Crystalline Representations for GL(2) over Quadratic Imaginary Fields

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Abstract

Let K be a quadratic imaginary field and π an irreducible regular algebraic cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_K)$. Under the assumption that the central character χ_π is isomorphic to its complex conjugate, Taylor et al. associated p-adic Galois representations $\rho_{\pi,p} : G_K \to \mathrm{GL}(2, \overline{\mathbb{Q}}_p)$ which are unramified except at finitely many places, and such that the truncated L-function of the Galois representation equals the truncated L-function of π. We extend this result to include crystallinity of the Galois representation at p, under some restrictions on π (we require distinct Satake parameters).

We first follow Kisin and Lai in constructing geometric families of finite slope overconvergent Siegel modular forms. This is achieved by defining overconvergent Siegel modular forms geometrically, and then showing that an Atkin-Lehner operator acts completely continuously on the space of such forms. Geometric families are then defined using eigenvarieties. We exhibit, in a rigid neighborhood of a theta lift of the representation π, a dense set of classical Siegel modular forms whose associated Galois representations are crystalline at p. Using this dense set of classical points we construct an analytic Galois representation in the chosen neighborhood of the theta lift. Finally, we appeal to a theorem of Kisin to show that crystalline periods at the dense set of classical points extend to the theta lift of π.

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I will forever cherish Princeton in my heart as the place where I serendipitously met my soulmate, Diana, thousands of miles away from our home. It is with deepest pleasure that I dedicate this thesis to her and our yet unborn child. Thank you for bringing spring into my life.

To Diana.

Contents

Chapter 1

Introduction

Let $K/\mathbb{Q}$ be a quadratic imaginary field and let π be a regular algebraic irreducible cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_K)$. It is expected that associated to π there is a compatible system of Galois representations $\rho_{\pi,p} : G_K \to \mathrm{GL}(2, \overline{\mathbb{Q}}_p)$ for each prime p, such that at almost all places v, the representation $\rho_{\pi,p}|_{G_{K_v}}$ is unramified and the roots of the characteristic polynomial of $\rho_{\pi,p}(\mathrm{Frob}_v)$ are the Satake parameters of π_v (cf. [Tay02, p. 445]). For a finite place v of K, one gets local Galois representations $\rho_{\pi,p}|_{G_{K_v}} \to \mathrm{GL}(2, \overline{\mathbb{Q}}_p)$.

Such Galois representations have been constructed in the following cases.

1. $\pi \otimes \delta \cong \pi$ for a quadratic character δ of K; in that case π is the automorphic induction of a character of the splitting field of δ, and the Galois representation is the induction of the Galois representation associated, via class field theory, to this character.

2. $\pi \otimes \nu \cong (\pi \otimes \nu)^c$ for a finite order character ν of K; in that case π is a twist of the base change of a cuspidal representation of $\mathrm{GL}(2, \mathbb{A}_\mathbb{Q})$, and the Galois representation is a twist of the restriction of the Galois representation associated to a classical modular form.

3. $\pi^\vee \cong \pi^c$ for π a regular algebraic irreducible cuspidal representation of $\mathrm{GL}(n)$, such that π_v is square-integrable at some place v; in that case the Galois representations exist by [HT01, Theorem C]. The assumption of π being conjugate self-dual is essential for their method, and it is part of the "Book Project" to construct Galois representations for such π such that π_∞ is regular, but without the assumption of square integrability at some finite place. Note that in the case of GL(2), which is the setting of this thesis, this case can be recovered from the previous one. Indeed, if $\pi^\vee \cong \pi^c$ then on the level of central characters we have $\chi_\pi^{-1} \cong \chi_{\pi^c}$. Therefore, $\chi_\pi = \nu^c/\nu$ for some character ν, in which case $\pi \otimes \nu \cong (\pi \otimes \nu)^c$, since $\pi^\vee \cong \pi \otimes \chi_\pi^{-1}$.

4. $\chi_\pi \cong \chi_\pi^c$ and π_∞ has Langlands parameter $z \mapsto \begin{pmatrix} z^{1-k} & \\ & \overline{z}^{1-k} \end{pmatrix}$ where $k \geq 2$; in that case the Galois representations are constructed in a series of papers ([HST93], [Tay91], [Tay93], [BH07]).

Once Galois representations are constructed, one next step is to analyze the behavior of the local Galois representations at p. It is expected that $\rho_{\pi,p}|_{G_{K_v}}$ is de Rham for all $v \mid p$ and that it is crystalline whenever the local representation π_v is an unramified principal series.

In the first three situations, the Galois representations at p are checked to be de Rham or crystalline: in case (1) it follows from [Ser89, p. III-7] (since the representation is the automorphic induction of a character which is locally algebraic as defined in loc. cit.), in case (2) from [Del, §3] and [Fal89], and in case (3) from [HT01, Theorem VII.1.9]. In case (4), under the added assumptions that $p = v \cdot v^c$ splits in K and π_v is ordinary, it follows from [Urb05, Corollary 2] that $\rho_{\pi,p}|_{G_{K_v}}$ is ordinary. When $k \geq 2$, this implies, using [PR94, Proposition 3.1], that the local representation is semistable (this is done by studying extensions in the category of weakly admissible filtered φ, N-modules); if moreover $k \geq 3$ it follows that it is also crystalline.

The focus of this thesis is to prove crystallinity more generally in case (4), under some restrictions on the Galois representation. This is done in Chapter 5, the main theorem of the chapter being the following:

Theorem. *Let π be a cuspidal automorphic representation of* $\mathrm{GL}(2, \mathbb{A}_K)$ *with infinite component π_∞ having Langlands parameter* $z \mapsto \begin{pmatrix} z^{1-k} & \\ & \overline{z}^{1-k} \end{pmatrix}$ *where $k \geq 2$, and such that the central character χ_π satisfies $\chi_\pi \cong \chi_\pi^c$. Let p be a prime number such that $K/\mathbb{Q}$ is not ramified at p.*

- *If $p = v$ is inert in K, assume π_v is an unramified principal series with distinct Satake parameters α_v, β_v;*

- *If $p = v \cdot v^c$ splits, assume that π_v and π_{v^c} are unramified principal series with Satake parameters α_v, β_v and $\alpha_{v^c}, \beta_{v^c}$ respectively, such that $\{\alpha_v, \beta_v, \alpha_{v^c}, \beta_{v^c}\}$ are all distinct.*

Then $\rho_{\pi,p}|_{G_{K_v}}$ (as well as $\rho_{\pi,p}|_{G_{K_{v^c}}}$ in the split case) is a crystalline representation.

Our proof uses the construction of the Galois representation via congruences. Proving crystalline results when the Galois representation is constructed by congruences has been done in other contexts, for example in the case of Hilbert modular forms in [Tay95] using p-divisible groups and in [Bre99] using the following theorem of integral p-adic Hodge theory: a p-adic representation of G_K, for $K/\mathbb{Q}_p$ finite, which is congruent modulo p^n to a torsion crystalline representation of Hodge-Tate weights bounded independently of n, must itself be torsion crystalline (cf. [Liu07, 1.0.1]).

However, the case of Hilbert modular forms is amenable to p-adic Hodge theory methods because the Galois representations involved have *bounded* Hodge-Tate weights, which is one of the hypotheses of the previously mentioned theorem. In the context of modular forms over $\mathrm{GL}(2)_{/K}$ the Galois representations are constructed using congruences with Siegel modular forms of increasing Hodge-Tate weight, therefore

[Liu07, 1.0.1] is no longer applicable.

To deal with this problem, one can use global methods. We construct p-adic families of finite slope overconvergent Siegel modular forms and then use a theorem of Kisin to obtain results about crystallinity at p. In general, p-adic families of finite slope overconvergent forms are represented by rigid analytic varieties, called *eigenvarieties*. Eigenvarieties have been extensively studied since the seminal paper [CM98] where the tame level 1 eigencurve for $\mathrm{GL}(2)_{/\mathbb{Q}}$ is constructed. Subsequent developments include [Buz07] where an "eigenvariety machine" is formalized and the eigencurve is extended to general tame level, [KL05] where a one dimensional eigenvariety for Hilbert modular forms is constructed, [Eme06] which constructs a cohomogical eigenvariety, [Urb] which constructs eigenvarieties associated to reductive groups whose real points have discrete series representations, again using cohomological methods.

The construction in this thesis is tailored on that of [KL05]. It was stated in the introduction to [KL05] that their methods may be applied in the context of Siegel modular forms. In fact, the brief note [LZ05] offers a statement of intent in this direction, but one of the authors (Lai) told us that the intent was never pursued. Some of the arguments involving moduli spaces of abelian varieties of level N were used in [Til06a] and [Til06b] to study families of ordinary Siegel modular forms.

Much as in the case of [KL05] one crucial defect of the p-adic family constructed is that, unlike in the case of the [CM98] eigenvariety, it is not a priori clear that classical points are dense; this result in the case of $\mathrm{GL}(2)_{/\mathbb{Q}}$ is given by Coleman's theorem that small slope forms are classical. Such a density is necessary to apply Kisin's theorem, and in Chapter 4 we follow [KL05, Theorem 4.5.6] to find a dense set of such classical points in a neighborhood of the given Siegel modular eigenform.

Recently, Andreatta, Iovita, Pilloni and Stevens announced a work in progress to construct geometric p-adic families of Siegel-Hilbert modular forms, although it is not clear whether their method yields information about the density of classical points.

Chapter 2

Notation

A sincere apology is in order for the sometimes unsightly and stuffy notation. It is a feature of any geometric method applied to the study of overconvergent modular forms that one deals with different level moduli spaces of abelian varieties (or elliptic curves), moduli spaces which are manifest as schemes, formal schemes and rigid spaces, both with and without cusps. Erring on the side of caution, we chose to label geometric objects unambiguously, with a detrimental effect on aesthetics and, perhaps, legibility.

(2.1) The algebraic groups involved:

- Let $\mathrm{GSp}(2n)$ be the algebraic group of similitude invariants of a symplectic vector space $(V, \langle \cdot, \cdot \rangle)$ of dimension $2n$. We choose the representation of $\mathrm{GSp}(2n) \subset \mathrm{GL}(2n) \times \mathrm{GL}(1)$ of pairs $(g, \nu(g))$ consisting of matrices $g \in \mathrm{GL}(2n)$ such that $\langle gx, gy \rangle = \nu(g)\langle x, y \rangle$ for all $x, y \in V$.

- To fix notation, we choose the standard symplectic vector space, with symplectic form

$$\langle x, y \rangle = x^T \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix} y$$

- The Lie algebra $\mathfrak{gsp}(2n)$ consists of matrices of the form

$$\begin{pmatrix} A & B \\ C & \nu(g) - A^T \end{pmatrix}$$

where B and C are symmetric matrices.

- With respect to the diagonal Cartan subalgebra, the root system of $\mathfrak{gsp}(2n)$ has the roots $e_i - e_j$ and $\pm(e_i + e_j - e_0)$ for $i, j \in \{1, 2, \ldots, n\}$. Among these roots, $\pm(e_i - e_j)$ are compact, while the rest are not compact. Here e_i is the character of the maximal torus which takes the element

$$\begin{pmatrix} x_1 & & & & & \\ & \ddots & & & & \\ & & x_n & & & \\ & & & x_0 x_1^{-1} & & \\ & & & & \ddots & \\ & & & & & x_0 x_n^{-1} \end{pmatrix}$$

to x_i.

(2.2) Formal and rigid geometry (cf. [KL05, Section 2]):

- X^{rig} is the rigidification of a scheme $X_{/\mathbb{Q}_p}$.

- $\mathfrak{X}^{\mathrm{rig}}$ is the rigidification of a formal scheme $\mathfrak{X}_{/\mathbb{Z}_p}$.

- $]\mathfrak{U}_{/\mathbb{F}_p}[\subset \mathfrak{X}^{\mathrm{rig}}$ is the tube of $\mathfrak{U}_{/\mathbb{F}_p} \subset \mathfrak{X}_{/\mathbb{F}_p}$ (for definition, see §3.3.2).

- $U \Subset_V X$ means that U is relatively compact in X relative to V ([KL05, 2.1.1]).

- $X^\dagger$ is the set of rigid objects $X \subset Y \subset \overline{X}$ such that $X \Subset_{\overline{X}} Y$, where X is an admissible open relatively quasi-compact space contained in a rigid space $\overline{X}$.

(2.3) Moduli spaces of abelian varieties:

- $\mathcal{M}_N$ is the moduli space of abelian varieties with principal polarization and level N structure.

- $\mathcal{M}_{N,T}$ is the moduli space of isogenies of type T over $\mathcal{M}_N$, where T is a double coset representing a Hecke operator (see §3.1.5).

- $\mathcal{M}_{p^n,N}$ is the moduli space of abelian varieties with principal polarization, level N structure and $\Gamma_{00}(p^n)$ level structure (for definition, see §3.1.1).

- $\overline{\mathcal{M}}_N$ is a choice of proper toroidal compactification (see §3.1.2).

- $\mathfrak{M}_N$ and $\overline{\mathfrak{M}}_N$ are the formal completions of the generic fibers of $\mathcal{M}_N$ and $\overline{\mathcal{M}}_N$, respectively.

- $\mathfrak{M}_N^{\mathrm{ord}}$ and $\overline{\mathfrak{M}}_N^{\mathrm{ord}}$ are the ordinary loci in $\mathfrak{M}_N$ and $\overline{\mathfrak{M}}_N$, respectively.

- $\mathcal{M}_N^{\mathrm{rig}}$, $\overline{\mathcal{M}}_N^{\mathrm{rig}}$, $\mathcal{M}_N^{\mathrm{ord,rig}}$ and $\overline{\mathcal{M}}_N^{\mathrm{ord,rig}}$ are rigidifications of the generic fibers of the appropriate formal schemes.

- $\overline{\mathfrak{M}}_{p^n,N}^{\mathrm{ord}}$ is the formal Igusa tower.

- $\mathcal{M}_N^{\dagger}$ and $\overline{\mathcal{M}}_N^{\dagger}$, $\mathcal{M}_{p^n,N}^{\dagger}$ and $\overline{\mathcal{M}}_{p^n,N}^{\dagger}$ are systems of rigid neighborhoods of the ordinary loci.

- $\mathcal{K}_{p,N}(r)$ is the level 1 canonical subgroup over the rigid domain $\overline{\mathcal{M}}_N^{\mathrm{rig}}(r)$.

(2.4) p-adic Hodge theory:

- B_{dR} is the period ring for de Rham representations.

- B_{cris} is Fontaine's period ring for crystalline representations.

- D_{dR} and D_{cris} are the associated Dieudonne modules.

- $t \in B_{\mathrm{dR}}^{+}$ is the usual uniformizer.

Chapter 3

Geometric Families of Siegel Modular Forms

3.1 Classical Siegel Modular Forms

(3.1.1) We begin with a review of the main results of [FC90]. Let $g \geq 2$ and $N \geq 3$ be positive integers.

Let $\mathcal{M}_N$ be the category fibered in groupoids whose objects over a $\mathbb{Z}\left[\frac{1}{N}\right]$-scheme S consist of triples $(A, \lambda, \phi_N)_{/S}$ where A is an abelian scheme of relative dimension g over S, $\lambda : A \to A^\vee$ is a principal polarization, and $\phi_N : (\mathbb{Z}/N\mathbb{Z})^g \times \mu_N^g \to A[N]$ is a symplectic similitude level structure, where the symplectic pairing on $A[N]$ is given by the Weil pairing coming from the principal polarization λ. Let $\mathcal{M}_{p^n,N}$ be the category fibered in groupoids whose objects over a $\mathbb{Z}\left[\frac{1}{pN}\right]$-scheme S consist of $(A, \lambda, \phi_N, \phi_{p^n})_{/S}$ where (A, λ, ϕ_N) are as before, and ϕ_{p^n} is a $\Gamma_{00}(p^n)$-structure, i.e., an injection $\phi_{p^n} : \mu_{p^n}^g \hookrightarrow A[p^n]$. (We remark that over $\operatorname{Spec} \mathbb{F}_p$ the existence of a $\Gamma_{00}(p)$ level structure would force the abelian variety to be ordinary.)

We note that the congruence subgroups of $\mathrm{GSp}(2g, \mathbb{Z})$ for these two moduli problems are

$$\{g \in \mathrm{GSp}(2g, \mathbb{Z}) | g \equiv I_{2g} \pmod N\}$$

and

$$\left\{ g \in \mathrm{GSp}(2g, \mathbb{Z}) | g \equiv \begin{pmatrix} I_g & * \\ O_g & I_g \end{pmatrix} \pmod{p^n} \right\}$$

In general $\mathcal{M}_N$ is represented by an algebraic stack, but having assumed that $N \geq 3$, the stack $\mathcal{M}_N$ is in fact a scheme over $\mathbb{Z}\left[\frac{1}{N}\right]$. The scheme $\mathcal{M}_N$ is smooth, but not proper, and has an action of $\mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ ([FC90, IV.6.2 (c)]). Since $\mathcal{M}_N$ is a scheme, $\mathcal{M}_{p^n,N}$ is represented by a smooth scheme over $\mathbb{Q}_p$, and it is finite over $(\mathcal{M}_N)_{/\mathbb{Q}_p}$.

