N_6
X	$\pi \rtimes \sigma$	$\sigma\omega_\pi, \sigma\phi_\pi, \sigma$	0
XIa	$\nu^{1/2}\pi \rtimes \nu^{-1/2}\sigma$	$\nu^{1/2}\sigma, \sigma\phi_\pi, \nu^{-1/2}\sigma$	N_2

Here

$$N_1 = \begin{pmatrix} 0 & & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{pmatrix} \qquad N_2 = \begin{pmatrix} 0 & & & 1 \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \qquad N_3 = \begin{pmatrix} 0 & & & 1 \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

$$N_4 = \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & -1 \\ & & & 0 \end{pmatrix} \qquad N_5 = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & -1 \\ & & & 0 \end{pmatrix} \qquad N_6 = \begin{pmatrix} 0 & & y & z \\ & 0 & t & y \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

In class II we need that $\chi^2 \neq \nu^{\pm 1}$ and $\chi \neq \nu^{\pm 3/2}$; in class III we need $\chi \notin \{1, \nu^{\pm 2}\}$; in class V the character ξ is any nontrivial quadratic character; in class I are all the other principal series induced from the Borel, not covered by cases II-VI; in class IX a the character ξ is a nontrivial character such that $\xi\pi = \pi$ (in particular, if $\pi = \mathrm{Ind}_E^F \psi$ for a quadratic extension E/F and ψ is a character of $E^\times$, then $\xi = \xi_{E/F}$ is the character of the quadratic extension); in class VII are all the principal series induced from the Klingen parabolic not covered by cases VIII and IX; in class XI we need that the central character of π be trivial; in class X are the other principal series induced from the Siegel parabolic. Whenever it occurs in the above table, π is a supercuspidal representation of $\mathrm{GL}(2,F)$.

Table 2. Standard representations of nonsupercuspidal L-parameters

Class	$\mathrm{std} \circ \mathrm{rec}_{\mathrm{GT}}(\pi)$		ker of monodromy
I	$\chi_1, \chi_1^{-1}, 1, \chi_2, \chi_2^{-1}$	0	$\chi_1, \chi_1^{-1}, 1, \chi_2, \chi_2^{-1}$
IIa	$\nu^{1/2}\chi, \nu^{-1/2}\chi, 1, \nu^{1/2}\chi^{-1}, \nu^{-1/2}\chi^{-1}$	M_1	$\nu^{1/2}\chi, 1, \nu^{1/2}\chi^{-1}$
IIIa	$\chi, \nu, 1, \nu^{-1}, \chi^{-1}$	M_4	χ, ν, χ^{-1}
IVa	$\nu^2, \nu, 1, \nu^{-1}, \nu^{-2}$	M_5	ν^2
Va	$\nu\xi, \xi, 1, \xi, \nu^{-1}\xi$	M_3	$\nu\xi, \xi, 1$
VIa	$\nu, 1, 1, 1, \nu^{-1}$	M_3	$\nu, 1, 1$
VII	$\chi, \mathrm{Ad} \circ \phi_\pi, \chi^{-1}$	0	$\chi, \mathrm{Ad} \circ \phi_\pi, \chi^{-1}$
VIIIa	$1, \mathrm{Ad} \circ \phi_\pi, 1$	0	$1, \mathrm{Ad} \circ \phi_\pi, 1$
IXa	$\nu\xi, \mathrm{Ad} \circ \phi_\pi, \nu^{-1}\xi$	M_6	$\nu\xi, \mathrm{Ad} \circ \phi_\pi / \xi$
X	$\lambda\phi_\pi\lambda^{-1}, 1, \omega_\pi^{-1}\lambda'\phi_\pi(\lambda')^{-1}$	0	$\lambda\phi_\pi\lambda^{-1}, 1, \omega_\pi^{-1}\lambda'\phi_\pi(\lambda')^{-1}$
XIa	$\nu^{1/2}\lambda\phi_\pi\lambda^{-1}, 1, \nu^{-1/2}\lambda'\phi_\pi(\lambda')^{-1}$	M_2	$\nu^{1/2}\lambda\phi_\pi\lambda^{-1}, 1$

$$M_1 = \begin{pmatrix} 0 & 2 & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & -2 \\ & & & & 0 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & & & -1 & \\ & 0 & & & 1 \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \qquad M_3 = \begin{pmatrix} 0 & 2 & & -1 & \\ & 0 & & & 1 \\ & & 0 & & \\ & & & 0 & -2 \\ & & & & 0 \end{pmatrix}$$

$$M_4 = \begin{pmatrix} 0 & & & & \\ & 0 & -1 & & \\ & & 0 & 1 & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \qquad M_5 = \begin{pmatrix} 0 & 2 & & & \\ & 0 & -1 & & \\ & & 0 & 1 & \\ & & & 0 & -2 \\ & & & & 0 \end{pmatrix} \qquad M_6 = \begin{pmatrix} 0 & 2t & 2y & -z & \\ & 0 & & & z \\ & & 0 & & -2y \\ & & & 0 & -2t \\ & & & & 0 \end{pmatrix}$$

(5.4) As already mentioned, the numerator of the γ-factors equals the L-function of the generic constituent in each induction. Therefore we compute, and tabulate, the L-functions obtained from the γ-factors by computing the L-functions of the Weil-Deligne representations $\mathrm{rec}_{\mathrm{GT}}(\pi)$ for generic π. In the following table, for a character χ, $n(\chi)$ is the integer f such that the conductor of χ is $1 + \varpi_F^f \mathcal{O}_F$. In this table, for a supercuspidal representation π of $\mathrm{GL}(2)$ we write $\mathrm{Ind}_E^F \psi_\pi$ for the L-parameter associated to π and $\xi_{E/F}$ for the character of a quadratic extension E/F. We note that $\mathrm{Ad} \circ \mathrm{Ind}_E^F \psi_\pi = \xi_{E/F} \oplus \mathrm{Ind}_E^F(\psi_\pi/\psi_\pi^c)$ where c is complex conjugation on E/F. Also note that the only time $L(\psi_\pi/\psi_\pi^c)$ appears in the above table is when the extension E/F is ramified, and therefore totally ramified, in which case the L-function is still a polynomial in q^{-s} since the residue fields of E and F are the same.

One final piece of notation about the quadratic extension E/F. There are three such extensions up to isomorphism: $E_0 = F(\sqrt{u})$ is the unique unramified quadratic extension, $E_1 = F(\sqrt{\varpi_F})$ and $E_2 = F(\sqrt{u\varpi_F})$ are the two ramified extensions, where $u \in \mathcal{O}_F^\times$ such that $\sqrt{u} \notin F$.

(5.5) We now compute the standard L-functions of the supercuspidal representations of $\mathrm{GSp}(4, F)$. These representations are tempered, so the L-functions will be equal to the reciprocals of the numerators of the associated γ-factors. Since we assume that the residual characteristic is not 2, every supercuspidal L-parameter is of the form $\mathrm{Ind}_E^F \sigma$ where σ is an irreducible 2-dimensional representation of W_E. By [GT10b, Theorem 6.5] we get the following cases (cases a1 and b2 would occur only if the residual characteristic were 2)

Here the standard L-function depends on whether the extension E/F is ramified or not.

The reason is that if $\mathrm{rec}_{\mathrm{GT}}(\pi) = \mathrm{Ind}_E^F \sigma$ then $\mathrm{std} \circ \mathrm{rec}_{\mathrm{GT}}(\pi)$ is

Table 3. Standard L-functions for non-supercuspidal

Class	Ramification Cases	$L(\mathrm{rec}_{\mathrm{GT}}(\pi)\otimes 1, \mathrm{std}\otimes\mathrm{std})$	Degree
I	all	$L(\chi_1)L(\chi_1^{-1})L(\chi_2)L(\chi_2^{-1})L(1)$	
	(i) $n(\chi_1)=n(\chi_2)=0$	$L(\chi_1)L(\chi_1^{-1})L(\chi_2)L(\chi_2^{-1})L(1)$	5
	(ii) $n(\chi_1)=0, n(\chi_2)>0$	$L(\chi_1)L(\chi_1^{-1})L(1)$	3
	(iii) $n(\chi_1), n(\chi_2)>0$	$L(1)$	1
IIa	all	$L(\nu^{1/2}\chi)L(1)L(\nu^{1/2}\chi^{-1})$	
	(i) $n(\chi)=0$	$L(\nu^{1/2}\chi)L(1)L(\nu^{1/2}\chi^{-1})$	3
	(ii) $n(\chi)>0$	$L(1)$	1
IIIa	all	$L(\chi)L(\nu)L(\chi^{-1})$	
	(i) $n(\chi)=0$	$L(\chi)L(\nu)L(\chi^{-1})$	3
	(ii) $n(\chi)>0$	$L(\nu)$	1
IVa		$L(\nu^2)$	1
Va	all	$L(\nu\xi)L(\xi)L(1)$	
	(i) $n(\xi)=0$	$L(\nu\xi)L(\xi)L(1)$	3
	(ii) $n(\xi)>0$	$L(1)$	1
VIa		$L(\nu)L(1)^2$	3
VII	all	$L(\chi)L(\chi^{-1})L(\xi_{E/F})L(\psi_\pi/\psi_\pi^{\mathrm{c}})$	
	(i) $n(\chi)=0, n(\xi_{E/F})=0, n(\psi_{E/F}/\psi_{E/F}^{\mathrm{c}})>0$	$L(\chi)L(\chi^{-1})L(\xi_{E/F})$	3
	(ii) $n(\chi)=0, n(\xi_{E/F})>0, n(\psi_{E/F}/\psi_{E/F}^{\mathrm{c}})=0$	$L(\chi)L(\chi^{-1})L(\psi_\pi/\psi_\pi^{\mathrm{c}})$	3
	(iii) $n(\chi)=0, n(\xi_{E/F})>0, n(\psi_{E/F}/\psi_{E/F}^{\mathrm{c}})>0$	$L(\chi)L(\chi^{-1})$	2
	(iv) $n(\chi)>0, n(\xi_{E/F})=0, n(\psi_{E/F}/\psi_{E/F}^{\mathrm{c}})>0$	$L(\xi_{E/F})$	1
	(v) $n(\chi)>0, n(\xi_{E/F})>0, n(\psi_{E/F}/\psi_{E/F}^{\mathrm{c}})=0$	$L(\psi_\pi/\psi_\pi^{\mathrm{c}})$	1
	(vi) $n(\chi)>0, n(\xi_{E/F})>0, n(\psi_{E/F}/\psi_{E/F}^{\mathrm{c}})>0$	1	0
VIIIab	all	$L(1)^2L(\xi_{E/F})L(\psi_\pi/\psi_\pi^{\mathrm{c}})$	
	(i) $n(\xi_{E/F})=0$	$L(1)^2L(\xi_{E/F})$	3
	(ii) $n(\xi_{E/F})>0, n(\psi_{E/F}/\psi_{E/F}^{\mathrm{c}})=0$	$L(1)^2L(\psi_\pi/\psi_\pi^{\mathrm{c}})$	3
	(iii) $n(\xi_{E/F})>0, n(\psi_{E/F}/\psi_{E/F}^{\mathrm{c}})>0$	$L(1)^2$	2
IXa	all	$L(\nu\xi)L(\psi_\pi/\psi_\pi^{\mathrm{c}})$	
	(i) $n(\xi_{E/F})=0, n(\psi_{E/F}/\psi_{E/F}^{\mathrm{c}})>0$	$L(\nu\xi)$	1
	(ii) $n(\xi_{E/F})>0, n(\psi_{E/F}/\psi_{E/F}^{\mathrm{c}})=0$	$L(\psi_\pi/\psi_\pi^{\mathrm{c}})$	1
	(iii) $n(\xi_{E/F})>0, n(\psi_{E/F}/\psi_{E/F}^{\mathrm{c}})>0$	1	0
X		$L(1)$	1
XIa		$L(1)$	1