Let $\pi : \mathcal{A}_N \to \mathcal{M}_N$ be the universal abelian variety. We will denote $\underline{\omega} = \pi_* \left(\Omega^1_{\mathcal{A}_N/\mathcal{M}_N}\right)$ and let $\det \underline{\omega} = \pi_* \left(\Omega^g_{\mathcal{A}_N/\mathcal{M}_N}\right)$.

(3.1.2) Since the schemes $\mathcal{M}_N$ are not proper, their rigid analytifications do not have good properties. This paragraph reviews the toroidal compactifications of [FC90, IV]. Fixing a smooth projective polyhedral admissible cone decomposition Σ, there exists an associated projective scheme $\overline{\mathcal{M}}_N$, which depends on Σ. Note that without the projective assumption on the polyhedral decomposition, $\overline{\mathcal{M}}_N$ might only be an algebraic space, and not a scheme (cf. [FC90, V.5]). The scheme $\overline{\mathcal{M}}_N$ contains $\mathcal{M}_N$, whose complement is a divisor D_N with normal crossings. The universal abelian scheme extends to a semiabelian scheme $\pi : \overline{\mathcal{A}}_N \to \overline{\mathcal{M}}_N$.

This construction extends the sheaves of differentials to the boundary and we obtain $\overline{\Omega}^\bullet$ and $\underline{\overline{\omega}}$. The Köcher principle says that

$$H^0(\mathcal{M}_N, (\det \underline{\omega})^{\otimes k}) = H^0(\overline{\mathcal{M}}_N, (\det \underline{\overline{\omega}})^{\otimes k})$$

([FC90, Proposition 1.5 (ii), p. 140]). We will use the Köcher principle to show that sections of a certain sheaf on a rigid space form a p-adic Banach space.

The scheme $\mathcal{M}_N^* = \operatorname{Proj}\left(\bigoplus_{k\geq 1} H^0(\overline{\mathcal{M}}_N, (\det \overline{\underline{\omega}})^{\otimes k})\right)$ is the minimal compactification over $\mathbb{Z}\left[\frac{1}{N}\right]$. There is a map $\overline{\mathcal{M}}_N \to \mathcal{M}_N^*$ and let $\det \underline{\omega}^*$ be the push-forward of $\det \overline{\underline{\omega}}$. When k is large $(\det \underline{\omega}^*)^{\otimes k}$ is very ample by construction. (The reason this compactification is needed is to lift the Hasse invariant to characteristic 0.)

(3.1.3) In this paragraph we define classical Siegel modular forms of level N and classical weight κ. Let P_{2g} be the Siegel parabolic in $\mathrm{GSp}(2g)$ and let M_{2g} be its standard Levi consisting of matrices of the form $\begin{pmatrix} A & \\ & z(A^T)^{-1} \end{pmatrix}$.

Let (ρ_κ, W_κ) be the algebraic representation of highest weight κ of the Levi M_{2g}, where κ is a suitable dominant weight. Unlike in the case of Siegel modular forms of parallel weight, which can be thought of as global sections of $(\det \underline{\omega})^{\otimes k}$ for the parallel weight k, Siegel modular forms of weight κ will be global sections of a vector bundle.

Let $\underline{\omega}^\kappa$ be defined by

$$\underline{\omega}^\kappa = \mathrm{Isom}_{\mathcal{M}_N}(\mathcal{O}^g_{\mathcal{M}_N}, \underline{\omega}) \times^L \rho_\kappa$$

where $\times^L$ is the contracted product $(f \circ g, w) \sim (f, g \cdot w)$ as in [Til06a, p. 1139]. This extends to $\overline{\underline{\omega}}^\kappa$ on $\overline{\mathcal{M}}_N$ (cf. [Hid02, Section 3.2]). Let $\overline{\underline{\omega}}^\kappa(-D_N)$ be the sheaf of sections of $\overline{\underline{\omega}}^\kappa$ which vanish along the complement divisor D_N. We note that, with this notation, $(\det \underline{\omega})^{\otimes k} = \underline{\omega}^\kappa$ where $\kappa = (k, \ldots, k; -gk)$, where by $(a_1, \ldots, a_g; a_0)$ we mean the character $\sum_{i=0}^{g} a_i e_i$.

Finally, we define the space of classical Siegel modular forms of level N and weight κ as global sections

$$H^0\left(\overline{\mathcal{M}}_N, \overline{\underline{\omega}}^\kappa\right)$$

The subspace of cusp forms is $H^0\left(\overline{\mathcal{M}}_N, \overline{\underline{\omega}}^\kappa(-D)\right)$.

(3.1.4) No theory of modular forms is complete without Hecke operators, and our exposition of Hecke operators for Siegel modular forms is based on [FC90, VII].

For a prime q and an algebraic group G the local Hecke algebra is

$$\mathcal{H}_q(G) = \mathrm{Hom}_{\mathrm{cont}}(G(\mathbb{Z}_q)\backslash G(\mathbb{Q}_q)/G(\mathbb{Z}_q), \mathbb{Z})$$

which becomes an algebra under the convolution product. As before, let $G = \mathrm{GSp}(2g)$, $M = M_{2g}$ the Levi of the Siegel parabolic, and $T = T_{2g}$ the diagonal maximal torus. The Satake isomorphisms (cf. [Gro98]) show that there exist injections over $\mathbb{Z}[q^{\pm 1/2}]$

$$\mathcal{H}_q(G) \hookrightarrow \mathcal{H}_q(M) \hookrightarrow \mathcal{H}_q(T)$$

and that the algebra $\mathcal{H}_q(G)$ is generated by the characteristic functions of the double cosets

$$T_{q,1} = \mathrm{GSp}(2g, \mathbb{Z}_q) \begin{pmatrix} I_g & \\ & qI_g \end{pmatrix} \mathrm{GSp}(2g, \mathbb{Z}_q) \tag{3.1}$$

$$T_{q,i} = \mathrm{GSp}(2g, \mathbb{Z}_q) \begin{pmatrix} P_{g,i} & \\ & q^2 P_{g,i}^{-1} \end{pmatrix} \mathrm{GSp}(2g, \mathbb{Z}_q) \tag{3.2}$$

for $i \in \{2, \ldots, g\}$, where $P_{g,i}$ is the diagonal matrix with the first $g - i + 1$ elements equal to 1 and the last $i - 1$ equal to q ([FC90, VII.1]).

Finally, we mention that the fraction field of the Hecke algebra $\mathcal{H}_q(M)$ is generated over the fraction field of the algebra $\mathcal{H}_q(G)$ by the Frobenius element, which corresponds to the double coset $M(\mathbb{Z}_p) \begin{pmatrix} pI_g & \\ & I_g \end{pmatrix} M(\mathbb{Z}_p)$ ([FC90, p. 247]). It is this congruence relation that is manifest in the computation of the local L-factors associated to the Galois representations of Siegel modular forms.

(3.1.5) We would like to give a geometric description of the Hecke operators, using correspondences. For a prime q let $\mathrm{Isog}_N(q)$ be the stack whose objects over a scheme U are q-isogenies $(A, \lambda_A, \phi_{N,A}) \to (B, \lambda_B, \phi_{N,B})$ of abelian varieties with principal polarization and level structure ([FC90, VII.3]), i.e., isogenies $f : A \to B$ of degree q^{gr} for some positive integer r, such that $f^*\lambda_B = q^r \lambda_A$, and such that f preserves the level structure.

If $\mathcal{M}_N$ is representable by a scheme, and it is, having assumed that $N \geq 3$, then $\mathrm{Isog}_N(q)$ is representable by a scheme, which for consistency with the notation of [KL05], we denote $\mathcal{M}_{N,(q)}$ and therefore comes with a universal isogeny $\mathcal{A}_q \to \mathcal{B}_q$ over $\mathcal{M}_{N,(q)}$. There are two projections $\pi_1, \pi_2 : \mathcal{M}_{N,(q)} \to \mathcal{M}_N$ given by $\pi_1(A \to B) = A$ and $\pi_2(A \to B) = B$.

Assume that q is invertible over the base scheme S. Then the two projection maps π_1 and π_2 are finite and étale. Moreover, the connected components of $\mathcal{M}_{N,(q)}$ are in one-to-one correspondence with the set of double cosets in the Hecke algebra $\mathcal{H}_q(G)$, the double coset representing the "type" of the universal isogeny over the connected component.

If $S = \mathrm{Spec}(\mathbb{F}_q)$, let $\mathcal{M}^{\mathrm{ord}}_{N,(q)}$ be the moduli space of isogenies of *ordinary* abelian varieties (which is a well-defined notion, since isogenies preserve the property of being ordinary). Here, an abelian variety A/S is ordinary if the p-divisible group $A[p^\infty]$ is an extension of an étale p-divisible group by a multiplicative one. By [FC90, VII.4] the connected components of $\mathcal{M}^{\mathrm{ord}}_{N,(q)}$ are in one-to-one correspondence with double cosets of $\mathcal{H}_q(M)$. By [FC90, p. 263] it follows that

$$\mathbb{Z}\left[\mathcal{M}_{N,(q)}\right] \cong \mathbb{Z}\left[\mathcal{M}^{\mathrm{ord}}_{N,(q)}\right]$$

in the sense that the connected components of $\mathcal{M}_{N,(q)}$ and $\mathcal{M}^{\mathrm{ord}}_{N,(q)}$ are in one-to-one correspondence.

Let $T \in \mathcal{H}_q(G)$ represent a double coset. Then, by the above, associated to T there is a connected component $\mathcal{M}_{N,T}$ of $\mathcal{M}_{N,(q)}$ over any base scheme S and let $\mathcal{A}_T \to \mathcal{B}_T$ be the universal isogeny over $\mathcal{M}_{N,T}$. The two projection maps $\pi_1, \pi_2 : \mathcal{M}_{N,T} \to \mathcal{M}_N$ are finite étale away from p and finite flat over the ordinary points in $(\mathcal{M}_N)_{/\mathbb{F}_p}$.

Finally, note that over $\mathbb{Q}_p$ we can extend the above to the case of the scheme $\mathcal{M}_{p^n,N}$ to get a scheme $\mathcal{M}_{p^n,N,T}$ parametrizing isogenies of type T between points of $\mathcal{M}_{p^n,N}$.

(3.1.6) We now define the action of Hecke operators on the space of Siegel modular forms. Consider the structure morphisms $\pi : \mathcal{A}_N \to \mathcal{M}_N$ and $\pi_T : (\mathcal{A}_T \to \mathcal{B}_T) \to \mathcal{M}_{N,T}$ where the latter is the universal isogeny. This induces

$$\begin{aligned}
\pi_2^*\underline{\omega} &= \pi_2^*\pi_*\Omega^1_{\mathcal{A}_N/\mathcal{M}_N} \\
&\to \pi_{T*}\Omega^1_{(\mathcal{A}_T\to\mathcal{B}_T)/\mathcal{M}_{N,T}} \\
&\cong \pi_{T*}\pi_1^*\Omega^1_{\mathcal{A}_N/\mathcal{M}_N} \\
&\cong \pi_1^*\underline{\omega}
\end{aligned}$$

For a weight κ this induces a natural map $\theta : \pi_2^*\underline{\omega}^\kappa \to \pi_1^*\underline{\omega}^\kappa$. This gives a morphism $\pi_{1*}\pi_2^*\underline{\omega}^\kappa \to \pi_{1*}\pi_1^*\underline{\omega}^\kappa$. Composing this map with the trace map $\pi_{1*}\pi_1^*\underline{\omega}^\kappa \to \underline{\omega}^\kappa$ gives action of the Hecke operator T on the classical Siegel modular forms $H^0(\mathcal{M}_N, \underline{\omega}^\kappa)$.

Explicitly, for a Siegel modular form f (which is a function of $(A, \lambda_A, \phi_{N,A}, o)$ where o is a basis of $\underline{\omega}^\kappa$ and a Hecke operator T

$$Tf(A, \lambda_A, \phi_{N,A}, o) = \sum f(B, \lambda_B, \phi_{N,B}, \pi_* o)$$

where the sum runs over isogenies $(A, \lambda_A, \phi_{N,A}) \xrightarrow{\pi} (B, \lambda_B, \phi_{N,B})$ of type T.

(3.1.7) A better way to test for ordinarity of an abelian variety over $\mathbb{F}_p$ is to use the Hasse invariant. Let R be an $\mathbb{F}_p$-algebra and let X be an abelian scheme over $\mathrm{Spec}(R)$

with principal polarization λ and level structure. (In fact, [Kat70, 7] only requires X to be smooth.) Then we get a commutative diagram

$$\begin{tikzcd} X \arrow[r, "F_{X/R}"] \arrow[dr] & X^{(p)} \arrow[r, "F_R"] \arrow[d] & X \arrow[d] \\ & \mathrm{Spec}(R) \arrow[r, "F_R"] & \mathrm{Spec}(R) \end{tikzcd}$$

where F_R is the absolute Frobenius on R and $F_{X/R}$ is the relative Frobenius map.

By [Kat70, Theorem 7.2] there exists a map of $\mathcal{O}_{X^{(p)}}$-modules

$$\mathcal{C}_{X/R} : (F_{X/R})_*(\wedge^g \Omega^1_{X/R}) \to F_R^*(\wedge^g \Omega^1_{X/R})$$

called the Cartier operator. Given a choice of basis $o = \{o_1, \ldots, o_g\}$ of $\Omega^1_{X/R}$ there exists a constant $h(X, \{o_i\}) \in R$, called the Hasse invariant, such that

$$\mathcal{C}_{X/R}(\wedge o_i) = h(X, \{o_i\}) F_R^*(\wedge o_i)$$

For any invertible matrix T it is apparent that $h(X, To) = (\det T)^{1-p} h(X, o)$ so the Hasse invariant h defines a global section $h \in H^0((\mathcal{M}_N)_{/R}, (\det \underline{\omega})^{\otimes(p-1)})$. By the Köcher principle, h has unique extensions to $H^0((\overline{\mathcal{M}}_N)_{/R}, (\det \underline{\overline{\omega}})^{\otimes(p-1)})$ and $H^0((\mathcal{M}_N^*)_{/R}, (\det \underline{\omega}^*)^{\otimes(p-1)})$.

Proposition 3.1.8. *Let R be an $\mathbb{F}_p$-algebra and X/R a semi-abelian scheme whose abelian part is A. Then $h(X, o) = 0$ if and only if A is not ordinary.*

Proof. See [Hid02, p. 22]. □

Lemma 3.1.9. *There exists a positive integer k_0 and a lift*

$$E = \widetilde{h^{k_0}} \in H^0((\mathcal{M}_N)_{/\mathbb{Z}_p}, (\det \underline{\omega})^{\otimes k_0(p-1)})$$

of $h^{k_0} \in H^0((\mathcal{M}_N)_{/\mathbb{F}_p}, (\det \underline{\omega})^{\otimes k_0(p-1)})$.

Proof. For a sufficiently large integer k_0 the sheaf $(\det \underline{\omega}^*)^{\otimes k_0(p-1)}$ is very ample, as already observed. Then the lift E exists by [Tig06, Lemma 2.3.5.1]. $\square$

Lemma 3.1.10. *Let T be a Hecke operator and let $\pi_1, \pi_2 : \mathcal{M}_{N,T} \to \mathcal{M}_N$ be the two projections. Then the map*

$$\theta : \pi_2^*(\det \omega)^{\otimes(p-1)} \to \pi_1^*(\det \omega)^{\otimes(p-1)}$$

induced by the Hecke operator T as in §3.1.6 has the property that $\theta(\pi_2^* h) = \pi_1^* h$.

Proof. Let $f : (A, \lambda_A, \phi_{N,A}) \to (B, \lambda_B, \phi_{N,B})$ be an isogeny over an $\mathbb{F}_p$-algebra R and choose o_A a basis of $\Omega^1_{A/R}$ and a basis o_B of $\Omega^1_{B/R}$ such that $f^* o_B = o_A$. If $f^{(p)}$ is the pullback of f via F_R then

$$\begin{aligned} h(B, o_B) F_R^*(\det o_A) &= h(B, o_B) F_R^*(f^{(p)*} \det o_B) \\ &= h(B, o_B) f^{(p)*} F_R^*(\det o_B) \\ &= f^{(p)*} \left(h(B, o_B) F_R^*(\det o_B)\right) \\ &= f^{(p)*} \mathcal{C}_{B/R}(\det o_B) \\ &= \mathcal{C}_{A/R}(f^* \det o_B) \\ &= h(A, o_A) F_R^*(f^* \det o_B) \\ &= h(A, o_A) F_R^*(\det o_A) \end{aligned}$$

Therefore

$$\theta(\pi_2^* h)(f, o_A, o_B) = h(B, o_B) = h(A, o_A) = \pi_1^*(h)(f, o_A, o_B)$$

Here we used that $\mathcal{C}_{B/R}$ commutes with étale localisation on B and that $A \to B$ is an étale map. $\square$

3.2 Automorphic Representations of $\mathrm{GSp}(4)$

In this section we reconcile the definition of genus 2 holomorphic Siegel modular forms given in §3.1.3 with that of holomorphic vectors in irreducible admissible automorphic representations of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$.