Table 4. Standard L-functions for supercuspidal

Class	Conditions	Standard L-function
a2	$\sigma^c \not\cong \sigma\otimes\chi, \sigma\cong \mathrm{Ind}_K^E\,\psi, \mathrm{Gal}(K/F)\cong \mathbb{Z}/4\mathbb{Z}$	$L(\xi_{E/F})$ or 1
a3	$\sigma^c \not\cong \sigma\otimes\chi, \sigma\cong \mathrm{Ind}_K^E\,\psi, \mathrm{Gal}(K/F)\cong (\mathbb{Z}/2\mathbb{Z})^2$	
a4	$\sigma^c \not\cong \sigma\otimes\chi, \sigma\cong \mathrm{Ind}_K^E\,\psi, \lvert\mathrm{Gal}(K/F)\rvert = 8$	
b1	$\sigma^c \cong \sigma\otimes\chi, \chi^c\not\cong\chi$	

(a2) $\xi_{E/F}\oplus$ an irreducible 4-dimensional representations,
(a3) $\xi_{E/F}\oplus$ a direct sum of two irreducible 2-dimensional representations,
(a4) $\xi_{E/F}\oplus$ a 4-dimensional representation whose restriction to a quadratic extension of E is a direct sum of two irreducible 2-dimensional representations,
(b1) $\xi_{E/F}\oplus$ a direct sum of two irreducible 2-dimensional representations.

Table 5. Standard L-functions by degree

$d(L)$	#	L	Repr.	Poles	Conditions
0	1	1	VII vi IX a iii SC, E/F r.	—	
1	1	$L(1)$	I iii II a ii V a ii X XI a	1	
	2 3 4	$L(\nu)$ $L(\nu^2)$ $L(\nu\xi)$	III a ii IV a IX a i	q q^2 $-q$	
	5	$L(\xi_{E/F})$	SC, E/F ur. VII iv	-1	
	6	$L(\psi_\pi/\psi_\pi^c)$	IX a ii VII v	b	$b \neq 1, \lvert b\rvert = 1$
2	1	$L(\chi)L(\chi^{-1})$ $L(1)^2$	VII iii VIII ab iii	a, a^{-1} $1, 1$	$a \neq 1, \pm q^{\pm 1}$
3	1 2 3 4 5 6 7 8 9	$L(\chi_1)L(\chi_1^{-1})L(1)$ $L(\nu^{1/2}\chi)L(1)L(\nu^{1/2}\chi^{-1})$ $L(\chi)L(\nu)L(\chi^{-1})$ $L(\nu\xi)L(\xi)L(1)$ $L(\nu)L(1)^2$ $L(\chi)L(\chi^{-1})L(\xi_{E/F})$ $L(1)^2L(\xi_{E/F})$ $L(\chi)L(\chi^{-1})L(\psi_\pi/\psi_\pi^c)$ $L(1)^2L(\psi_\pi/\psi_\pi^c)$	I ii II a i III a i V a i VI ab VII i VIII ab i VII ii VIII ab ii	$a, a^{-1}, 1$ $a\sqrt{q}, a^{-1}\sqrt{q}, 1$ a, a^{-1}, q $-q, -1, 1$ $q, 1, 1$ $a, a^{-1}, -1$ $1, 1, -1$ a, a^{-1}, b $1, 1, b$	$a \neq q^{\pm 1}$ $a^2 \neq q^{\pm 1}, a \neq q^{\pm 3/2}$ $a \neq 1, q^{\pm 2}$ $a \neq 1, \pm q^{\pm 1}, b \neq 1, \lvert b\rvert = 1$ $b \neq 1, \lvert b\rvert = 1$
5	1	$L(\chi_1)L(\chi_1^{-1})L(\chi_2)L(\chi_2^{-1})L(1)$	I i	$a, a^{-1}, b, b^{-1}, 1$	$a^{\pm 1}, b^{\pm 1}, a^{\pm 1}b^{\pm 1} \neq q$

(5.6) We compile the list of standard L-functions in a table according to the degree of the L-function as a rational function of q^{-s}.

Note that for quadratic $\xi_{E/F}$ the L-function is $(1+q^{-s})^{-1}$ when E/F is unramified ($\mathrm{Ind}_E^F 1 = 1 \oplus \xi_{E/F}$ and since E/F is unramified, $L(\mathrm{Ind}_E^F 1) = (1-q^{-2s})^{-1}$). Also, if $b = (\psi_\pi/\psi_\pi^c)(\varpi_F)$ then $b \neq 1$ (or else $\mathrm{Ind}_E^F \psi_\pi$ would split). To show that $|b| = 1$ note that if $f = n(\psi_\pi)$ then ψ_π is a finite order character on $\mathcal{O}_F^\times$, since it factors through $\mathcal{O}_F^\times/1+\varpi_F^f\mathcal{O}_F$ which is a finite group. Therefore $\psi_\pi(\varpi_F)$ and $\psi_\pi^c(\varpi_F)$ differ by the value of ψ_π at an element in $\mathcal{O}_F^\times$, which has absolute value 1.

6. A local converse theorem for GSp(4)

We start with a useful lemma:

Lemma 6.1. *Let π and π' be two supercuspidal representations of* $\mathrm{GL}(2,F)$*, where F is a finite extension of $\mathbb{Q}_p$ for p odd, such that $\gamma(s, \mathrm{Ad}\circ\pi, \eta) = \gamma(s, \mathrm{Ad}\circ\pi', \eta)$ for every character η. Then $\pi \cong \pi' \otimes \tau$ where τ is a character.*

Proof. Since $\mathrm{Ad}\circ\pi$ and $\mathrm{Ad}\circ\pi'$ are representations of $\mathrm{GL}(3,F)$ we may apply the local converse theorem for GL(3) ([JPSS79, Lemma 7.5.3]) to deduce that $\mathrm{Ad}\circ\pi \cong \mathrm{Ad}\circ\pi'$. Let E/F

and E'/F be quadratic characters such that $\pi = \operatorname{Ind}_E^F \psi$ and $\pi' = \operatorname{Ind}_{E'}^F \psi'$. Then we get that $\xi_{E/F} \oplus \operatorname{Ind}_E^F \psi/\psi^c = \xi_{E'/F} \oplus \operatorname{Ind}_{E'}^F \psi'/(\psi')^c$.

If $E = E'$ we deduce that $\psi/\psi^c = (\psi'/(\psi')^c)^{\pm 1}$, say $\psi/\psi^c = \psi'/(\psi')^c$ (the other case being analogous). Then $\tau = \psi'/\psi$ has the property that $\tau = \tau^c$ so there exists a character τ on W_F such that $\psi' = \psi\tau|_{W_E}$. It follows that $\pi' = \tau \otimes \pi$.

If $E \neq E'$ then we deduce that $\operatorname{Ind}_E^F \psi/\psi^c = \xi_{E'/F} \oplus \xi_{E'/F}^{c_E}$ and $\operatorname{Ind}_{E'}^F \psi'/(\psi')^c = \xi_{E/F} \oplus \xi_{E/F}^{c_{E'}}$ are reducible where c_E and $c_{E'}$ are the conjugations in E and E'. Simplifying we deduce that $\xi_{E/F}^{c_{E'}} \cong \xi_{E'/F}^{c_E}$.

If E is unramified then E' is ramified so $I_E = I_F \neq I_{E'}$. Since inertia subgroups are normal in Galois groups it follows that $c_{E'} I_F c_{E'} = I_F$ and $c_E I_F c_E = I_F$. Therefore $\xi_{E/F}^{c_{E'}}$ is trivial on I_F and so $\xi_{E'/F}^{c_E}$ iss trivial on I_F which would imply that $\xi_{E'/F}$ is trivial on I_F, which cannot be since E'/F is ramified.

If E and E' are the two ramified quadratic extensions of F, as before we get that both $\xi_{E/F}$ and $\xi_{E'/F}$ are trivial on I_E and $I_{E'}$. But this would imply that they are trivial on I_F which cannot be since E and E' are ramified over F. $\square$

Proposition 6.2. *For an irreducible smooth representation of* $\mathrm{GSp}(4, F)$ *where* F *is a p-adic field we write* $\mathrm{ST}(\pi)$ *for the Sally-Tadic class of* π*, where if* π *is supercuspidal then* $\mathrm{ST}(\pi) = SC$.

Let π_1 *and* π_2 *be two tempered or generic irreducible smooth representations of* $\mathrm{GSp}(4, F)$. *Suppose* $\gamma(s, \pi_1, \eta, \mathrm{std}) = \gamma(s, \pi_2, \eta, \mathrm{std})$ *for all characters* η. *If* $\{\mathrm{ST}(\pi_1), \mathrm{ST}(\pi_2)\} \neq \{X, XIa\}$ *then* $\mathrm{ST}(\pi_1) = \mathrm{ST}(\pi_2)$.

Proof. – Degrees ≥ 2*:* The preceding table shows that if π_1 belongs to a Sally-Tadic class such that the degree of the standard L-function is ≥ 2, or if it belongs to one of the following classes: III a ii, IV a, IX a i, then π_2 must also belong to the same Sally-Tadic class, or else the two standard L-functions would differ.

– Degree 1, cases 5 and 6:

(1) (IX a ii and VII v) To tell apart the classes VII v and IX a ii let π_1 of type VII v be $\chi \rtimes \operatorname{Ind}_E^F \psi$ and let π_2 of type IX a ii be $\nu\xi_{E'/F} \rtimes \nu^{-1/2} \operatorname{Ind}_{E'}^F \psi'$, where E and E' are ramified over F. We will do this first case in extra detail, as the following cases are done similarly. For a character η the equality of L-functions twisted by η is
$$L(\chi\eta)L(\chi^{-1}\eta)L(\xi_{E/F}\eta)L(\psi/\psi^c\eta|_{W_E}) = L(\nu\xi_{E'/F}\eta)L(\psi'/(\psi')^c\eta|_{W_{E'}})$$
Twisting by $\xi_{E'/F}$ we get
$$L(\chi\xi_{E'/F})L(\chi^{-1}\xi_{E'/F})L(1)L(\psi/\psi^c\xi_{E'/F}|_{W_E}) = L(\nu)L(\psi'/(\psi')^c)$$
Since a pole of $L(\psi/\psi^c\xi_{E'/F}|_{W_E})$ has absolute value 1 it follows that the factor $L(\nu)$ has to be equal to either $L(\chi\xi_{E'/F})$ or $L(\chi^{-1}\xi_{E'/F})$. Therefore $\chi^{\pm 1} = \nu\xi_{E'/F}$ which is a contradiction.

(2) (IX a ii and VII iv) Let $\pi_1 = \chi \rtimes \operatorname{Ind}_E^F \psi$ be of class VII iv and $\pi_2 = \nu\xi_{E'/F} \rtimes \nu^{-1/2} \operatorname{Ind}_{E'}^F \psi'$ be of class IX a ii with E/F unramified and E'/F ramified. Twisting by $\xi_{E'/F}$ we get
$$L(\chi\xi_{E'/F})L(\chi^{-1}\xi_{E'/F})L(\xi_{E/F}\xi_{E'/F'})L(\psi/\psi^c\xi_{E'/F}|_{W_E}) = L(\nu)L(\psi'/(\psi')^c)$$

Because $L(\nu)$ is a polynomial in q^{-s} it cannot equal $L(\psi/\psi^c \otimes \xi_{E'/F}|_{W_E})$, a polynomial in q^{-2s}. Therefore, without loss of generality we may assume that $\chi = \nu\xi_{E'/F}$. Twisting by $\xi_{E/F}$ now shows that $\psi'/(\psi')^c = \xi_{E'/F}|_{W_E}$. But then twisting by $\xi_{E'/F}$ again gives a contradiction.