(3.2.1) The roots of the Lie algebra $\mathfrak{g} = \mathfrak{gsp}(4)$ with respect to the diagonal Cartan subalgebra $\mathfrak{h}$ are $\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta)$ of which α and $-\alpha$ (shaded black in the figure below) are compact. With respect to the standard Cartan involution, the maximal compact Lie algebra is

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \right\}$$

where B is symmetric and A is antisymmetric, while the complement in $\mathfrak{g}$ is

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ B & z - A \end{pmatrix} \right\}$$

where A and B are symmetric. If for a root ψ, s_ψ represents reflection across the vanishing hyperplane of ψ, then the Weyl groups are

$$W_{\mathfrak{g},\mathfrak{h}} = \{1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, s_\alpha s_\beta s_\alpha, s_\beta s_\alpha s_\beta, s_\beta s_\alpha s_\beta s_\alpha\}$$
$$W_{\mathfrak{k},\mathfrak{h}} = \{1, s_\alpha\}$$

(3.2.2) By Harish-Chandra's classification of discrete series representations for reductive groups, discrete series representations for $\mathrm{GSp}(4, \mathbb{R})$ are in one-to-one correspondence with *non-singular* elements of the coweight lattice λ, up to the action of $W_{\mathfrak{k},\mathfrak{h}}$ ([Sch97, p. 95]). Therefore, for $\lambda \in \mathcal{C}^{3,0} \cup \mathcal{C}^{2,1} \cup \mathcal{C}^{1,2} \cup \mathcal{C}^{0,3}$ (where $\mathcal{C}^{3,0}$, $\mathcal{C}^{2,1}$, $\mathcal{C}^{1,2}$, $\mathcal{C}^{0,3}$ are the Weyl chambers from the figure) there exists a discrete series representation

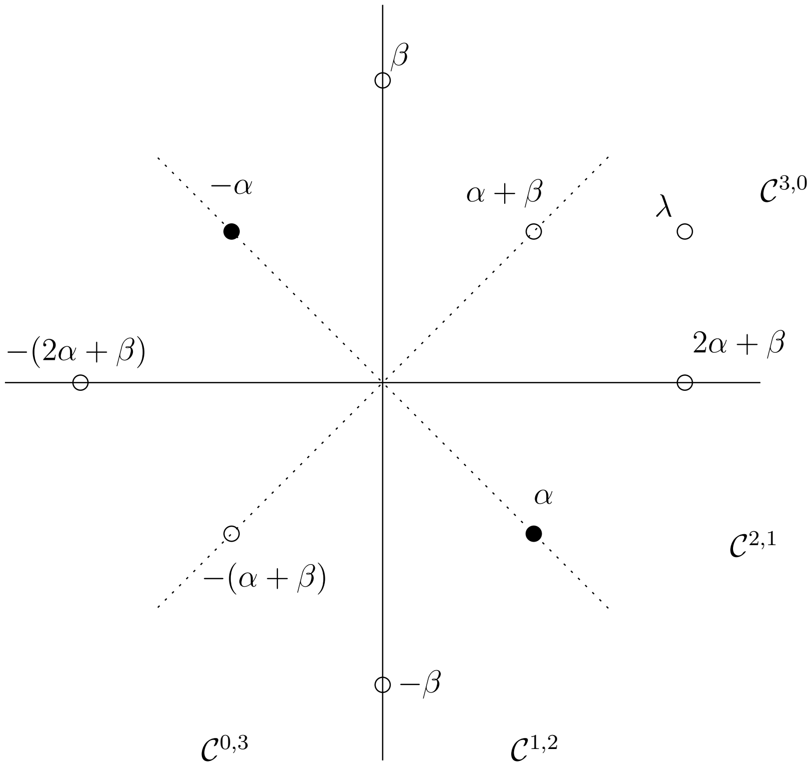

Figure 3.1: Roots of GSp(4)

π_λ; among the K-types of π_λ the smallest one is $\lambda+\rho_n-\rho_c$ (where $\rho_c=\frac{1}{2}\alpha$ and $\rho_n=\frac{1}{2}\left((2\alpha+\beta)+(\alpha+\beta)+\beta\right)=\frac{3}{2}(\alpha+\beta)$) and occurs with multiplicity 1 among the K-types (this is the Blattner conjecture, proven by Hecht and Schmid).

One may describe discrete series representations either by λ, known by the name of Harish-Chandra parameter, or by the lowest K-type $\lambda+\rho_n-\rho_c$, known by the name of Blattner parameter. Explicitly, if $\lambda=(a,b;c)=ae_1+be_2+ce_0$ is the Harish-Chandra parameter, then the Blattner parameter is

$$\lambda+\left(1,2,-\frac{3}{2}\right)=\left(a+1,b+2;c-\frac{3}{2}\right)$$

This is a recurring point of confusion in the literature, both parametrizations often being used interchangeably. We will use the Harish-Chandra parameters for describing automorphic representations of GSp(4), although classically, the weight of a Siegel modular form is the Blattner parameter of the infinite component of the automorphic representation coming from the Siegel modular form.

If λ is a *singular* element of the coweight lattice, and $\mathcal{C} \in \{\mathcal{C}^{3,0}, \mathcal{C}^{2,1}, \mathcal{C}^{1,2}, \mathcal{C}^{0,3}\}$ is a choice of Weyl chamber such that λ lies in the closure of $\mathcal{C}$, then there exists a limit of discrete series representation π_λ, which is no longer square integrable. It is not zero if and only if no positive simple compact root relative to $\mathcal{C}$ vanishes on λ (cf. [Kna86, 12.26]). If π_λ is not zero, it is a tempered representation, and its lowest K-type is $\lambda + \rho_n - \rho_c$. In the special case of GSp(4), limits of discrete series exist unless $\alpha(\lambda) = 0$ and $\mathcal{C} \in \{\mathcal{C}^{3,0}, \mathcal{C}^{0,3}\}$.

It is an important consequence of the computations in [Tay93, p. 293] that if $\lambda \in \mathcal{C}^{3,0}$, the (limit of) discrete series π_λ is holomorphic and if $\lambda \in \mathcal{C}^{0,3}$ it is antiholomorphic. This allows a choice of holomorphic Siegel modular form in the automorphic representation of GSp(4) as long as the representation of $\mathrm{GSp}(4, \mathbb{R})$ is a holomorphic (limit of) discrete series. An often quoted, but nontrivial, further result is that if $\lambda \in \mathcal{C}^{1,2} \cup \mathcal{C}^{2,1}$ then π_λ has an associated Whittaker model (π_λ is generic). We do not give a detailed proof of this fact but remark the following: by a combination of [Vog78, 6.7] and [Kos78, 6.8.1] π_λ has a Whittaker model if and only if the Gel'fand-Kirillov dimension of π_λ, which can be computed as the dimension of $[\mathfrak{k}, \mathfrak{p}_-]$ (where $\mathfrak{p}_-$ is the complex conjugate of the sum of the eigenspaces of positive noncompact roots) is equal to the dimension of $\mathfrak{u}$ (where $\mathfrak{u}$ is the nilpotent Lie subalgebra associated to the Weyl chamber $\mathcal{C}$); it is a simple exercise to check this is indeed the case in $\mathcal{C}^{1,2} \cup \mathcal{C}^{2,1}$, but not so in $\mathcal{C}^{3,0} \cup \mathcal{C}^{0,3}$. (A brief overview of the Gel'fand-Kirillov dimension: if $X = \{X_1, \ldots, X_s\}$ is a finite generating set of π_λ, and $U(\mathfrak{g})_n \subset U(\mathfrak{g})$ is the subset of degree at most n elements of the universal enveloping algebra of $\mathfrak{g}$, then $U(\mathfrak{g})_n X$ is a finite dimensional space of dimension $P(n) \in \mathbb{Z}$. For n large, the function $P(n)$ is in fact a polynomial in n, and the Gel'fand-Kirillov dimension of π_λ is the degree of the polynomial P.)

(3.2.3) It is described in [Tay88, §2.2] how to relate a holomorphic Siegel modular form with an automorphic form on GSp(4). Briefly, let $\mathrm{GSp}(4,\mathbb{R})^+$ be the subgroup of positive determinant matrices, and let $\mathcal{Z}_2$ be the set of symmetric complex 2×2 matrices with positive definite imaginary part. Starting with the algebraic representation κ of the Levi subgroup M_4 on gets a automorphy factor $J_\kappa : \mathrm{GSp}(4,\mathbb{R})^+ \times \mathcal{Z}_2 \to W_\kappa$. Let $U \subset \mathrm{GSp}(4,\mathbb{A}_{\mathbb{Q}}^f)$ be an open compact subgroup. Then from a Siegel modular form f of level U we get an automorphic form ϕ_f defined by

$$\phi_f(g) = J_\kappa(g, \sqrt{-1}I_g)^{-1} f(g\sqrt{-1}I_g)$$

where $g = \gamma u g^+$ with $\gamma \in \mathrm{GSp}(\mathbb{Q})$, $u \in U$ and $g^+ \in \mathrm{GSp}(4,\mathbb{R})^+$, while from an automorphic form ϕ we get a Siegel modular form

$$f_\phi(z) = J_\kappa(g, \sqrt{-1}I_g)\phi(g)$$

where $g\sqrt{-1}I_g = z$.

An automorphic form ϕ for GSp(4) generates, under the left regular action of $\mathrm{GSp}(4,\mathbb{A}_{\mathbb{Q}})$ an irreducible automorphic representation π_ϕ of $\mathrm{GSp}(4,\mathbb{A}_{\mathbb{Q}})$. If the holomorphic Siegel modular form f giving rise to ϕ_f has level given by an open compact $U \subset \mathrm{GSp}(4,\mathbb{A}_{\mathbb{Q}}^f)$, then $\phi_f \in \pi_f^U$.

(3.2.4) We end with a discussion of the level structures introduced in §3.1.1 from the perspective of representation theory. Let f be a cuspidal holomorphic Siegel modular form of level N. If $U_N \subset \mathrm{GSp}(4,\mathbb{A}_{\mathbb{Q}}^f)$ is the open compact group consisting of matrices $\equiv I_4 \pmod N$, and if π_f is the irreducible automorphic representation generated by ϕ_f, then $\phi_f \in \pi_f^{U_N}$. Similarly, if f is a Siegel modular form of level $p^n N$, and $U_{p^n,N} \subset \mathrm{GSp}(4,\mathbb{A}_{\mathbb{Q}}^f)$ is the open compact group consisting of matrices $\equiv I_4$

(mod N) and $\equiv \begin{pmatrix} I_2 & * \\ & I_2 \end{pmatrix}$ (mod p^n), then $\phi_f \in \pi_f^{U_{p^n,N}}$.

We would like to analyze how an eigenform of level N breaks up into eigenforms which are old at level pN. Before delving into the case of GSp(4), it is instructive to recall the case of classical modular forms for $\mathrm{GL}(2)_{/\mathbb{Q}}$. Let g be an eigenform of weight k and level $\Gamma_1(N)$ and let α and β be the eigenvalues of the Hecke operator T_p, i.e., the roots of the polynomial $x^2 - a_p x + \varepsilon(p)p^{k-1}$, where ε is the nebentypus. Then the space of oldforms of level $\Gamma_1(N) \cap \Gamma_0(p)$ is two dimensional, generated by the forms $g(z)$ and $g(pz)$, and the U_p Hecke operator acts via the matrix

$$\begin{pmatrix} a_p & -\varepsilon(p)p^{k-1} \\ 1 & \end{pmatrix}$$

If $\alpha \neq \beta$ the action of U_p is diagonalizable, obtaining two eigenforms for the U_p operator and all the other Hecke operators: $g_\alpha(z) = g(z) - \alpha g(pz)$ and $g_\beta(z) = g(z) - \beta g(pz)$. If $\alpha = \beta$, the U_p operator need not be diagonalizable, but what one can still say is that the form $g_\alpha = g(z) - \alpha g(pz)$ is an eigenform for U_p with eigenvalue α.

Returning to the case of GSp(4), let p be a prime not dividing N. Let $K_p = \mathrm{GSp}(4, \mathbb{Z}_p)$ be the standard maximal compact of $\mathrm{GSp}(4, \mathbb{Q}_p)$, and let f be a Siegel modular form of level N giving rise to an irreducible automorphic representation π_f as before. Since $\pi_{f,p}$, the local component of π_f at p, is an unramified principal series, it follows that $\pi_{f,p}^{K_p}$ is one dimensional, generated by f. Let $\alpha_p, \beta_p, \gamma_p, \delta_p$ be the Satake parameters of $\pi_{f,p}$. It follows from [Tay88, p. 40] using computations that, if K_p^P is the Iwahori subgroup with respect to the Siegel parabolic P, then $\pi_{f,p}^{K_p^P}$ is four dimensional; if the eigenvalues $\alpha_p, \beta_p, \gamma_p, \delta_p$ are all distinct, then the action of $U_{p,1}$ is diagonalizable, the eigenvectors being forms whose $U_{p,1}$-eigenvalues are α_p, β_p, γ_p and δ_p. If the eigenvalues are not distinct, the action of $U_{p,1}$ need not be diagonalizable, but for each distinct eigenvalue $\lambda \in \{\alpha_p, \beta_p, \gamma_p, \delta_p\}$ there exists an eigenform with $U_{p,1}$-eigenvalue λ. In fact Taylor's computations give an explicit basis of $\pi_{f,p}^{K_p^P}$ in the style of Bruhat (cf. [Cas, p. 62]) and the matrix of $U_{p,1}$ with respect to this basis.

Alternatively, one could use [Cas, Theorem 3.3.3] to deduce that $\pi_{f,p}^{K_p^P}$ surjects onto the Levi invariants of the Jacquet module of $\pi_{f,p}$ with respect to the unipotent subgroup of the Siegel parabolic P. To make things explicit, let

$$P = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix} \right\}$$

be the Siegel parabolic having the Levi decomposition $P = M_P N_P$ where

$$M_P = \left\{ \begin{pmatrix} * & * & & \\ * & * & & \\ & & * & * \\ & & * & * \end{pmatrix} \right\}$$

and

$$N_P = \left\{ \begin{pmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\}$$

and let

$$B = \left\{ \begin{pmatrix} * & & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \right\}$$

be the Borel, with the Levi decomposition $B = M_B N_B$, where

$$M_B = T_4 = \left\{ \begin{pmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{pmatrix} \right\}$$

is the diagonal maximal torus, and

$$N_B = \left\{ \begin{pmatrix} 1 & & * & * \\ * & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \right\}$$

In that case $K_p^P = \{g \in K_p | g \pmod p \in P(\mathbb{F}_p)\}$. Let χ be a character of T_4 such that $\pi_{f,p} = \mathrm{Ind}_B^G \chi$. Then by Theorem 3.3.3 loc. cit. there exists a surjection

$$\pi_{f,p}^{K_p^P} \twoheadrightarrow \pi_{f,p,N_P}^{K_p^P \cap M_P}$$

where π_{f,p,N_P} is the Jacquet module with respect to N_P. By [Cas, Proposition 6.4.1],

$$\pi_{f,p,N_P} \cong \bigoplus_w e^{\rho_P} \left(w^{-1} \mathrm{Ind}_{B \cap M_P}^{M_P} \chi \right)$$

where w runs over the list of Weyl group elements list on [Tay88, p. 40], and ρ_P is, as usual, half the sum of the positive roots of P. Since $\mathrm{Ind}_{B \cap M_P}^{M_P} \chi$ is unramified and $K_p^P \cap M_P$ is maximal compact, it follows that $\pi_{f,p,N_P}^{K_p^P \cap M_P}$ is four dimensional. As in [Cas, Proposition 9.2.3], it follows that $\pi_{f,p}^{K_p^P}$ is four dimensional. Surjectivity of the map shows that for each distinct eigenvalue λ there exists an eigenform whose $U_{p,1}$-eigenvalue is λ.

The combinatorics of level N eigenforms factoring into old level pN eigenforms can be analyzed analogously for various notions of "level p". If K_p^B is the Iwahori subgroup with respect to the Borel B, using the surjection

$$\pi_{f,p}^{K_p^B} \twoheadrightarrow \pi_{f,p,N_B}^{K_p^B \cap M_B}$$

it follows that $\pi_{f,p}^{K_p^B}$ is eight dimensional, the set of $U_{p,1}$ eigenvalues being the set of Satake parameters, each with multiplicity 2. We stress that $U_{p,1}$ need not be diagonalizable, but there exists an explicit map $\pi_{f,p}^{K_p} \to \pi_{f,p}^{K_p^B}$ (described in [Til06a, 3.1 (2)]), similar to the map $g(z) \mapsto g(z) - \lambda g(pz)$ in the case of $\mathrm{GL}(2)_{/\mathbb{Q}}$, constructing a eigenform of old level K_p^B whose $U_{p,1}$-eigenvalue is λ. (In general, for genus g Siegel modular forms, $\pi_{f,p}^{K_p^P}$ has dimension 2^g, while $\pi_{f,p}^{K_p^B}$ has dimension $2^g g!$, a fact that follows from the corresponding combinatorics of their associated Weyl groups.)

We use Siegel modular forms of level $\Gamma_{00}(p)$ which, at p, corresponds to the compact $K_p^1 = \{g \pmod p) \in N_P(\mathbb{F}_p)\} \subset K_p^B$. In this case, the surjection

$$\pi_{f,p}^{K_p^1} \twoheadrightarrow \pi_{f,p,N_B}^{K_p^1 \cap M_B}$$

is no longer injective (a simple dimension count suffices for this). However, by surjectivity, for each Satake parameter α, we deduce the existence of an oldform having $U_{p,1}$ eigenvalue α. We will use this in §4.2.

3.3 The Rigid Analytic Picture

(3.3.1) We start with an overview of the interplay between schemes, formal schemes and rigid analytic spaces. By [Bos, 1.13.4] there exists a functor from that category of schemes which are locally of finite type over a finite extension $L/\mathbb{Q}_p$ to the category of rigid analytic spaces over L, sending X to the associated rigid analytic space X^{rig}. Rigid GAGA states that if X is proper over L and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module then $H^q(X, \mathcal{F}) \cong H^q(X^{\mathrm{rig}}, \mathcal{F}^{\mathrm{rig}})$ for all q, and that the rigidification $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{rig}}$ is an equivalence of categories of coherent sheaves ([Bos, 1.16.12]).

If X is a scheme which is locally of finite type over $\mathcal{O}_L$, let $\mathfrak{X}$ be the formal completion of X at the special fiber. By [Bos, 2.4] there is a functor from the category of formal schemes over $\mathcal{O}_L$ to the category of rigid L-spaces associating to $\mathfrak{X}$ the rigid fiber $\mathfrak{X}^{\mathrm{rig}}$. If, moreover, X is proper over $\mathcal{O}_L$ then the natural inclusion $\mathfrak{X}^{\mathrm{rig}} \hookrightarrow (X \times_{\operatorname{Spec} \mathcal{O}_L} \operatorname{Spec} L)^{\mathrm{rig}}$ is an isomorphism.

Finally, if X is a quasi-compact rigid space and $\mathcal{F}$ is a coherent sheaf on X, then $\mathcal{F}(X)$ has a natural structure of p-adic Banach space. Recall that X is quasi-compact if it has an admissible covering with affinoids. If $\operatorname{Sp} \mathcal{R}$ is an affinoid then $\mathcal{F}(\operatorname{Sp} \mathcal{R})$ is a p-adic Banach space by [BGR84, 9.4.3/3] and [BGR84, 3.7.3/3]. If $\mathcal{U}$ is an admissible covering of X with finitely many affinoids, then using the Čech complex (cf. bottom of page 324 [BGR84]) we get a p-adic Banach module structure on $\mathcal{F}(X)$ sitting inside the first term in the complex.

(3.3.2) We recall Raynaud's theory of formal models for rigid analytic spaces, together with the notation of [KL05, §2], which we use extensively. The main theorem of Raynaud theory is that the rigidification functor $\mathfrak{X} \mapsto \mathfrak{X}^{\mathrm{rig}}$ is an equivalence of categories between the category of quasi-compact admissible formal schemes over $\mathbb{Z}_p$, localized by admissible formal blow-ups, and the category of quasi-compact quasi-separated rigid spaces over $\mathbb{Q}_p$ (cf. [BL93, 4.1]). Implicit in this theorem is the fact that morphisms of rigid spaces admit formal models.

The standard example for illustrating this equivalence is that of the formal completion $\mathfrak{X} = \operatorname{Spf} \mathbb{Z}_p\{x, y\}$ of $\operatorname{Spec} \mathbb{Z}_p[x, y]$ at the special fiber. The admissible blow-up of $\mathfrak{X}$ at an open coherent ideal generated by functions $f_1, \ldots, f_n$ can be covered by charts of the form $\mathfrak{X}_i = \operatorname{Spf} \mathbb{Z}_p\{x, y\}[T_{ij}]/(f_i T_{ij} - f_j)$ whose rigid fiber is $\operatorname{Sp} \mathbb{Q}_p\langle x, y\rangle[f_j/f_i]$, which are the standard rational domains in $\mathfrak{X}^{\mathrm{rig}}$ (cf. [BL93, 2.2]).

Given an admissible formal scheme $\mathfrak{X}$ over $\mathbb{Z}_p$, there exists a specialization morphism $\mathfrak{X}^{\mathrm{rig}} \to \mathfrak{X}$, given by rig-points of the formal model $\mathfrak{X}$, and by [BL93, 3.5] it follows that the specialization morphism surjects onto the special fiber of $\mathfrak{X}$. This motivates the definition of the "tube" of a locally closed subspace $\mathfrak{U}_{/\mathbb{F}_p}$ of $\mathfrak{X}_{/\mathbb{F}_p}$, denoted by $]\mathfrak{U}_{/\mathbb{F}_p}[$, as the set of points of $\mathfrak{X}^{\mathrm{rig}}$ which specialize to $\mathfrak{U}_{/\mathbb{F}_p}$.

Finally, we define certain relative notions in the category of rigid analytic spaces. First, a morphism $f : X \to Y$ of rigid spaces over $\mathbb{Q}_p$ is relatively quasi-compact if there exists an admissible open cover of Y such that the preimage of each open in the cover is quasi-compact. If $f : X \to Y$ is a quasi-compact morphism of rigid spaces and $U \subset X$ is an admissible open relatively quasi-compact subset there exists a notion of relative compactness of U in X over Y, denoted by $U \Subset_Y X$. We do not repeat the definition here, but refer to [KL05, 2.1.1], and remark that if $X = \operatorname{Sp} A$ and $Y = \operatorname{Sp} B$ are affinoids, the subspace $U \Subset_Y X$ if there exists an affinoid generating system $f_1, \ldots, f_n$ of A over B such that

$$U \subset \{x \in X : |f_1(x)| < 1, \ldots, |f_n(x)| < 1\}$$

(cf. [BGR84, 9.6.2]).

(3.3.3) Let E be the lift of the Hasse invariant from Lemma 3.1.9 and let $\overline{\mathcal{M}}_N^{\mathrm{ord}} = \overline{\mathcal{M}}_N\left[\frac{1}{E}\right]$ be the subspace where we invert E. The special fiber of $\overline{\mathcal{M}}_N^{\mathrm{ord}}$ parametrizes ordinary points on $\left(\overline{\mathcal{M}}_N\right)_{/\mathbb{F}_p}$. Similarly let $\mathcal{M}_N^{*,\mathrm{ord}} = \mathcal{M}_N^*\left[\frac{1}{E}\right]$. Alternatively, we could have defined these spaces by removing the supersingular points from their special fibers.

We would like to define certain overconvergent rigid neighborhoods of these ordinary spaces. Let $\mathfrak{M}_N^*$ and $\overline{\mathfrak{M}}_N$ be the formal completions of $\mathcal{M}_N^*$ and $\overline{\mathcal{M}}_N$ at their fibers over p. Let $\mathfrak{M}_N^{*,\mathrm{ord}}$ and $\overline{\mathfrak{M}}_N^{\mathrm{ord}}$ be the formal subschemes where we invert E.

In the above definitions, it was essential that the schemes $\mathcal{M}_N$ are defined over $\mathbb{F}_p$, which assumes the fact that $p \nmid N$. In order to extend these definitions to level $p^n N$, one can use an analogue of the Igusa tower in the rigid setting. For consistency with the notation of [KL05, 3.2.2] define

$$\overline{\mathfrak{M}}_{p^n,N}^{\mathrm{ord}} = \mathrm{Isom}_{\overline{\mathfrak{M}}_N^{\mathrm{ord}}}\left(\mu_{p^n}^g, \overline{\mathfrak{A}}_N^{\mathrm{ord}}[p^n]^\circ\right)$$

where $\overline{\mathfrak{A}}_N^{\mathrm{ord}}[p]^\circ$ is the multiplicative part of the torsion subgroup, isomorphic to $\mu_{p^n}^g$ over the ordinary locus. Here, $\overline{\mathfrak{A}}_N$ is the formal completion of the universal semi-abelian scheme $\overline{\mathcal{A}}_N$ at its special fiber and $\overline{\mathfrak{A}}_N^{\mathrm{ord}}$ is the restriction to the ordinary part. (Note that in [Hid02] this space is denoted $T_{\infty,n}$.)

Then $\overline{\mathfrak{M}}_{p^n,N}^{\mathrm{ord}}$ is a formal scheme which is a Galois cover of $\overline{\mathfrak{M}}_N$ with Galois group $\mathrm{GL}_g(\mathbb{Z}/p^n\mathbb{Z})$. This space represents abelian varieties with principal polarization, symplectic level N and Γ_{00}-level p^n (cf. [Hid02, 3.2]), where recall that a $\Gamma_{00}(p^n)$-level structure for a (semi-)abelian scheme X is an injection $\mu_{p^n}^g \hookrightarrow X[p]$. Finally, let $\mathfrak{M}_{p^n,N}^{\mathrm{ord}}$ be the preimage of $\mathfrak{M}_N^{\mathrm{ord}}$ in $\overline{\mathfrak{M}}_{p^n,N}^{\mathrm{ord}}$

(3.3.4) Since the scheme $\overline{\mathcal{M}}_N$ is proper over $\mathbb{Z}_p$, by 3.3.1 if $\overline{\mathcal{M}}_N^{\mathrm{rig}}$ is the rigid fiber of $\overline{\mathfrak{M}}_N$, then $\overline{\mathcal{M}}_N^{\mathrm{rig}}$ will be isomorphic to the rigidification of the generic fiber of $\overline{\mathcal{M}}_N$. Similarly, let $\overline{\mathcal{M}}_N^{\mathrm{ord,rig}}$ be the rigid fiber of $\overline{\mathfrak{M}}_N^{\mathrm{ord}}$. By rigid GAGA, the lift E of the Hasse invariant transfers to a global section on the rigid space $H^0(\overline{\mathcal{M}}_N^{\mathrm{rig}}, (\det \underline{\omega})^{\otimes k_0(p-1)})$, having denoted by $\underline{\omega}$ the rigidified sheaf as well.

The complement of $\overline{\mathfrak{M}}_N^{\mathrm{ord}}$ in $\overline{\mathfrak{M}}_N$ is a divisor defined by the vanishing of the Hasse invariant. Using [KL05, 2.3] we define rigid domains $\overline{\mathcal{M}}_N^{\mathrm{rig}}(r) \subset \overline{\mathcal{M}}_N^{\mathrm{rig}}$, as the locus of points where the rigid E has p-adic norm at most r:

$$\overline{\mathcal{M}}_N^{\mathrm{rig}}(r) = \{x \in \overline{\mathcal{M}}_N^{\mathrm{rig}} : |E(x)| \geq r\}$$

By construction, with this notation, $\overline{\mathcal{M}}_N^{\mathrm{rig}}(1) = \overline{\mathcal{M}}_N^{\mathrm{ord,rig}}$. Note that since it is not in the scope of this work to analyze the *level* of overconvergence, and we only care about what happens when $r < 1$ is close to 1, the choice of lift E (i.e., of exponent k_0) is irrelevant. The choice of lift E and, implicitly, the choice of degree of overconvergence for the rigid domain $\overline{\mathcal{M}}_N^{\mathrm{rig}}(r)$ becomes relevant only if effective bounds for r are sought for the existence of canonical subgroups.

Alternatively, we could have used the inclusion $\overline{\mathcal{M}}_N^{\mathrm{ord,rig}} \subset \overline{\mathcal{M}}_N^{\mathrm{rig}}$ to define the system of neighborhoods $\overline{\mathcal{M}}_N^{\mathrm{rig},\dagger}$ of admissible open relatively quasi-compact subsets $X \subset \overline{\mathcal{M}}_N^{\mathrm{rig}}$ such that $\overline{\mathcal{M}}_N^{\mathrm{ord,rig}} \Subset_{\overline{\mathcal{M}}_N^{\mathrm{rig}}} X$. In that case, $\overline{\mathcal{M}}_N^{\mathrm{rig}}(r) \in \overline{\mathcal{M}}_N^{\mathrm{rig},\dagger}$ and for $r < s < 1$ sufficiently close to 1 we have

$$\overline{\mathcal{M}}_N^{\mathrm{rig}}(r) \Subset \overline{\mathcal{M}}_N^{\mathrm{rig}}(s)$$

where $X \Subset Y$ for rigid spaces X and Y over K means $X \Subset_{\mathrm{Sp}\,K} Y$ (this follows from

[KL05, 2.3.1 (2)]).

Finally, let $\overline{\mathcal{M}}^{\mathrm{ord,rig}}_{p^n,N}$ be the rigid fiber of $\overline{\mathfrak{M}}^{\mathrm{ord}}_{p^n,N}$ (although, from a purely notational perspective, it would have been more consistent to denote it $\overline{\mathfrak{M}}^{\mathrm{ord,rig}}_{p^n,N}$, since there is no $\mathbb{Z}_p$ scheme around). Then $\overline{\mathcal{M}}^{\mathrm{ord,rig}}_{p^n,N} \to \overline{\mathcal{M}}^{\mathrm{ord,rig}}_{N}$ is a Galois cover with Galois group $\mathrm{GL}(g, \mathbb{Z}/p^n\mathbb{Z})$.

(3.3.5) In order to define the overconvergent rigid domains of level p^nN and later to define the $U_{p,1}$-operator, we need the theory of the canonical subgroup for the p-torsion of an abelian variety.

Canonical subgroups have first been constructed for elliptic curves by Katz based on the work of Lubin (cf. [Kat73, 3.7]). If E is an elliptic curve over $\mathbb{Z}_p$ with supersingular reduction at p (in the case of ordinary reduction at p, the canonical subgroup is $E[p]^\circ$) and $\mathcal{E}$ is the associated formal group (i.e., the formal completion of E along the identity section), then $\mathcal{E}$ is a one-parameter formal group whose special fiber has height 2. Multiplication by p in the formal group is given by the equation

$$[p](x) = px + ax^p + \sum_{m\geq 2} c_m x^{m(p-1)+1}$$

where $a \equiv E_{p-1}(E,\omega) \pmod p$, where ω is an invariant differential on E. Setting $g(t) = p + at + \sum_{m\geq 2} c_m t^m$ (so that $[p](x) = xg(x^{p-1})$), there exists a canonical zero t_{can} of $g(t)$ constructed using Newton's method in [Kat73, p. 119], whenever $|E_{p-1}(E,\omega)|$ is sufficiently close to 1. In that case, the canonical subgroup is defined to be the finite flat rank p subscheme of $\ker[p]$ given by $x^p - t_{\mathrm{can}}x$.

Going back to the case of abelian varieties, we will use the following theorem, due to Abbes and Mokrane:

Theorem 3.3.6. *For r close enough to 1 there is a canonical subgroup $\mathcal{K}_{p,N}(r)$ of $\overline{\mathcal{A}}^{\mathrm{rig}}_N[p]$ over $\overline{\mathcal{M}}^{\mathrm{rig}}_N(r)$, which is locally free of rank p^g. Over the ordinary locus $\overline{\mathcal{M}}^{\mathrm{rig}}_N(1)$, we have $\mathcal{K}_{p,N}(1) \cong \overline{\mathcal{A}}^{\mathrm{rig}}_N[p]^\circ$.*

Proof. This is the content of Proposition 8.2.3 from [AM04] with $Q = \overline{\mathfrak{M}}_N^{\mathrm{ord}}$ and $P = \overline{\mathfrak{M}}_N$. $\square$

Remark 3.3.7. *We remark that [AM04] construct only level one canonical subgroups $\mathcal{K}_{p,N}(r)$. This is a technical point; in fact higher level canonical subgroups of abelian varieties have been constructed in [Con], but since the extension to the cusps is not yet in the literature*

we will assume from now on that $n = 1$.

but we keep the notation p^n because our proofs carry over to the construction of geometric families of level $p^n N$ finite slope overconvergent Siegel modular forms, as long as one assumes the existence of higher level canonical subgroups.

We would like to define overconvergent domains $\overline{\mathcal{M}}_{p^n,N}^{\mathrm{rig}}(r)$ without recourse to an analysis of "unramified cusps" of $(\mathcal{M}_{p^n,N})_{/\mathbb{Q}_p}$ (as it is done in [KL05, 3.2.1]). For this, observe that $\overline{\mathfrak{M}}_{p^n,N}^{\mathrm{ord}}$ is the Galois cover of $\overline{\mathfrak{M}}_N^{\mathrm{ord}}$ which trivializes the étale sheaf $\overline{\mathfrak{A}}_N[p]/\overline{\mathfrak{A}}_N[p]^\circ \cong \overline{\mathfrak{A}}_N^{\text{ét}}$. Define $\overline{\mathcal{M}}_{p^n,N}^{\mathrm{rig}}(r)$ to be the Galois cover of $\overline{\mathcal{M}}_N^{\mathrm{rig}}(r)$ which trivializes the finite flat sheaf $\overline{\mathcal{A}}_N^{\mathrm{rig}}[p]/\mathcal{K}_N(r)$. The fact that this sheaf is finite flat follows, for example, from Lemma 3.4.2. Restricting to the ordinary part $\overline{\mathcal{M}}_N^{\mathrm{ord,rig}}$, this space is the rigidification of $\overline{\mathfrak{M}}_{p^n,N}^{\mathrm{ord}}$ coming from the Igusa tower, since $\overline{\mathcal{A}}_N^{\mathrm{rig}}[p]/\mathcal{K}_N(r)$ extends $\overline{\mathcal{A}}_N^{\mathrm{ord,rig}}[p]/\overline{\mathcal{A}}_N^{\mathrm{ord,rig}}[p]^\circ$.