(3) (VII iv and supercuspidal with E/F unramified) let $\pi_1 = \chi \rtimes \pi$ and π_2 be supercuspidal with E/F unramified. Twist by $\eta = \chi^{-1}$ in which case the γ-factor for π_1 will have a factor of $\gamma(1)$, while the γ-factor for the supercuspidal will still be equal to 1.

(4) (IX a ii or VII v and supercuspidal with E/F unramified) Twisting by $\xi_{E/F}$ or χ gives a contradiction as before.

(5) (VII v and VII iv) Let $\pi_1 = \chi \rtimes \mathrm{Ind}_E^F \psi$ and $\pi_2 = \chi' \rtimes \mathrm{Ind}_{E'/F} \psi'$ where χ and χ' are ramified, E/F is unramified and ψ/ψ^c is ramified, and E'/F is ramified and $\psi'/(\psi')^c$ is unramified. Twisting by χ we obtain

$$L(\chi^2)L(1)L(\chi\xi_{E/F})L(\psi/\psi^c\chi|_{W_E}) = L(\chi'\chi)L((\chi')^{-1}\chi)L(\xi_{E'/F}\chi)L(\psi'/(\psi')^c\chi|_{W_{E'}})$$

The first possibility is that $\chi' = \chi^{\pm 1}$. Without loss of generality we may assume that $\chi' = \chi$ in which case the equality of γ-factors becomes

$$\gamma(s, \mathrm{Ad} \circ \pi, \eta) = \gamma(s, \mathrm{Ad} \circ \pi', \eta)$$

for all characters η. By Lemma 6.1 we deduce that there exists a character τ such that $\pi' \cong \pi \otimes \tau$ which contradicts the fact that E is unramified while E' is ramified.

The second possibility is that $\chi = \xi_{E'/F}$. Twisting by $\xi_{E'/F}$ gives (as we have already treated the case $\chi = \chi'$) that $\psi' = (\psi')^c$ which cannot be since π' is supercuspidal.

The final possibility is that $\psi'/(\psi')^c = \chi^{-1}|_{W_{E'}}$. Since ψ' is unramified it follows that $\chi = \xi_{E'/F}\chi_1$ where χ_1 is unramified. Then twisting by ξ_1^{-1} we get

$$L(\xi_{E'/F})L(\xi_{E'/F}\chi_1^2)L(\chi_1\xi_{E/F})L(\psi/\psi^c\xi_1|_{W_E}) = L(\chi'\chi_1^{-1})L((\chi'\chi)^{-1})L(\xi_{E'/F}\chi_1)L(1|_{W_{E'}})$$

Since $\xi_{E'/F}$ and χ' are ramified and χ_1 and $\xi_{E/F}$ are unramified it follows that the above equation can be rewritten as $L(\chi_1\xi_{E/F})L(\psi/\psi^c\xi_1|_{W_E}) = L(1|_{W_{E'}})$ which gives $\chi_1 = \xi_{E/F}$. Twisting by $\xi_{E'/F}$ we get

$$L(\xi_{E/F})^2 L(\psi/\psi^c\xi_{E'/F}|_{W_E}) = L(\chi'\xi_{E'/F})L((\chi')^{-1}\xi_{E'/F})L(1)L(\xi_E|_{W_{E'}})$$

which implies that $\psi/\psi^c = \xi_{E'/F}|_{W_E}$ and then that $\chi' = \xi_{E/F}\xi_{E'/F} = \chi$ which was already treated before.

– *Degree 1, case 1:* We want to tell apart classes I iii, II a ii, V a ii, X and XI a.

(1) (I iii and II a ii) Twisting by χ_i gives that $\chi_i = \nu^{-1/2}\chi^{\pm 1}$ while twisting by χ_i^{-1} gives $\chi_i = \nu^{1/2}\chi^{\pm 1}$. These contradict the irreducibility conditions for II.

(2) (I iii and V a ii) Twisting by χ_i it follows that either $\chi_i = \nu\xi$ or $\chi = \xi$. In either case twisting by ξ gives a contradiction.

(3) (I iii and X or XI a) Twisting by χ_i gives a contradiction.

(4) (II a ii and V a ii) Twisting by ξ we get a contradiction.

(5) (II a ii and X or XI a) Twisting by χ gives a contradiction.

(6) (V a ii and X or XI a) Twisting by ξ gives a contradition.

– *Degree 0:*

(1) (VII vi and supercuspidal with E/F ramified) Let $\pi_1 = \chi \rtimes \pi$ and $\pi_2 = \operatorname{Ind}_{E'}^F \sigma$. Twisting by χ^{-1} we get that χ must be $\xi_{E'/F}$ and the L-function for the supercuspidal is of degree 1 while the one for VII vi is of degree at least 2.
(2) (IX a iii and supercuspidal with E/F ramified) Let $\pi_1 = \nu\xi \rtimes \nu^{-1/2}\pi$ and $\pi_2 = \operatorname{Ind}_{E'}^F \sigma$. Twisting by $\eta = \xi_{E/F}$ where E/F is the ramified field for IX a iii we get a contradiction.
(3) (VII vi and IX a iii) Let $\pi_1 = \chi \rtimes \operatorname{Ind}_E^F \psi$ and let $\pi_2 = \nu\xi_{E'/F} \rtimes \nu^{-1/2} \operatorname{Ind}_{E'}^F \psi'$. Twisting by $\eta = \xi_{E'/F}$ give

$$L(\chi\xi_{E'/F})L(\chi^{-1}\xi_{E'/F})L(\xi_{E/F}\xi_{E'/F})L(\psi/\psi^c\xi_{E'/F}|_{W_E}) = L(\nu)L(\psi'/(\psi')^c)$$

Assume first that $\chi = \nu^{\pm 1}\xi_{E'/F}$. Then E and E' are the two ramified extensions of F, or else we get a contradiction. Thus $L(\nu^{-1}) = L(\psi'/(\psi')^c)$ which cannot be since we've already seen that the pole of the RHS has absolute value 1. The only possibility left is that $L(\psi/\psi^c\xi_{E'/F}|_{W_E}) = L(\nu)$ which again cannot occur since the pole of the LHS has absolute value 1.