Whereas to define the family of neighborhoods $\overline{\mathcal{M}}_N^{\mathrm{rig},\dagger}$ we looked at the inclusion $\overline{\mathcal{M}}_N^{\mathrm{ord,rig}} \subset \overline{\mathcal{M}}_N^{\mathrm{rig}}$, in order to define the family of subsets $\overline{\mathcal{M}}_{p^n,N}^{\mathrm{ord,rig},\dagger}$ we look at the inclusion $\overline{\mathcal{M}}_{p^n,N}^{\mathrm{ord,rig}} \subset \overline{\mathcal{M}}_{p^n,N}^{\mathrm{rig}}(r)$. Thus, there exists a finite étale map $\overline{\mathcal{M}}_{p^n,N}^{\mathrm{rig},\dagger} \to \overline{\mathcal{M}}_N^{\mathrm{rig},\dagger}$ (in the language of [KL05, Proposition 2.2.1]). Moreover, by [KL05, 2.1.8] it follows that for $r < s < 1$ sufficiently close to 1, we have

$$\overline{\mathcal{M}}_{p^n,N}^{\mathrm{rig}}(r) \Subset \overline{\mathcal{M}}_{p^n,N}^{\mathrm{rig}}(s)$$

Let

$$\mathcal{M}^{\mathrm{ur,ord,rig}}_{p^n,N} = \overline{\mathcal{M}}^{\mathrm{ord,rig}}_{p^n,N} \cap (\mathcal{M}_{p^n,N})^{\mathrm{rig}}_{/\mathbb{Q}_p}$$

where $(\mathcal{M}_{p^n,N})^{\mathrm{rig}}_{/\mathbb{Q}_p}$ is the rigidification of the generic fiber of $\mathcal{M}_{p^n,N}$. Similarly define

$$\mathcal{M}^{\mathrm{ur,rig}}_{p^n,N}(r) = \overline{\mathcal{M}}^{\mathrm{rig}}_{p^n,N}(r) \cap (\mathcal{M}_{p^n,N})^{\mathrm{rig}}_{/\mathbb{Q}_p}$$

Example 3.3.8. *It is instructive to see an example that highlights the difference between $\mathcal{M}^{\mathrm{ur,ord,rig}}_{p^n,N}$ and $\mathcal{M}^{\mathrm{ord,rig}}_{p^n,N}$. Let $X = \mathbb{G}_m \hookrightarrow \overline{X} = \mathbb{A}^1$ be schemes over $\mathbb{Z}_p$. The formal completion of X at the special fiber is $\mathfrak{X} = \operatorname{Spf} \mathbb{Z}_p\{x, x^{-1}\}$ while the formal completion of $\overline{X}$ at its own special fiber is $\overline{\mathfrak{X}} = \operatorname{Spf} \mathbb{Z}_p\{x\}$. Then the intersection $\mathfrak{X}^{\mathrm{ur}} = \overline{\mathfrak{X}} \cap X$ is $\operatorname{Spf} \mathbb{Z}_p\{x\}[x^{-1}]$.*

Let T be a double coset. Rigidifying the projection maps $\pi_1, \pi_2 : \mathcal{M}_{p^n,N,T} \to \mathcal{M}_{p^n,N}$ we get two projection maps $\pi_1, \pi_2 : \mathcal{M}^{\mathrm{rig}}_{p^n,N,T} \to \mathcal{M}^{\mathrm{rig}}_{p^n,N}$. Define

$$\mathcal{M}^{\mathrm{ur,ord,rig}}_{p^n,N,T} = \pi_1^{-1}\left(\mathcal{M}^{\mathrm{ur,ord,rig}}_{p^n,N}\right)$$

and

$$\mathcal{M}^{\mathrm{ur,rig}}_{p^n,N,T}(r) = \pi_1^{-1}\left(\mathcal{M}^{\mathrm{ur,rig}}_{p^n,N}(r)\right)$$

(3.3.9) In this paragraph we study a rigid Köcher principle, which allows us to endow the space of analytic functions on $\mathcal{M}^{\mathrm{ur,rig}}_{p^n,N,T}(r)$ with the structure of a p-adic Banach module. To begin with, let

$$\overline{\mathcal{Z}}^{\mathrm{rig}} = \left]\left(\overline{\mathfrak{M}}^{\mathrm{ord}}_{p^n,N}\right)_{/\mathbb{F}_p} \setminus \left(\mathfrak{M}^{\mathrm{ord}}_{p^n,N}\right)_{/\mathbb{F}_p}\right[\subset \left(\overline{\mathfrak{M}}^{\mathrm{ord}}_{p^n,N}\right)^{\mathrm{rig}} = \overline{\mathcal{M}}^{\mathrm{ord,rig}}_{p^n,N}$$

and

$$\mathcal{Z}^{\mathrm{ur,rig}} = \overline{\mathcal{Z}}^{\mathrm{rig}} \cap \mathcal{M}^{\mathrm{ur,ord,rig}}_{p^n,N}$$

Lemma 3.3.10. *For r sufficiently close to 1*

$$H^0(\overline{\mathcal{M}}^{\mathrm{rig}}_{p^n,N}(r),\underline{\omega}^\kappa)\cong H^0(\mathcal{M}^{\mathrm{ur,rig}}_{p^n,N}(r),\underline{\omega}^\kappa)\cong H^0(\mathcal{M}^{\mathrm{ur,rig}}_{p^n,N}(r)\setminus\mathcal{Z}^{\mathrm{ur,rig}},\underline{\omega}^\kappa)$$

Proof. The proof of [KL05, Lemma 4.1.4] reduces this problem to extending sections $H^0(\mathfrak{M}^{\mathrm{ur,ord}}_{p^n,N},\underline{\omega}^\kappa)$ to $H^0(\overline{\mathfrak{M}}^{\mathrm{ord}}_{p^n,N},\overline{\underline{\omega}}^\kappa)$, where $\mathfrak{M}^{\mathrm{ur,ord}}_{p^n,N}$ is the preimage of

$$\mathfrak{M}^{\mathrm{ur,ord}}_N=\overline{\mathfrak{M}}^{\mathrm{ord}}_N\cap\mathcal{M}^{\mathrm{ord}}_N$$

under the covering map. Using the Galois cover $\overline{\mathcal{M}}^{\mathrm{ord}}_{p^n,N}\to\mathcal{M}^{\mathrm{ord}}_N$ it suffices to show this for level N, in which case it follows as in [Rap78, 4.9] □

Lemma 3.3.11. *For $f\in\mathcal{O}\left(\mathcal{M}^{\mathrm{ur,rig}}_{p^n,N,T}(r)\right)$ define*

$$|f|_r=\sup_{x\in\mathcal{M}^{\mathrm{ur,rig}}_{p^n,N,T}(r)}|f(x)|$$

For r sufficiently close to 1, $|f|_r$ is a norm which turns $\mathcal{O}\left(\mathcal{M}^{\mathrm{ur,rig}}_{p^n,N,T}(r)\right)$ into a p-adic Banach space.

Proof. Let

$$|f|^\circ_r=\sup_{x\in\mathcal{M}^{\mathrm{ur,rig}}_{p^n,N,T}(r)\setminus\pi_1^{-1}(\mathcal{Z}^{\mathrm{ur,rig}})}|f(x)|$$

The proof of [KL05, Lemma 4.1.6] carries over without change by showing that $|f|_r=|f|^\circ_r$. The main ingredients are Lemma 3.3.10 and finite étalenesses of the composite map

$$\mathcal{M}^{\mathrm{ur,ord,rig}}_{p^n,N,T}\xrightarrow{\pi_1}\mathcal{M}^{\mathrm{ur,ord,rig}}_{p^n,N}\longrightarrow\mathcal{M}^{\mathrm{ord}}_N$$

which follows from [FC90, VII.4 p. 259] and the definition of $\mathcal{M}^{\mathrm{ur,ord,rig}}_{p^n,N}$.

The main reason Lemma 3.3.10 is necessary is that sections of coherent sheaves on a rigid space form a p-adic Banach space if the rigid space is quasi-compact. This can be checked for $\mathcal{M}^{\mathrm{ur,rig}}_{p^n,N,T}(r) \setminus \pi_1^{-1}(\mathcal{Z}^{\mathrm{ur,rig}})$, being finite over $\overline{\mathcal{M}}^{\mathrm{rig}}_{p^n,N}(r) \setminus \overline{\mathcal{Z}}^{\mathrm{rig}}$. $\square$

3.4 Overconvergence

(3.4.1) Fundamental to showing the fact that the $U_{p,1}$ operator acts completely continuously on a suitable space of overconvergent Siegel modular forms is the existence of an overconvergent lift of the Frobenius operator (cf. [AM04, 1.1]). Again, due to restrictions in available literature, we must assume $n = 1$ (cf. 3.3.7), which suffices for our purposes.

The setup is the following: if X is an S-point of $\mathfrak{M}^{\mathrm{ord}}_N$, then the multiplicative part $X[p]^\circ$ is isomorphic to μ_p^g and $X/X[p]^\circ$ has a natural structure of an S-point of $\mathfrak{M}^{\mathrm{ord}}_N$ (see the first paragraph on [AM04, p. 148]) and we denote by ψ_N the morphism $X \mapsto X/X[p]^\circ$. This morphism extends to a morphism $\overline{\psi}_N : \overline{\mathfrak{M}}^{\mathrm{ord}}_N \to \overline{\mathfrak{M}}^{\mathrm{ord}}_N$, given by $\overline{\mathfrak{A}}_N \mapsto \overline{\mathfrak{A}}_N/\overline{\mathfrak{A}}_N[p]^\circ$. Note that over the special fiber $\mathfrak{A}_N/\mathfrak{A}_N[p]^\circ$ is isomorphic to $\mathfrak{A}^{\mathrm{ord,Frob}}_N$ where $\mathrm{Frob} : \mathfrak{A}^{\mathrm{ord,Frob}}_N \to \mathfrak{A}^{\mathrm{ord}}_N$ is the absolute Frobenius morphism over $\operatorname{Spec} \mathbb{F}_p$.

Lemma 3.4.2. *The map $\overline{\psi}^{\mathrm{ord}}_N : \overline{\mathcal{M}}^{\mathrm{ord,rig}}_N \to \overline{\mathcal{M}}^{\mathrm{ord,rig}}_N$ given by the rigidification of the map $\overline{\psi}_N$ extends to a map $\overline{\psi}^{\dagger}_N : \overline{\mathcal{M}}^{\mathrm{rig},\dagger}_N \to \overline{\mathcal{M}}^{\mathrm{rig},\dagger}_N$ of degree p^g. For r close to 1 this induces a finite flat morphism $\overline{\psi}_N(r) : \overline{\mathcal{M}}^{\mathrm{rig}}_N(r) \to \overline{\mathcal{M}}^{\mathrm{rig}}_N(r^p)$ of degree p^g.*

Proof. The existence of the map $\overline{\psi}^{\dagger}_N : \overline{\mathcal{M}}^{\mathrm{rig},\dagger}_N \to \overline{\mathcal{M}}^{\mathrm{rig},\dagger}_N$ of degree p^g is the content of [AM04, Theorem 8.1.1]. The statement about the map $\overline{\psi}_N(r)$ is proven as in [KL05, Lemma 3.1.7], see for example [Til06a, Proposition 4.5]. $\square$

Proposition 3.4.3. *There exists a map $\overline{\psi}^{\dagger}_{p^n,N} : \overline{\mathcal{M}}^{\mathrm{rig},\dagger}_{p^n,N} \to \overline{\mathcal{M}}^{\mathrm{rig},\dagger}_{p^n,N}$ lifting $\overline{\psi}^{\dagger}_N$. For r close to 1 this induces a finite flat map $\overline{\psi}^{\dagger}_{p^n,N}(r) : \overline{\mathcal{M}}^{\mathrm{rig}}_{p^n,N}(r) \to \overline{\mathcal{M}}^{\mathrm{rig}}_{p^n,N}(r^p)$ of degree p^g.*

Proof. The projection map $\overline{\mathfrak{M}}^{\mathrm{ord}}_{p^n,N} \to \overline{\mathfrak{M}}^{\mathrm{ord}}_N$ is finite étale so by [sga71, I 5.5] the morphism $\overline{\psi}_N : \overline{\mathfrak{M}}^{\mathrm{ord}}_N \to \overline{\mathfrak{M}}^{\mathrm{ord}}_N$ lifts to a morphism $\overline{\psi}_{p^n,N} : \overline{\mathfrak{M}}^{\mathrm{ord}}_{p^n,N} \to \overline{\mathfrak{M}}^{\mathrm{ord}}_{p^n,N}$ as long as it does so at a closed point; this point can be taken to be an abelian variety and then the existence of the lifting follows from [KL05, 3.2.5]. By [KL05, Corollary 2.2.2] this shows the existence of the morphism $\overline{\psi}^{\dagger}_{p^n,N}$ such that the diagram commutes

$$\begin{array}{ccc} \overline{\mathcal{M}}^{\mathrm{rig},\dagger}_{p^n,N} & \xrightarrow{\overline{\psi}^{\dagger}_{p^n,N}} & \overline{\mathcal{M}}^{\mathrm{rig},\dagger}_{p^n,N} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}^{\mathrm{rig},\dagger}_{N} & \xrightarrow{\overline{\psi}^{\dagger}_{N}} & \overline{\mathcal{M}}^{\mathrm{rig},\dagger}_{N} \end{array}$$

As in [KL05, Proposition 3.2.6] we conclude the for r close enough to 1 we get

$$\overline{\psi}^{\dagger}_{p^n,N}(r) : \overline{\mathcal{M}}^{\mathrm{rig}}_{p^n,N}(r) \to \overline{\mathcal{M}}^{\mathrm{rig}}_{p^n,N}(r^p)$$

which is finite flat of degree p^g. $\square$

(3.4.4) [CM98] define overconvergent modular forms of p-adic weight using the Eisenstein series E_{p-1}. Such a simple choice is not available to us, but since we only care about overconvergence in a neighborhood of the ordinary locus, we make do with some lift of a power of the Hasse invariant E. In order to compensate for the explicit computations in [CM98] involving E_{p-1}, we follow [KL05] in analyzing the interplay between the lift E and Hecke operators.

We have seen that for some large enough integer k_0 the section h^{k_0} lifts to a global section $E = \widetilde{h^{k_0}} \in H^0((\mathcal{M}_N)_{\mathbb{Z}_p}, (\det \underline{\omega})^{\otimes k_0(p-1)})$ and we will choose k_0 such that $p \nmid k_0$ (this condition is necessary in order to define an *analytic* action of Hecke operators on the space of overconvergent Siegel modular forms)

Lemma 3.4.5. *Let T be a double coset. There exists a $f_{k_0} \in \mathcal{O}(\mathcal{M}^{\mathrm{ur,rig}}_{p^n,N,T}(r))$ such that*

$$\theta\left(\pi_2^* E\right) = f_{k_0} \pi_1^* E$$

where $f_{k_0} - 1$ is topologically nilpotent. Moreover, for any $\varepsilon > 0$ there exists r suf-

ficiently close to 1 such that $|f_{k_0} - 1|_r < |p|^{\varepsilon}$, where $|\cdot|_r$ is the norm on the p-adic Banach space $\mathcal{O}(\mathcal{M}^{\mathrm{ur,rig}}_{p^n,N,T}(r))$ defined in Lemma 3.3.11.

Proof. The proof [KL05, 4.1.7] carries over, the main ingredients being Lemmas 3.1.10 and 3.3.11. □

(3.4.6) We now define the space of overconvergent Siegel modular forms of weight κ and level p^nN. Whereas classical such forms were defined as global sections over $\overline{\mathcal{M}}_N$, overconvergent forms are global sections over the overconvergent neighborhoods. For a field $L/\mathbb{Q}_p$

$$M^{\dagger}_{p^n,N,\kappa}(L) = \lim_{r\to 1^-} H^0\left(\mathcal{M}^{\mathrm{ur,rig}}_{p^n,N}(r)_L, \underline{\omega}^{\kappa}\right)$$

For the individual spaces

$$M_{p^n,N,\kappa}(L,r) = H^0\left(\mathcal{M}^{\mathrm{ur,rig}}_{p^n,N}(r)_L, \underline{\omega}^{\kappa}\right)$$

Lemma 3.3.10 shows that they are p-adic Banach spaces (since they are also the space of global sections of a coherent sheaf on a quasi-compact rigid space, cf. [KL05, 2.4]). Finally, by [KL05, Proposition 2.4.1] it follows that the transition maps in the limit are completely continuous. Thus $M^{\dagger}_{p^n,N,\kappa}(L)$ is a p-adic Fréchet space; this detail will be important to ascertaining the independence of the characteristic series of the $U_{p,1}$ operator from the particular rigid domain chosen.