□

Remark 1. The classes X and XIa cannot be differentiated on the basis of standard γ-factors with twists by characters. For example: let π be a supercuspidal representation of GL(2) with trivial central character. Let $\pi_1 = (\nu^{3/2}\pi) \rtimes \nu^{-1}\sigma$ and π_2 be the generic constituent of $\nu^{1/2}\pi \rtimes \nu^{-1/2}\sigma$. Then π_1 is in class X and π_2 is in class XIa, their L-functions are equal because character twists of irreducible representations are still irreducible representations. Finally, for the ε-factors note that (let $\rho = \operatorname{rec}(\pi)$, then $\operatorname{rec}(\pi\chi) = \rho\chi$; we use that $\det \rho = 1$)

$$\begin{aligned}
\varepsilon(\pi_1, \eta, \mathrm{std}) &= \varepsilon(\eta\nu^{3/2}\rho)\varepsilon(\eta)\varepsilon(\eta\nu^{-3/2}(\det\rho)^{-1}\rho) \\
&= \varepsilon(\eta\rho)\varepsilon(\eta)\varepsilon(\eta\rho) \\
&= \varepsilon(\eta\nu^{1/2}\rho)\varepsilon(\eta)\varepsilon(\eta\nu^{-1/2}\rho) \\
&= \varepsilon(\pi_2, \eta, \mathrm{std})
\end{aligned}$$

The second and third equalities follow from the explicit formula for ε-factors using Gauss sums. Moreover, the central characters of π_1 and π_2 are both equal to $\nu^{-1/2}\sigma^2$.

Corollary 6.3. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A}_F)$ as in Conjecture 1.1. If v is a nonarchimedean place such that $\mathrm{ST}(\pi_v)$ is not X or XIa and such that π_v is either tempered or generic, then the monodromy operator of $\mathrm{WD}(\rho_{\pi,\iota}|_{W_{F_v}})$ is the same as the monodromy operator of $\operatorname{rec}_{\mathrm{GT}}(\pi_v \otimes |\nu|^{-3/2})$*

Proof. By Proposition 6.2 the representations π_v and π_v^G (the latter being generic) are in the same Sally-Tadic class, then use Conjecture 1.1 for π^G. □

Corollary 6.4 (Skinner-Urban Conjecture)**.** *If π is the cuspidal automorphic representation associated to a holomorphic Siegel modular form and v is a finite place such that π_v is a generic para-spherical representation then the rank of the monodromy operator is 1. This is* [SU06, Conjecture 3.1.7].

Proof. Generic para-spherical representations are of class II a so the result follows from Corollary 6.3. □

Proposition 6.5. *Let π_1 and π_2 be irreducibile admissible representations of* $\mathrm{GSp}(4,F)$*. If π_1 and π_2 are generic and $\mathrm{ST}(\pi_1)=\mathrm{ST}(\pi_2)\notin\{X, XIa, SC\}$ and if for all characters ω we have $\gamma(s,\pi_1,\omega,\mathrm{std})=\gamma(s,\pi_2,\omega,\mathrm{std})$, then there exists a quadratic character η such that $\pi_1\cong\pi_2\otimes\eta$ with one exception: let E/F be a ramified quadratic character and let ξ be the character of this extension, let $\chi^2=\xi$, ω a quadratic character $\neq 1,\xi$, $\psi/\psi^c=\omega\chi$ and $\psi'/(\psi')^c=\chi$; then $\chi\rtimes\mathrm{Ind}_E^F\psi$ and $\omega\chi\rtimes\mathrm{Ind}_E^F\psi'$ have the same γ-factors for all character twists.*