(3.4.7) One of the reasons p-adic families of modular forms are so useful is that they interpolate modular forms of different, even p-adic, weight. We would like to define overconvergent forms of general weight in an admissible affinoid subspace of the weight space. For an affinoid algebra $\mathcal{R}$ over L, with submultiplicative semi-norm $|\cdot|$ and $Y \in \mathcal{R}$ with

$$|Y| < |p|^{\frac{2-p}{p-1}}$$

(see [Urb, Lemma 3.3.5] for the origin of the bound on $|Y|$), we define the space of overconvergent Siegel modular form over $\mathcal{R}$ of level $p^n N$ and weight $\kappa + Y$ as

$$M^{\dagger}_{p^n,N,\kappa+Y}(L) = \lim_{r \to 1^-} H^0\left(\mathcal{M}^{\mathrm{ur,rig}}_{p^n,N}(r)_L, \underline{\omega}^{\kappa}\right) \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{R}$$

The space $M^{\dagger}_{p^n,N,\kappa+Y}(L)$ of overconvergent forms is endowed with a Hecke action, concocted in a way that makes it compatible with the previously defined Hecke algebra action at different classical weights.

Let T be a double coset. Recall from Lemma 3.4.5 that for a sufficiently large integer k_0, not divisibile by p, for which we have a lift E of h^{k_0}, we got a function $f_{k_0} \in \mathcal{O}\left(\mathcal{M}^{\mathrm{ur,rig}}_{p^n,N,T}(r)\right)$ such that $f_{k_0} - 1$ is topologically nilpotent. Therefore, the function

$$h_Y = \exp\left(\frac{\log(f_{k_0})}{k_0(p-1)} \otimes Y\right)$$

is well defined as a rigid analytic function on $\mathcal{M}^{\mathrm{ur,rig}}_{p^n,N,T}(r) \otimes \mathcal{R}$. To see this, note that $\log(f_{k_0})$ is well-defined since $f_{k_0} - 1$ is topologically nilpotent. Analyticity now follows by [Urb, Lemma 3.3.5] and the fact that $|Y| < |p|^{\frac{2-p}{p-1}}$. (Note that we may apply this lemma because $p \nmid k_0$.)

Finally, the action of the Hecke operator is given by the map of sheaves

$$\pi_{1*}\pi_2^*\underline{\omega}^{\kappa} \longrightarrow \pi_{1*}\pi_1^*\underline{\omega}^{\kappa} \xrightarrow{\pi_{1*}(\cdot h_Y)} \pi_{1*}\pi_1^*\underline{\omega}^{\kappa} \xrightarrow{\mathrm{Tr}} \underline{\omega}^{\kappa}$$

as in the classical setting.

(3.4.8) We motivated the construction of the above Hecke action by positing compatibility with the Hecke actions at different weights. Concretely, consider a specialization map $e_t : \mathcal{R} \to L'$, where L'/L is a finite extension, such that $e_t(Y) = (p-1)k_0 t$ for some $t \in \mathbb{Z}$. Then $\mathrm{id} \otimes e_t$ induces a map

$$M^{\dagger}_{p^n,N,\kappa+Y}(\mathcal{R}) \to M^{\dagger}_{p^n,N,\kappa}(L')$$

where the second space is relative to the affinoid L' with the choice of $Y = 0$. Multiplication by $E^t \otimes 1$ gives a map

$$M^{\dagger}_{p^n,N,\kappa}(L') \to M^{\dagger}_{p^n,N,\kappa+(p-1)k_0 t}(L')$$

and the composition of the two maps gives Siegel modular forms of various classical weights.

We remark that this way of defining p-adic families of modular forms is due to Coleman, and is labeled "awkward" by [CM98, p.7 (a)], where a more direct definition of overconvergent forms is asked for. (In fact, such a more direct definition has been announced by Andreatta et al., by constructing a suitable generalization of $\underline{\omega}^{\kappa}$ for p-adic weights κ.)

The main result of this paragraph is that the Hecke action defined above is compatible with specialization. In the context of modular forms over $\mathrm{GL}(2)_{/\mathbb{Q}}$, the equivalent of this statement is [Col97, Lemma B 5.4].

Proposition 3.4.9. *For a Hecke operator T there is a commutative diagram*

$$\begin{tikzcd} M^{\dagger}_{p^n,N,\kappa+Y}(\mathcal{R}) \arrow[r, "\mathrm{id}\otimes e_t"] \arrow[d, "T"] & M^{\dagger}_{p^n,N,\kappa}(L') \arrow[r, "\cdot E^t\otimes 1"] & M^{\dagger}_{p^n,N,\kappa+(p-1)k_0 t}(L') \arrow[d, "T"] \\ M^{\dagger}_{p^n,N,\kappa+Y}(\mathcal{R}) \arrow[r, "\mathrm{id}\otimes e_t"] & M^{\dagger}_{p^n,N,\kappa}(L') \arrow[r, "\cdot E^t\otimes 1"] & M^{\dagger}_{p^n,N,\kappa+(p-1)k_0 t}(L') \end{tikzcd} \tag{3.3}$$

where the left vertical map is with respect to $\mathcal{R}$ and Y, while the right vertical map is with respect to L' and 0.

Proof. For an element $g \otimes r \in M^{\dagger}_{p^n,N,\kappa+Y}(\mathcal{R})$ we have

$$T(g \otimes r)(A) = \sum g(B) \otimes h_Y r$$

where the sum runs over isogenies $A \to B$ of type T (for simplicity of notation we only include the abelian variety). We need to show that

$$\sum g(B) f^t_{k_0} \smile E^t \otimes e_t(r) = T\left(\sum g(B) \smile E^t \otimes e_t(r)\right)$$

which follows from Lemma 3.4.5. Here $\smile$ is the cup product on cohomology. □

(3.4.10) The main ingredient in our geometric construction of p-adic families of Siegel modular forms is the complete continuity of the action of $U_{p,1}$ on spaces of overconvergent forms. In order to show this, we need to show complete continuity of restriction of sections to overconvergent subdomains.

Lemma 3.4.11. *If $r < s < 1$ are sufficiently close to 1, then the restriction map*

$$\mathrm{Res}_{r,s} : M_{p^n,N,\kappa}(L, r) \to M_{p^n,N,\kappa}(L, s)$$

is completely continuous.

Proof. By Lemma 3.3.10 it is enough to show that the restriction map

$$H^0(\overline{\mathcal{M}}^{\mathrm{rig}}_{p^n,N}(r), \underline{\overline{\omega}}^{\kappa}) \to H^0(\overline{\mathcal{M}}^{\mathrm{rig}}_{p^n,N}(s), \underline{\overline{\omega}}^{\kappa})$$

is completely continuous, which follows from [KL05, 2.4.1], since $\underline{\overline{\omega}}^{\kappa}$ is coherent. □

(3.4.12) We now show that the action of the $U_{p,1}$ operator is completely continuous.

By Proposition 3.4.3 for r sufficiently close to 1 we get a map

$$\overline{\psi}_{p^n,N}(r) : \overline{\mathcal{M}}^{\mathrm{ord,rig}}_{p^n,N}(r) \to \overline{\mathcal{M}}^{\mathrm{ord,rig}}_{p^n,N}(r^p)$$

which is finite flat of degree p^g. Therefore $\overline{\psi}_{p^n,N}(r)$ gives a trace map

$$\mathrm{Tr}_{\overline{\psi}_{p^n,N}(r)^*} : M_{p^n,N,\kappa+Y}(L,r) \to M_{p^n,N,\kappa+Y}(L,r^p)$$

Since $r > r^p$ for $r < 1$ by Lemma 3.4.11 the restriction map

$$\mathrm{Res}_{r^p,r} : M_{p^n,N,\kappa+Y}(L,r) \to M_{p^n,N,\kappa+Y}(L,r^p)$$

is completely continuous and we can define

$$U_{p,1} = p^{-3}\,\mathrm{Res}_{r^p,r} \circ \mathrm{Tr}_{\overline{\psi}_{p^n,N}(r)^*}$$

Since $U_{p,1}$ is the composition of a completely continuous map with a continuous one, it is completely continuous acting on $M_{p^n,N,\kappa+Y}(L,r)$ for r sufficiently close to 1.

By extending scalars, it follows that for r close to 1 the action of $U_{p,1}$ on $M^{\dagger}_{p^n,N,\kappa+Y}(\mathcal{R},r)$ is completely continuous for an affinoid $\mathcal{R}$.

(3.4.13) Via the rigidification functor, classical Siegel modular forms are also overconvergent: the natural inclusion $M_{p^n,N,\kappa}(L) \subset M^{\dagger}_{p^n,N,\kappa}(L)$ is given by restriction of global sections since by definition $\mathcal{M}^{\mathrm{ur,rig}}_{p^n,N}(r)$ is contained in the rigidification of the generic fiber of $\mathcal{M}_{p^n,N}$. By construction this inclusion is equivariant with respect to the action of the Hecke operators. The fact that the action of $U_{p,1}$ agrees with the geometric action can be seen as follows: it is enough to check on the ordinary locus over which the map ψ_N is defined as $X \mapsto X/X[p]^{\circ}$. The claim follows from the observation that the fiber of ψ consists precisely of the isogenies of type $U_{p,1}$, which results from the definition of $U_{p,1}$ (cf. [KL05, 4.3.3]).

(3.4.14) Let $\mathbb{T}^{\dagger}_{p^n,N,\kappa+Y}$ be the Banach closure of the ring of endomorphisms of $M^{\dagger}_{p^n,N,\kappa+Y}$ generated by the operators $T_{p,1}$ and $T_{p,2}$ away from Np. The main theorem of this section is that the Hecke algebra $\mathbb{T}^{\dagger}_{p^n,N,\kappa+Y}$ is *commutative*, which will allow us to define an eigenvariety using the rigid spectrum of a constant slope part of this ring (see 3.5.6). In the case of [CM98] the proof made use of a classical result of Hida on the Hecke algebra acting on Katz modular forms. Since in our case we fix the level p^nN not only the tame level N, we can get by with less.

Proposition 3.4.15. *The Hecke algebra* $\mathbb{T}^{\dagger}_{p^n,N,\kappa+Y}$ *is commutative.*

Proof. The main ideas from [KL05, 4.4.2] are applicable in our context and the proof of [Tig06, 3.0.6.2] works. The only ingredient is that the Hecke algebra acting on classical Siegel modular forms is commutative and the fact that by the top row of (3.3) we get infinitely many specializations to classical Siegel modular forms. □

3.5 p-adic Families

(3.5.1) We now specialize to the case of Siegel modular forms of genus 2, which correspond to automorphic representations of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$. There are several strategies to construct p-adic families of modular forms. The first, pioneered by Hida involves p-adic interpolation of coefficients of Eisenstein series ([Wil88] for ordinary Hilbert modular forms and [Tay88] for ordinary Siegel modular forms). The second, based on [CM98] involves constructing rigid objects parametrizing not necessarily ordinary, but finite slope, overconvergent modular forms.

Here too, there are variations. In the original paper constructing the eigencurve for $\mathrm{GL}(2)_{/\mathbb{Q}}$ the eigencurve is constructed as the rigidification of the generic fiber of a universal pseudodeformation ring with maps to spectral varieties; in [Buz07] eigenvarieties are constructed as admissible covers of spectral varieties. Another area of variation is the construction of spectral varieties. The "geometric" method, defines spectral varieties as Fredholm hypersurfaces associated to completely continuous operators acting on spaces of overconvergent modular forms defined as sections over overconvergent rigid neighborhoods. The "cohomological" method uses completely continuous operators acting on cohomologies of arithmetic groups, in the style of the Eichler-Shimura map. We will take the geometric approach, which is of independent interest.

The outline of this section is the following: we define a rigid weight space which contains the weights of classical Siegel modular forms. Over a subspace of the weight space we construct a rigid variety parametrizing systems of Hecke eigenvalues of constant finite slope overconvergent Siegel modular forms. This variety comes equipped with finite projections to spectral varieties.

(3.5.2) Let $\mathcal{W}$ be the rigid weight space, whose points over a field $L/\mathbb{Q}_p$ are the continuous characters $\mathrm{Hom}_{\mathrm{cont}}(T_4(\mathbb{Z}_p), L^\times)$ where T_4 is the torus in GSp_4. Such continuous homomorphisms are in fact locally analytic (cf. [Urb, Lemma 3.1.4]). The weight space is a finite union of rigid balls of dimension 3. However, since our definition of overconvergent forms of arbitrary weight involves multiplication by a power of an Eisenstein series, given a fixed weight we can only reach, via this definition, a subset of the whole weight space.

To account for this problem, for a classical weight $\kappa \in \mathcal{W}$ let $\mathcal{W}_\kappa \subset \mathcal{W}$ be the subspace of $\mathcal{W}$ whose points over a p-adic field L are of the form $\kappa + (t, t; -2t)$ where t is sufficiently close to 0. That this choice is made allows one to construct families of level $p^n N$, not only of tame level N (cf. [KL05, 4.5]).

The parameter t can be recovered rigid analytically as follows: let $\chi : T_4(\mathbb{Z}_p) \to$

$\mathcal{O}(\mathcal{W}_\kappa)^\times$ be the universal character of the form specified above. Then

$$Y = \frac{\log(\chi\kappa^{-1}(x))}{\log((1,1;-2)(x))}$$

is rigid analytic, and independent of the choice of $x \in T_4(\mathbb{Z}_p)$ close to 1. Here, $(1,1;-2)$ represents the character

$$\begin{pmatrix} x & & & \\ & y & & \\ & & zx^{-1} & \\ & & & zy^{-1} \end{pmatrix} \mapsto xyz^{-2}$$

(3.5.3) In order to construct p-adic families of finite slope overconvergent Siegel modular forms we will use the action of the $U_{p,1}$ operator on certain spaces of overconvergent forms to cut out a subspace of the rigidification of the Hecke algebra. This paragraph is modeled on [CM98, Chapter 6].

By Proposition 3.4.15, there exists a map from the polynomial algebra

$$\mathcal{H}^\circ = \mathbb{Z}_p[T_{q,1}, T_{q,2}]_{q\nmid pN}$$

to the Hecke algebra

$$\iota : \mathcal{H}^\circ \to \mathbb{T}^\dagger_{p^n,N,\kappa+Y}$$

For r sufficiently close to 1 the action of $U_{p,1}$ on $M_{p^n,N,\kappa+Y}(\mathcal{R}, r)$ is completely continuous. For $\alpha \in \mathcal{H}$ such that $\iota(\alpha)$ is topologically unipotent (for example, anything of the form $1+p\tau$, where $\tau \in \mathcal{H}$), the operator $\iota(\alpha)U_{p,1}$ is still completely continuous.

Consider the characteristic series

$$P_\alpha(\kappa, T) = \det\nolimits_{\mathcal{R}}\left(1 - \iota(\alpha)U_{p,1}T | M_{p^n,N,\kappa}(L, r)\right)$$

We would like to deduce that $P_\alpha(\kappa, T)$ varies analytically with κ, as in [CM98, 4.3]. To achieve this, we will use [CM98, Proposition 4.3.3], which we recall below.

Proposition 3.5.4 (Coleman-Mazur)**.** *Suppose given the following data*

- *a sequence* $\{A_m\}_{m\geq 1}$ *of Banach algebras with contractive ring homomorphisms* $A_{m+1} \to A_m$*,*
- *for each* m *consider a sequence of Banach* A_m*-modules* $\{M_{m,i}\}_{i\geq s_m}$ *(for some integer* s_m*), such that* $|M_{m,i} - \{0\}| = |A_m - \{0\}|$ *and for* $q \geq m$ *and* i *sufficiently large* $M_{q,i} = M_{m,i}\widehat{\otimes} A_q$
- A_m*-module homomorphisms* $f_{m,i} : M_{m,i} \to M_{m,i+1}$ *and an orthogonal basis* $B_{m,i}$ *of* $M_{m,i}$ *such that* $f(B_{m,i})$ *is an orthogonal basis,*
- *completely continuous operators* $V_{m,i}$ *on* $M_{m,i}$ *compatible with the data of* $\{A_m\}$ *and* $\{M_{m,i}\}$*, and such that for* i *sufficiently large* $V_{q,i} = V_{m,i} \otimes 1$*.*

Then there exists a power series in $\varprojlim A_m[\![T]\!]$ *whose projection in* $A_m[\![T]\!]$ *equals the characteristic series*

$$\det(1 - TV_{m,i}|M_{m,i})$$

(3.5.5) Let $\mathrm{Sp}\,\mathcal{R}$ be a small affinoid around κ inside $\mathcal{W}_\kappa$. Take $A_m = \mathcal{R}$ and

$$M_{m,i} = M_{p^n,N,\kappa+Y}(\mathcal{R}, v)$$

where $v = \dfrac{1}{i}$. Let $f_{n,i}$ be the natural inclusion. Then the first two condition of the proposition are automatically satisfied. The third property follows from [KL05, 2.4.5] as in [KL05, 4.3.9]. Finally, let $V_{m,i} = \iota(\alpha)U_{p,1}$ acting completely continuously on the space of overconvergent modular forms $M_{p^n,N,\kappa+Y}(\mathcal{R}, v)$. We conclude from Proposition 3.5.4 that there is an analytic characteristic series

$$P_\alpha(T) \in \mathcal{R}[\![T]\!]$$

such that $e_{\kappa'} P_\alpha(T) = P_\alpha(\kappa', T)$ where $e_{\kappa'} : \mathcal{R} \to L$ is the morphism giving the point $\kappa' \in \mathcal{W}_\kappa(L)$. In particular, the series $P_\alpha(T)$ is independent of r.