Proof. – *Classes IV a, VI a, V a i, III a i:* Using the fact that the central characters are the same, it follows that there exists a quadratic character η such that $\mathrm{rec}_{\mathrm{GT}}(\pi_1)\cong\mathrm{rec}_{\mathrm{GT}}(\pi_2)\otimes\eta$.
– *Class I i:* Suppose $\pi_1=\chi_1\times\chi_2\rtimes\sigma$ and $\pi_2=\chi_1'\times\chi_2'\rtimes\sigma'$. It follows that $\{\chi_1,\chi_1^{-1},\chi_2,\chi_2^{-1}\}=\{\chi_1',(\chi_1')^{-1},\chi_2',(\chi_2')^{-1}\}$. In all possible cases we deduce the existence of a quadratic character η such that $\mathrm{rec}_{\mathrm{GT}}(\pi_1)\cong\mathrm{rec}_{\mathrm{GT}}(\pi_2)\otimes\eta$, from the fact that the central characters of π_1 and π_2 are equal.
– *Class I ii:* Suppose $\pi_1=\chi_1\times\chi_2\rtimes\sigma$ and $\pi_2=\chi_1'\times\chi_2'\rtimes\sigma'$. It follows that $\{\chi_1,\chi_1^{-1}\}=\{\chi_1',(\chi_1')^{-1}\}$. Moreover, $\varepsilon(s,\chi_2\omega,\psi)\varepsilon(s,\chi_2^{-1}\omega,\psi)=\varepsilon(s,\chi_2'\omega,\psi)\varepsilon(s,(\chi_2')^{-1}\omega,\psi)$ for all characters ω. This can be rewritten as $\varepsilon(s,\sigma\otimes\omega,\psi)=\varepsilon(s,\sigma'\otimes\omega,\psi)$ where $\sigma=\mathrm{Ind}\,\chi_2\otimes\chi_2^{-1}$ and $\sigma'=\mathrm{Ind}\,\chi_2'\otimes(\chi_2')^{-1}$. By [JL70, Corollary 2.19] it follows that $\sigma\cong\sigma'$ which implies that $\{\chi_2,\chi_2^{-1}\}=\{\chi_2',(\chi_2')^{-1}\}$. As before, we deduce the existence of a quadratic character η such that $\mathrm{rec}_{\mathrm{GT}}(\pi_1)\cong\mathrm{rec}_{\mathrm{GT}}(\pi_2)\otimes\eta$.
– *Class I iii:* If $\pi_1=\chi_1\times\chi_2\rtimes\sigma$ and $\pi_2=\chi_1'\times\chi_2'\rtimes\sigma'$ with $\chi_1,\chi_1',\chi_2,\chi_2'$ ramified characters then by twisting by χ_1 we deduce from the equality of L-functions that $\chi_1\in\{\chi_1',(\chi_1')^{-1},\chi_2',(\chi_2')^{-1}\}$. Without loss of generality we may assume that $\chi_1\in\{\chi_1',(\chi_1')^{-1}\}$; twisting by χ_2 shows that $\chi_2\in\{\chi_2',(\chi_2')^{-1}\}$ and again we deduce compatibility up to a quadratic twist.
– *Class II a i:* If $\pi_1=\nu^{1/2}\chi\times\nu^{-1/2}\chi\rtimes\sigma$ and $\pi_2=\nu^{1/2}\chi'\times\nu^{-1/2}\chi'\rtimes\sigma$ then $\{\chi,\chi^{-1}\}=\{\chi',(\chi')^{-1}\}$. Again we deduce $\mathrm{rec}_{\mathrm{GT}}(\pi_1)\cong\mathrm{rec}_{\mathrm{GT}}(\pi_2)\otimes\eta$.
– *Class II a ii:* If $\pi_1=\nu^{1/2}\chi\times\nu^{-1/2}\chi\rtimes\sigma$ and $\pi_2=\nu^{1/2}\chi'\times\nu^{-1/2}\chi'\rtimes\sigma$ then twisting by χ' shows that $\chi'\in\{\chi,\chi^{-1}\}$ and the conclusion follows.
– *Class III a ii:* As in case Iii we deduce $\mathrm{rec}_{\mathrm{GT}}(\pi_1)\cong\mathrm{rec}_{\mathrm{GT}}(\pi_2)\otimes\eta$ for some quadratic character η by going to $\mathrm{GL}(2)$.
– *Class V a ii:* Twisting by ξ does the job.
– *Classes VII i, ii, iii, VIII ab i, ii, iii:* For classes VII i,ii and iii let $\pi_1=\chi\rtimes\pi$ and $\pi_2=\chi'\rtimes\pi'$ where $\pi=\mathrm{Ind}_E^F\psi$ and $\pi'=\mathrm{Ind}_{E'}^F\psi'$, where E and E' are quadratic extensions of F. From the equality of L-functions we get that $\chi'=\chi^{\pm1}$. For classes VIII ab i, ii, iii let $\pi_1=1\rtimes\pi$ and $\pi_2=1\rtimes\pi'$. In both cases we get that for all characters η, $\gamma(s,\mathrm{Ad}\circ\pi,\eta)=\gamma(s,\mathrm{Ad}\circ\pi',\eta)$. By Lemma 6.1 we deduce that there exists a character τ such that $\pi'\cong\pi\otimes\tau$. Using the isomorphism $\chi\rtimes\pi\cong\chi^{-1}\rtimes\chi\pi$ ([ST93, Proposition 4.8 (ii)]) we have that $\pi_2=\chi\rtimes\tau\pi=\pi_1\otimes\tau$. Comparing central characters we again obtain that τ is quadratic.
– *Classes VII iv, v, vi:* Let $\pi_1=\chi\rtimes\pi$ and $\pi_2=\chi'\rtimes\pi'$ and $\pi=\mathrm{Ind}_E^F\psi$ and $\pi'=\mathrm{Ind}_{E'}^F\psi'$. Twisting the L-function equation by χ we obtain two cases

(1) $\chi'=\chi^{\pm1}$. In this case the argument from VII i, ii, iii applies.
(2) $\chi=\xi_{E'/F}$ which would imply that $\psi'/(\psi')^c=1$ which cannot be since π' is a supercuspidal representation.

(3) $L(\psi'/(\psi')^c\chi|_{W_{E'}}) = L(1)$. Analogously, twisting by χ^{-1} we get $\psi'/(\psi')^c = \chi|_{W_{E'}}$ so $\chi^2|_{W_{E'}} = 1$. We may conclude that

$$\operatorname{Ind}_E^F \psi/\psi^c = \chi' \oplus \chi'\xi_{E/F}$$
$$\operatorname{Ind}_{E'}^F \psi'/(\psi')^c = \chi \oplus \chi\xi_{E'/F}$$

The equality of L-functions can then be rewritten as

$$L(\chi\eta)L(\chi^{-1}\eta)L(\eta\xi_{E/F})L(\eta\chi')L(\eta\chi'\xi_{E/F}) = L(\chi'\eta)L((\chi')^{-1}\eta)L(\eta\xi_{E'/F})L(\eta\chi)L(\eta\chi\xi_{E'/F})$$
$$L(\chi^{-1}\eta)L(\eta\xi_{E/F})L(\eta\chi'\xi_{E/F}) = L((\chi')^{-1}\eta)L(\eta\xi_{E'/F})L(\eta\chi\xi_{E'/F})$$

Let $\chi^{-1} = \chi\tau$ and $(\chi')^{-1} = \chi'\tau'$. From $\chi^2|_{W_{E'}} = 1$ and $(\chi')^2|_{W_E} = 1$ it follows that $\tau \in \{1, \xi_{E'/F}\}$ and $\tau' \in \{1, \xi_{E/F}\}$. Suppose $\tau = \tau' = 1$. Twisting by $\chi^{-1}\xi_{E'/F}$ and using that $\chi \neq \chi'$ it would follow that $\chi = \xi_{E/F}\xi_{E'/F}$. Analogously we'd also get $\chi' = \xi_{E/F}\xi_{E'/F}$ contradicting $\chi \neq \chi'$. If $\tau = 1$ and $\tau' = \xi_{E/F}$ but then twisting by $\xi_{E'/F}$ gives a contradiction. Assume now that $\tau = \xi_{E'/F}$ and $\tau' = \xi_{E/F}$ in which case we get

$$L(\chi\xi_{E'/F}\eta)L(\xi_{E/F}\eta)L(\chi'\xi_{E/F}\eta) = L(\chi'\xi_{E/F}\eta)L(\xi_{E'/F}\eta)L(\chi\xi_{E'/F}\eta)$$

which gives that $E = E'$ ramified over F and $\chi^2 = (\chi')^2 = \xi_{E/F}$.

– *Class IX a i, ii, iii:* Letting $\pi = \nu\xi_{E/F} \rtimes \nu^{-1/2} \operatorname{Ind}_E^F \psi$ and $\pi = \nu\xi_{E'/F} \rtimes \nu^{-1/2} \operatorname{Ind}_{E'}^F \psi'$ and twisting by $\xi_{E'/F}$ we get that $E = E'$. Then the result follows as in class VII i. $\square$

7. Local-global compatibility for holomorphic Siegel modular forms

Theorem 7.1. *Let π be as in Conjecture 1.1 and let v be a nonarchimedean place such that π_v is in the Sally-Tadic classes I, II, III, IV, V, VI, VII, VIII and IX (in particular any of Iwahori level). Then, unless π_v is the exception described in Proposition 6.5, there exists a quadratic character η such that*

- *We get local-global compatibility up to semisimplification*

$$\iota\,\mathrm{WD}(\rho_{\pi,\iota}|_{W_{F_v}})^{\mathrm{ss}} \cong \mathrm{rec}_{\mathrm{GT}}(\pi_v \otimes |\nu|^{-3/2})^{\mathrm{ss}} \otimes \eta$$

- *If, moreover, we assume that π_v is generic or tempered then*

$$\iota\,\mathrm{WD}(\rho_{\pi,\iota}|_{W_{F_v}})^{\text{Fr-ss}} \cong \mathrm{rec}_{\mathrm{GT}}(\pi_v \otimes |\nu|^{-3/2}) \otimes \eta$$

- *If π is unitary and π_v is generic of the above type then π_v is tempered.*

Proof.
- Applying Proposition 6.5 to π_v and π_v^G to get the desired local-global compatibility for π_v generic (π_v^G is automatically generic); if we no longer assume that π_v is generic, but we assume it is a constituent of an induction containing a generic representation as above, then we get local-global compatibility up to semisimplification; of course, it is expected this situation does not occur, as all π_v are expected to be tempered, and all nonsupercuspidal tempered π_v lie in the same L-packet as a generic tempered representation.
- For the temperedness result: if the local representation π_v is generic and as in the statement, it is a quadratic twist of π_v^G which is tempered.

$\square$

Remark 2. Tempered implies generic in the cases above except when π_v is of type VIb or VIIIb, in which case it lies in the same L-packet as something tempered generic.

We would like to give a more precise result when the local representation is an irreducible principal series.

Proposition 7.2. *Let π be as in the previous Theorem where the number field $F = \mathbb{Q}$. If $\pi_v = \chi_1 \times \chi_2 \rtimes \sigma$ is an irreducible principal series induced from the Borel subgroup then*

$$\iota\,\mathrm{WD}(\rho_{\pi,\iota}|_{W_{F_v}})^{\text{Fr-ss}} \cong \mathrm{rec}_{\mathrm{GT}}(\pi_v \otimes |\nu|^{-3/2})$$

Proof. By Proposition 6.5 there exists a quadratic character μ such that $\pi_v^G = \chi_1 \times \chi_2 \rtimes \sigma\mu$ and the statement of the proposition is equivalent to the triviality of μ. To tackle this we need to use the spin L-function instead of the standard one. Recall that one reason we did not use the spin L-function from the beginning was that in order to define it for nongeneric representations one had to forego reducing the statement of rigidity to that of a local converse theorem. To make up for this shortcoming we use a transcendence argument.

For a representation π of $\mathrm{GSp}(4, K)$ where K is a p-adic field, a character η of $K^\times$ and a character of K let $\gamma_{\mathrm{B}}(\pi, \eta, \psi, s)$ be the γ-factor defined using Bessel models in [PS97]. If the residue field of F has q elements then $\gamma_{\mathrm{B}}(\pi, \eta, \psi, s)$ is a rational function in q^{-s}, and we write it as $A \cdot P_{\pi,\eta,\psi}(q^{-s})$ where $P_{\pi,\eta,\psi}$ is a monic rational function. For an automorphic representation π of $\mathrm{GSp}(4, \mathbb{A}_F)$ the local γ-factors satisfy a global functional equation ([PS97, Theorem 5.3])

$$\prod_v \gamma_{\mathrm{B}}(\pi_v, \eta_v, \psi_v, s) = 1$$

Knowing that $\pi_v \cong \pi_v^G$ unless v is contained in S, a finite set of places, we deduce, as before, that

$$\prod_{v\in S} \gamma_{\mathrm{B}}(\pi_v, \eta_v, \psi_v, s) = \prod_{v\in S} \gamma_{\mathrm{B}}(\pi_v^G, \eta_v, \psi_v, s)$$

and by a transcendence argument we deduce that for each prime number $v = q$ we have

$$P_{\pi_v,\eta_v,\psi_v}(q^{-s}) = P_{\pi_v^G,\eta_v,\psi_v}(q^{-s}))$$

But we know that $\pi_v = \chi_1 \times \chi_2 \rtimes \sigma$ and $\pi_v^G = \chi_1 \times \chi_2 \rtimes \sigma\mu$ so combining [PSS81, Theorem 3.1] with [GI, Theorem A.10 (vi)] we deduce that

$$L(\sigma\eta)L(\sigma\chi_1\eta)L(\sigma\chi_2\eta)L(\sigma\chi_1\chi_2\eta) = L(\sigma\mu\eta)L(\sigma\mu\chi_1\eta)L(\sigma\mu\chi_2\eta)L(\sigma\mu\chi_1\chi_2\eta)$$

Twisting by σ^{-1} we get either that $\mu = 1$ or, without loss of generality, that $\chi_1 = \mu$. In the latter case the associated L-parameters are the same and the result follows. ☐

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