(3.5.6) By [CM98, 4.1] (in particular Theorem 4.1.1 on the unique factorization of Fredholm characteristic series of compact operators into linear terms over $\mathbb{C}_p$) the space $M^\dagger_{p^n,N,\kappa}$ has a slope filtration: for $\sigma \in \mathbb{Q}^+$ get a finitely generated subspaces $M^\dagger_{p^n,N,\kappa,\sigma} \subset M^\dagger_{p^n,N,\kappa}$ consisting of generalized eigenforms for the operator $U_{p,1}$ with eigenvalue of slope σ. Note that since $\iota(\alpha)$ is invertible, the slope decompositions with respect to $U_{p,1}$ and $\iota(\alpha)U_{p,1}$ are the same [CM98, 4.1.2]. Since the arguments of [KL05, 4.5.6] are simpler on a constant slope rigid family, we will use the slope filtration to adapt the result to our setting.

By analyticity of the characteristic series $P_\alpha(T)$, it follows that, by shrinking $\mathcal{R}$, the subspace $M^\dagger_{p^n,N,\kappa+Y,\sigma}(\mathcal{R}) \subset M_{p^n,N,\kappa+Y}(\mathcal{R})$ of overconvergent slope σ Siegel modular forms is finitely generated over $\mathcal{R}$. Since $\iota(\mathcal{H}^\circ)$ commutes with $U_{p,1}$, it follows that $\iota(\mathcal{H}^\circ)$ acts on $M^\dagger_{p^n,N,\kappa+Y,\sigma}(\mathcal{R})$ and therefore ι gives a map $\mathcal{H}^\circ \to \mathbb{T}^\dagger_{p^n,N,\kappa+Y,\sigma}$ where $\mathbb{T}^\dagger_{p^n,N,\kappa+Y,\sigma}$ is the image of the Hecke algebra in the ring of endomorphisms over $\mathcal{R}$ of $M^\dagger_{p^n,N,\kappa+Y,\sigma}(\mathcal{R})$.

Let $\mathbb{T}^{\dagger,\mathrm{red}}_{p^n,N,\kappa+Y,\sigma}$ be the nilreduction of $\mathbb{T}^\dagger_{p^n,N,\kappa+Y,\sigma}$. This ring is commutative, reduced and it is a subalgebra of a finite Banach algebra over $\mathcal{R}$. Therefore it is an affinoid space and we may define

$$\mathcal{X}_\sigma = \mathrm{Sp}\, \mathbb{T}^{\dagger,\mathrm{red}}_{p^n,N,\kappa+Y,\sigma}$$

(3.5.7) The main feature of eigenvarieties as defined below is that they are endowed with finite maps to spectral varieties associated to $\iota(\alpha)U_{p,1}$. The slope of an overconvergent form is independent of the chosen $\alpha = 1 + p\tau$, but the rigid varieties cut out by them are not. The spectral variety associated to $\iota(\alpha)U_{p,1}$ over the affinoid neighborhood $\mathrm{Sp}\,\mathcal{R}$ of κ is the Fredholm hypersurface

$$\mathcal{Z}_\alpha \subset \mathrm{Sp}\,\mathcal{R} \times \mathbb{A}^{1,\mathrm{rig}}$$

which is the zero locus of the characteristic series $P_\alpha(T)$. It comes with a structure morphism $f : \mathcal{Z}_\alpha \to \mathcal{R}$. It is explained in [Buz07, p. 26] that the spectral variety also comes with an admissible covering $\mathcal{C}$ consisting of affinoid subdomains $Y \subset \mathcal{Z}_\alpha$ such that $Y \subset f^{-1}(X)$ where X is an affinoid in $\mathcal{W}_\kappa$ such that $f : Y \to X$ is finite surjective.

Now we have two options in defining an eigenvariety: the construction of [Buz07, 5.7] or that of [CM98, 6.1]. We choose the latter to define the eigenvariety

$$\mathcal{E}_\sigma \subset \mathcal{X}_\sigma \times \mathbb{G}_m^{\mathrm{rig}}$$

First, let $\mathcal{H} = \mathcal{H}^\circ[U_{p,1}]$ and consider the natural extension $\iota : \mathcal{H} \to \mathcal{O}(\mathcal{X}_\sigma \times \mathbb{G}_m^{\mathrm{rig}})$ which sends $U_{p,1}$ to the parameter x_p of $\mathbb{G}_m^{\mathrm{rig}}$. (This can be done since $U_{p,1}$ commutes with the Hecke operators away from p.) Let $\mathcal{E}_\sigma$ be nilreduction of the subspace defined by the ideal

$$\mathcal{I} = \left(P_\alpha\left(\frac{1}{x_p\iota(\alpha)}\right)\right)_{\alpha=1+p\tau}$$

Since x_p and $\iota(\alpha)$ are invertible in $\mathcal{O}(\mathcal{X}_\sigma \times \mathbb{G}_m^{\mathrm{rig}})$, $\mathcal{I}$ is indeed an ideal of $\mathcal{O}(\mathcal{X}_\sigma \times \mathbb{G}_m^{\mathrm{rig}})$.

Alternatively, following the notation of [CM98, 6.1], we could have defined $r_\alpha : \mathcal{X}_\sigma \times \mathbb{G}_m^{\mathrm{rig}} \to \mathrm{Sp}\,\mathcal{R} \times \mathbb{A}^{1,\mathrm{rig}}$ given by

$$r_\alpha(x,t) = \left(\pi(x), \frac{t}{\phi_x(\iota(\alpha))}\right)$$

where $\pi : \mathcal{X}_\sigma \to \mathrm{Sp}\,\mathcal{R}$ is the structure map, and $\phi_x : \mathbb{T}^{\dagger,\mathrm{red}}_{p^n,N,\kappa+Y,\sigma} \to L$ is the morphism giving the point $x \in \mathcal{X}_\sigma(L)$. Then $\mathcal{E}_\sigma$ is the nilreduction of

$$\bigcap_\alpha r_\alpha^{-1}(\mathcal{Z}_\alpha)$$

Therefore, the eigenvariety comes with maps $r_\alpha : \mathcal{E}_\sigma \to \mathcal{Z}_\alpha$.

Remark 3.5.8. *Since our ultimate goal is the study of Galois representations (which, by Chebotarëv's density theorem, are uniquely determined by the behaviour away from a finite set of primes) and not the parametrization of all overconvergent finite slope Siegel modular forms, we disregard the Hecke operators at primes dividing the level* N.

(3.5.9) We would now like to show that the natural projections $r_\alpha : \mathcal{E}_\sigma \to \mathcal{Z}_\alpha$ are finite maps. This is a rather involved process, but analogous to [CM98, 7]. To summarize, one uses [Buz07, 5.7] to construct a reduced rigid variety $\mathcal{D}$ equipped with finite projection maps to the spectral varieties, and then one shows that $\mathcal{D} \cong \mathcal{E}_\sigma$. Moreover, by [CM98, 7.2.2] it follows that, after taking the nilreduction of $\mathrm{Sp}\,\mathcal{R}$ the projection $\mathcal{E}_\sigma \to \mathrm{Sp}\,\mathcal{R}$ become finite flat.

3.6 Points on the Eigenvariety

(3.6.1) We would like to have an interpretation of the eigenvariety in the style of [CM98, Theorem 6.2.1], whose statement we recall: there is a one-to-one correspondence between the set of finite slope normalized overconvergent modular eigenforms of tame level N and $\mathbb{C}_p$ points on the eigencurve. This formulation of [CM98, Theorem 6.2.1] belies the fact that eigenvarieties, in general, parametrize not *modular forms* but systems of Hecke eigenvalues associated to modular forms. If one is in a setting where multiplicity one results are known, such as $\mathrm{GL}(2)_{/\mathbb{Q}}$, then systems of Hecke eigenvalues in fact determine the modular form. This is not so in the case of

GSp(4), and the issue already appears in [KL05, 4.5.5 (1)].

Consider f a finite slope σ overconvergent Siegel modular eigenform of level p^nN and weight κ. To f we can associate a morphism

$$\lambda_f : \mathbb{T}^{\dagger}_{p^n,N,\kappa+Y,\sigma} \to \mathbb{C}_p$$

given by mapping a Hecke operator to its eigenvalue acting on f, as well as an eigenvalue of the $U_{p,1}$ operator. We will consider such systems of Hecke eigenvalues with respect to the Hecke ring $\iota(\mathcal{H}^{\circ})$ and $U_{p,1}$. The datum of such a system of eigenvalues is the same as the datum of a morphism $\lambda : \mathbb{T}^{\dagger}_{p^n,N,\kappa+Y,\sigma} \to \mathbb{C}_p$ together with a $U_{p,1}$-eigenvalue. Systems of Hecke eigenvalues come from (not necessarily unique) finite slope overconvergent Siegel modular eigenforms.

The first direction, that of interpreting systems of Hecke eigenvalues as points on the eigenvariety, is not difficult. Consider a system of Hecke eigenvalues given by a morphism λ and a $U_{p,1}$-eigenvalue u. In that case λ is nothing more than a point in $\mathcal{X}_{\sigma}(\mathbb{C}_p)$, thus $\left(\lambda, \dfrac{1}{u}\right) \in \mathcal{X}_{\sigma}(\mathbb{C}_p) \times \mathbb{G}_m(\mathbb{C}_p)$. Since $\iota(\mathcal{H})$ is commutative, if f is a finite slope σ overconvergent Siegel eigenform giving rise to $\lambda = \lambda_f$, then f is also an eigenform for $\iota(\alpha)U_{p,1}$, for all $\alpha \in \mathcal{H}^{\circ}$. Therefore we get a point on $\mathcal{Z}_{\alpha}(\mathbb{C}_p)$. The second description of the eigenvariety shows that the point $\left(\lambda, \dfrac{1}{u}\right) \in \mathcal{X}_{\sigma}(\mathbb{C}_p) \times \mathbb{G}_m(\mathbb{C}_p)$, lying in each $\mathcal{Z}_{\alpha}(\mathbb{C}_p)$, also lies in $\mathcal{E}_{\sigma}(\mathbb{C}_p) \subset \mathcal{X}_{\sigma}(\mathbb{C}_p) \times \mathbb{G}_m(\mathbb{C}_p)$. We denote by z the point on $\mathcal{E}_{\sigma}(L)$ corresponding to the system of Hecke eigenvalues arising from the eigenform f.

We do not need the second direction, that of interpreting points on $\mathcal{E}_{\sigma}(\mathbb{C}_p)$ as systems of Hecke eigenvalues coming from finite slope σ overconvergent Siegel modular forms. However, we need to tackle this issue in order to produce a dense set of classical points on the eigenvariety. In particular, we need the conclusion of the first paragraph of the proof of [KL05, Theorem 4.5.6]: that if f is a finite slope classical eigenform giving rise to a point z of $\mathcal{E}_\sigma$ then there exists an invertible $\alpha \in \mathcal{H}^\circ$ such that

$$r_\alpha^{-1}(r_\alpha(z)) = z$$

What this condition says is that if one can find points x_t in $\mathcal{Z}_\alpha(\mathbb{C}_p)$ converging to $r_\alpha(z)$, then the preimages in $\mathcal{E}_\sigma(\mathbb{C}_p)$ of x_t will converge to z. We prove this fact in §3.6.2.

(3.6.2) Let $z \in \mathcal{E}_\sigma(\mathbb{C}_p)$, in which case σ is the p-adic valuation of the projection from $\mathcal{E}_\sigma$ to $\mathbb{G}_m$. By an argument similar to [CM98, 6.2.2, 6.2.3] there exists a $\tau \in \mathcal{H}^\circ$ such that there is a unique system of Hecke eigenvalues, arising from a slope σ overconvergent Siegel modular form (which may not be unique) which is an eigenform for the Hecke operator $\iota(1 + p\tau)U_{p,1}$ as well as for the Hecke algebra $\iota(\mathcal{H}^\circ)$, having the same $\iota(1 + p\tau)U_{p,1}$ eigenvalue as z. It is instructive to see this argument adapted to the case of Siegel modular forms:

Let z be defined over the finite extension $L/\mathbb{Q}_p$. First, we enumerate the Hecke operators $T_{q,1}$ and $T_{q,2}$ in some order as $T_1, T_2, \ldots$. Next, suppose we have constructed a Hecke operator $\widetilde{U}_k$ acting on the finite dimensional vector space $M_{p^n,N,\kappa,\sigma}(L)$ of slope σ overconvergent Siegel modular forms, such that:

1. there exists a positive integer a_k such that any two distinct eigenvalues of $\widetilde{U}_k$ acting on $M_{p^n,N,\kappa,\sigma}(L)$ are in p-adic distance more than p^{-a_k} apart (this can be done since $M_{p^n,N,\kappa,\sigma}(L)$ is a finite dimensional vector space).

2. if f_k is an eigenform with $\widetilde{U}_k$ eigenvalue equal to that of z, then the set of eigenvalues of the operators $U_{p,1}, T_1, \ldots, T_k$ acting on f_k is the same as the set of eigenvalues of the same Hecke operators acting on z.

For $k = 0$ we may take $\widetilde{U}_0 = U_{p,1}$. Now for the inductive step: define

$$\widetilde{U}_{k+1} = (1 + p^{a_k} T_{k+1})\widetilde{U}_k$$

Let f_{k+1} be an eigenform with $\widetilde{U}_{k+1}$ eigenvalue $u_{k+1,f}$, $\widetilde{U}_k$ eigenvalue $u_{k,f}$ and T_{k+1} eigenvalue $t_{k+1,f}$. Let $u_{k+1,z}$, $u_{k,z}$ and $t_{k+1,z}$ be the analogous eigenvalues for the eigenform z. Assuming that

$$u_{k+1,f} = u_{k+1,z}$$

and using that

$$u_{k+1,f} = (1 + p^{a_k} t_{k+1,f}) u_{k,f}$$
$$u_{k+1,z} = (1 + p^{a_k} t_{k+1,z}) u_{k,z}$$

we get that

$$u_{k,f} - u_{k,z} = p^{a_k} \left(t_{k+1,z} u_{k,z} - t_{k+1,f} u_{k,f} \right)$$

so $|u_{k,f} - u_{k,z}|_p < p^{-a_k}$. By construction of a_k, it follows that $u_{k,f} = u_{k,z}$, and plugging into the formula, we get that $t_{k+1,f} = t_{k+1,z}$. By the inductive hypothesis, it follows also that the eigenvalues of $U_{p,1}, T_1, \ldots, T_k$ of f_{k+1} and z agree.

In the notation of [CM98, p. 86], let $\mathcal{F}_{\widetilde{U}_k}$ be the set of systems of Hecke eigenvalues whose $\widetilde{U}_k$-eigenvalue is equal to that of z. The above argument shows that

$$\bigcap_{k \geq 0} \mathcal{F}_{\widetilde{U}_k} = \{z\}$$

and so for some k it must be that $\mathcal{F}_{\widetilde{U}_k} = \{z\}$.

This proves that for

$$\alpha = 1 + p\tau = \prod_{i=1}^{k-1} (1 + p^{a_i} T_i)$$

we have

$$r_\alpha^{-1}(r_\alpha(z)) = z$$

(3.6.3) Rigid analytic interpolation of classical modular forms would be an incomplete theory in the absence of some result on the distribution of classical points on the rigid family. In the case of modular forms over $\mathrm{GL}(2)_{/\mathbb{Q}}$, classical points are dense on the eigencurve; this result follows from Coleman's theorem, which states that overconvergent modular forms whose slope is less than the weight minus one are necesarily classical. (In fact Coleman's result also treats the case of equality, as long as the overconvergent form is not in the image of a certain operator.)

We do not obtain such a result for overconvergent Siegel modular forms; however, following [KL05, 4.5.6] (which is based on a suggestion of Matthew Emerton) we show that in a neighborhood of a classical point there are infinitely many classical points converging to it. This, in turn, is necessary to analytically interpolate Galois representations over the eigenvariety.

Proposition 3.6.4. *Let $f \in M^{\dagger}_{p^n,N,\kappa}(L)$ be a classical finite slope σ Siegel modular eigenform. It gives rise to a point $z \in \mathcal{E}_\sigma(L)$. Then for infinitely many integers t there exist classical Siegel modular forms f_t giving rise to points $z_t \in \mathcal{E}_\sigma$ such that $\lim z_t \to z$.*

Proof. Consider the maps

$$\mathcal{E}_\sigma \xrightarrow{r_\alpha} \mathcal{Z}_\alpha \to \mathcal{W}_\kappa$$

By §3.6.2 there exists an $\alpha = 1 + p\tau$ such that $r_\alpha^{-1}(r_\alpha(z)) = z$. Let $P_\alpha(T)$ be the analytic characteristic series defining the spectral variety $\mathcal{Z}_\alpha$. Specializing in weight κ gives the characteristic series $P_\alpha(\kappa, T)$, which, by [CM98, 1.3.7], has finitely many roots of slope σ. Let $z_0, z_1, \ldots, z_r$ be the other points of $\mathcal{Z}_\alpha$ corresponding to these finitely many roots, where $z_0 = r_\alpha(z)$. Let $\lambda_0, \lambda_1, \ldots, \lambda_r$ be the $U_\alpha = \iota(\alpha)U_{p,1}$ eigenvalues of $z_0, z_1, \ldots, z_r$.

As already observed in §3.5.9, by shrinking $\mathrm{Sp}\,\mathcal{R}$ we may assume that the projection $\mathcal{E}_\sigma \to \mathrm{Sp}\,\mathcal{R}$ is finite and flat. Therefore, we may assume that there exist $\mathcal{Z}_{\alpha,i} \subset \mathcal{Z}_\alpha$, disjoint, such that z_i is the only point of the set $\{z_0, z_1, \ldots, z_r\}$ lying in $\mathcal{Z}_{\alpha,i}$.

Consider the polynomial

$$R(T) = \frac{T(T-\lambda_1)\cdots(T-\lambda_r)}{\lambda_0(\lambda_0-\lambda_1)\cdots(\lambda_0-\lambda_r)}$$

Since $R(\lambda_i) = 0$ for $i \in \{1, 2, \ldots, r\}$ and $R(\lambda_0) = 1$ it follows that by shrinking $\mathrm{Sp}\,\mathcal{R}$ we may assume that for $x \in \cup\mathcal{Z}_{\alpha,i}$ the function $R(T)$, where T is the projection to $\mathbb{G}_m$, is topologically unipotent for $x \in \mathcal{Z}_{\alpha,0}$ and topologically nilpotent for $x \in \mathcal{Z}_{\alpha,1} \cup \ldots \cup \mathcal{Z}_{\alpha,r}$. Therefore, the expression

$$e = \lim_{n\to\infty} R(U_\alpha)^{n!}$$

is well-defined. It has the property that if h is an eigenform giving rise to a point $y \in \mathcal{E}_\sigma$ then

- if $r_\alpha(y) \in \mathcal{Z}_{\alpha,0}$ then $e(y) = y$, since $R(T)$ is topologically unipotent over $\mathcal{Z}_{\alpha,0}$,
- if $r_\alpha(y) \in \mathcal{Z}_{\alpha,1} \cup \ldots \mathcal{Z}_{\alpha,r}$ then $e(y) = 0$, since $R(T)$ is topologically nilpotent over these components.

The rest of the argument is an adaptation of [Tay91, Proposition 3]. Define

$$g_t = e(E^t f)$$

Then $g_0 = e(f) = f$ as $R(\lambda_0) = 1$ and therefore, as t approaches zero p-adically, $g_t \to g_0 = f$, since the action of the Hecke operators is continuous. More explicitly, if $F \in M^\dagger_{p^n,N,\kappa+Y,\sigma}$ such that the specialization at $Y = 0$ of F gives f, then by 3.4.9

it follows that the specialization at $Y = k_0(p-1)t$ is g_t, and therefore, for t going p-adically to 0, the sequence g_t converges to f.

Finally, decomposing $E^t f$ as a sum of classical Hecke eigenforms $\sum f_s$, where the eigenform f_s gives rise to the point y_s. Since for t close enough to 0, the form g_t is nonzero (since $g_0 = f$ is nonzero), it follows that $\sum e(f_s) \neq 0$. By the above, this (nonempty) sum contains the terms f_s such that $r_\alpha(y_s) \in \mathcal{Z}_{\alpha,0}$. For each t close to 0, choose one such y_s and denote it by z_t. By construction, $r_\alpha(z_t) \to r_\alpha(z)$, but since $r_\alpha^{-1}(r_\alpha(z)) = z$, it follows that $z_t \to z$.

$\square$

Chapter 4

Crystalline Representations

We exhibited a dense set of points in a neighborhood of a given classical point. Using the Galois representations constructed at the dense set of points we produce an analytic pseudorepresentation which specializes at traces of classically constructed Galois representations. Assuming the given classical point has an absolutely irreducible Galois representation, in a small neighborhood around it we lift the pseudorepresentation to a Galois representation. Finally, applying a theorem of Kisin, we deduce crystallinity of the Galois representation in certain circumstances.

4.1 Overview of p-adic Hodge Theory

(4.1.1) Given that the purpose of this thesis is to analyze certain p-adic Galois representations at the decomposition group $G_{\mathbb{Q}_p}$, it is useful to briefly describe the main objects of p-adic Hodge theory. Before proceeding, it is instructive to contrast this setting with that of ℓ-adic representations. Given a Galois representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$ where $p \neq \ell$, the wild inertia subgroup $I_{\mathbb{Q}_p}^{\mathrm{wild}} \subset G_{\mathbb{Q}_p}$, being a pro-$p$ subgroup, acts through a finite quotient. In contrast, the cyclotomic character $\chi_p : G_{\mathbb{Q}_p} \to \mathbb{Q}_p^\times$ takes wild inertia to $1 + p\mathbb{Z}_p$, and, in some sense, the scope of p-adic Hodge theory is to control the action of inertia under p-adic Galois representations.

(4.1.2) To achieve this, p-adic Hodge theory concocts certain $\mathbb{Q}_p$-algebra domains B, satisfying $B^{G_{\mathbb{Q}_p}} = \mathbb{Q}_p$ and a technical property and, in some cases, endowed with additional structure. For each such ring B there is a notion of admissibility of a p-adic Galois representation with respect to B.

For a p-adic Galois representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}(V)$ the associated Diedonné module relative to a domain B is $D_B(V) = (B \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$. This is a $\mathbb{Q}_p$-vector space of dimension at most $\dim V$ and the representation ρ is said to be B*-admissible* if, in fact, $\dim_{\mathbb{Q}_p} D_B(V) = \dim_{\mathbb{Q}_p} V$.

For the constructed rings B, the associated category of B-admissible p-adic Galois representations is supposed to be the target category of representations arising from certain geometric settings.

(4.1.3) The two basic rings of p-adic Hodge theory that we use are B_{dR} and B_{cris}. We will not define these rings, but refer to [BCb]. We will be content with remarking that de Rham representations, those which are admissible with respect to B_{dR}, typically arise in the cohomology of smooth proper varieties over $\mathbb{Q}_p$, while crystalline representations, those which are admissible with respect to B_{cris}, typically arise in the cohomology of smooth proper varieties over $\mathbb{Z}_p$.

The ring B_{dR} is filtered and has the property that $\mathrm{gr}\, B_{\mathrm{dR}} \cong \mathbb{C}_p[t, t^{-1}]$ where $t \in B_{\mathrm{dR}}$ is an element that will be used later in Theorem 4.3.2. The ring B_{cris} is a subring of B_{dR} (thus inheriting a filtration), and is endowed with an injective semilinear map $\varphi : B_{\mathrm{cris}} \to B_{\mathrm{cris}}$ called Frobenius. One final piece of notation: we write $B_{\mathrm{dR}}^+ = \mathrm{Fil}^0 B_{\mathrm{dR}}$ and $B_{\mathrm{cris}}^+ = \mathrm{Fil}^0 B_{\mathrm{cris}}$.

For simplicity, we have been deliberately deceiving. In general, for a finite extension $K/\mathbb{Q}_p$, we have $B_{\mathrm{dR}}^{G_K} = K$, whereas $B_{\mathrm{cris}}^{G_K} = K_0$, where $K_0 \subset K$ is the maximal unramified subfield. In that case, $B_{\mathrm{cris}} \otimes_{K_0} K \subset B_{\mathrm{dR}}$ are filtered K-vector spaces. We note, however, that we only treat crystallinity in settings where $K/\mathbb{Q}_p$ is an unramified extension, therefore $K_0 = K$.

4.2 An Analytic Galois Representation

(4.2.1) In some sense, this section adapts the proof of [Tay91, Theorem 3] to the context of eigenvarieties. In the aforementioned article, Taylor cites a theorem on Galois representations associated to regular Siegel modular forms due to Shimura, Deligne, Chai and Faltings. We find it more convenient to cite an amalgamation of more recent results. For consistency of notation (as promised in §3.2.2), we will denote by κ the Harish-Chandra parameter of the infinite component π_∞ of an irreducible automorphic representation π of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$, assuming π_∞ is a (limit of) discrete series, and by $\kappa^B = \kappa + (1, 2)$ the Blattner parameter of π_∞. If f is a holomorphic Siegel modular form giving rise to the representation π, then f has weight κ^B.

Theorem 4.2.2. *Let π be an irreducible admissible automorphic representation of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$ with π_∞ having Harish-Chandra parameter $\kappa = (k_1, k_2)$ with $k_1 \geq k_2+1 \geq 2$ (or equivalently, having Blattner parameter $\kappa^B = (k_1+1, k_2+2)$ with $k_1+1 \geq k_2+2 \geq 3$) and level N. Then there exists a number field E such that for all primes p and places $v \mid p$ of E there exist Galois representations $\rho_{\pi,p} : G_\mathbb{Q} \to \mathrm{GL}(4, \overline{E}_v)$ with the following properties:*

1. *$\rho_{\pi,p}|_{G_{\mathbb{Q}_\ell}}$ is unramified at primes $\ell \nmid Np$,*

2. *at primes $\ell \nmid Np$,*

$$L_\ell\left(\pi, s - \frac{w}{2}, \mathrm{spin}\right) = \det\left(1 - \rho_{\pi,p}(\mathrm{Frob}_\ell)\ell^{-s}\right)^{-1}$$

   where $w = k_1 + k_2$ and Frob_ℓ is the geometric Frobenius.

3. *If $p \nmid N$ then $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is crystalline and the crystalline Frobenius φ has characteristic polynomial*

$$(x - \alpha_p)(x - \beta_p)(x - \gamma_p)(x - \delta_p)$$

where $\alpha_p, \beta_p, \gamma_p, \delta_p$ are the Satake parameters associated to the unramified principal series π_p, where π_p is the component at p of π.

Proof. The existence and equality of local L-functions are guaranteed by [Wei05, Theorem I]. The last part follows from [Fal89] and [Urb05, Theorem 1]. □

Remark 4.2.3. *The first two parts of the theorem are based on the work of Taylor in [Tay93], who obtains the compatibility between the automorphic and Galois L-functions away from p at a set of primes of Dirichlet density 1. The last part of the theorem is subtle, and is achieved by extending Hecke correspondences to a chosen arithmetic toroidal compactification, where the methods of Katz and Messing (which require projectivity of the underlying variety) are applicable.*

(4.2.4) We now specialize to the case of interest to us. Let f be a holomorphic Siegel modular eigenform of level N and weight $\kappa^B = (k+1, 2)$, with $k \geq 1$, giving rise to an irreducible admissible cuspidal automorphic representation π of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ whose component at infinity π_∞ is a holomorphic limit of discrete series with Harish-Chandra parameter $\kappa = (k, 0)$. Let p be a prime such that π_p is an unramified principal series with Satake parameters $\alpha_p, \beta_p, \gamma_p, \delta_p$. Let α be one of these Satake parameters and let f_α be a level pN eigenform whose $U_{p,1}$ eigenvalue is α (cf. §3.2.4). Let σ be the slope of α and let z be the point on $\mathcal{E}_\sigma$ coming from f_α.

We now explain the interaction between Taylor's construction of the Galois representation $\rho_{f,p}$ ([Tay91, 3.6]) and the construction, in Proposition 3.6.4, of infinitely many points $z_t \in \mathcal{E}_\sigma$ in an affinoid neighborhood $\mathrm{Sp}\,\mathcal{R}$ of the point z. Let E be as in Lemma 3.1.9 be a lift of the Hasse invariant, and let π_n be the irreducible automorphic representation coming from the holomorphic Siegel modular form fE^{p^n}. Then $f_\alpha E^{p^n}$ gives rise to an automorphic form in π_n, while z_t corresponds to the Hecke eigensystem of a form f_t which gives rise to an automorphic form in π_n.

Let $\mathrm{Sp}\,\mathcal{R}_\alpha$ be an affinoid neighborhood of z such that z_t are dense in $\mathrm{Sp}\,\mathcal{R}_\alpha$. (If z_t

were not dense in $\mathrm{Sp}\,\mathcal{R}$, they would be contained in an analytic hypersurface, hence take an affinoid neighborhood in the intersection between $\mathrm{Sp}\,\mathcal{R}$ and this hypersurface.) For simplicity of notation, we will write $\mathcal{R}$ instead of $\mathcal{R}_\alpha$ from now on.

By Theorem 4.2.2, there are Galois representations $\rho_{z_t} : G_{\mathbb{Q}} \to \mathrm{GL}(4, \mathbb{C}_p)$ associated to z_t and the representation $\rho_{z_t}|_{G_{\mathbb{Q}_p}}$ is crystalline, since π_n has level N. (In effect, the points z_t come from oldforms of level pN.) By [Che04, 7.1.1], since the points z_t are dense in $\mathrm{Sp}\,\mathcal{R}$, there exists a pseudorepresentation $T : G_{\mathbb{Q}} \to \mathcal{R}$ specializing to $\mathrm{Tr}\,\rho_{z_t}$ at the points z_t, and the Galois representation $\rho_{\pi,p} = \rho_z : G_{\mathbb{Q}} \to \mathrm{GL}(4, E_z)$ associated to the form f is the Galois representation whose trace is $\phi_z \circ T$, where $\phi_z : \mathcal{R} \to E_z$ represents the point $z \in \mathcal{E}_\sigma(E_z)$.

(4.2.5) We now make the assumption that the Galois representation $\rho_{f,p} : G_{\mathbb{Q}} \to \mathrm{GL}(4, \mathbb{C}_p)$ is absolutely irreducible.

Proposition 4.2.6. *By shrinking* $\mathrm{Sp}\,\mathcal{R}$ *if necessary, there exists a Galois representation* $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(4, \mathcal{R})$ *whose trace is the pseudorepresentation* T.

Proof. This is a generalization of [CM98, 5.1.2], whose proof relies on the very explicit formulae used by Wiles in his original definition of two dimensional pseudorepresentations. For higher dimensional pseudorepresentations (for which explicit formulae are not available) an indirect proof is required.

We start with the observation that local rings at analytic points on rigid varieties are henselian ([dJvdP96, 2.1.1 (1)]), therefore the local ring $\mathcal{O}_{X,z}$ is henselian, where we write $X = \mathrm{Sp}\,\mathcal{R}$. The residue field of $\mathcal{O}_{X,z}$ has characteristic 0 and the representation ρ_z is absolutely irreducible. By [Rou96, 5.2] it follows that there exists a Galois representation $\rho_{\mathcal{O}_{X,z}} : G_{\mathbb{Q}} \to \mathrm{GL}(4, \mathcal{O}_{X,z})$ having trace $T_{\mathcal{O}_{X,z}}$, and such that $\phi_z \circ \rho_{\mathcal{O}_{X,z}} = \rho_z$. Since, by definition, $\mathcal{O}_{X,z} = \varinjlim \mathcal{O}_X(U)$ where the limit is over admissible open neighborhoods U of z in X, it follows that we may shrink $\mathrm{Sp}\,\mathcal{R}$ to get a Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(4, \mathcal{R})$ whose trace is the pseudorepresentation

T. $\square$

Remark 4.2.7. *We remark that a more general phenomenon occurs. If X is a rigid space and $T : G_{\mathbb{Q}} \to \mathcal{O}(X)$ is a pseudorepresentation, and at each closed point x of X, T is the trace of an absolutely irreducible representation, then there exists a Galois representation $\rho : G_{\mathbb{Q}} \to \mathcal{A}^{\times}$, where $\mathcal{A}$ is an Azumaya algebra over X, such that T is the reduced trace of ρ. This is the content of [Che04, 7.2.6], which follows the suggestion after [CM98, 5.1.2].*

4.3 Crystalline Eigenvalues

(4.3.1) We start this section with a theorem of Kisin, which is a concatenation of Proposition 5.14 and Corollary 5.15 of [Kis03]. Let $\mathcal{R}$ be an affinoid over $\mathbb{Q}_p$ and let M be a finite free $\mathcal{R}$-module with a continuous $G_{\mathbb{Q}_p}$-action. Let $Y \in \mathcal{R}^{\times}$ such that Y is $\mathcal{R}$-small. We will not repeat the definition of "$\mathcal{R}$-small" here, but we remark that this is satisfied for any sufficiently small affinoid $\operatorname{Sp}\mathcal{R}$ containing a closed point x (cf. [Kis03, 5.2]). Associated to M we take the Sen polynomial $P(T) \in \mathcal{R}[T]$, as in [Kis03, 2.2].

Theorem 4.3.2 (Kisin)**.** *Let $\{\mathcal{R}_i\}_{i\in I}$ be a collection of Tate $\mathcal{R}$-algebras such that for each $n > 0$ there exists a subset I_n of the set of indices I with the following properties:*

1. *For every $i \in I_n$ every $G_{\mathbb{Q}_p}$-equivariant map*

$$M^* \otimes_{\mathcal{R}} \mathcal{R}_i \to B^+_{\mathrm{dR}}/t^n B^+_{\mathrm{dR}} \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{R}_i$$

   factors through $(B^+_{\mathrm{cris}} \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{R}_i)^{\varphi=Y}$, where φ is the crystalline Frobenius.

2. *For each $i \in I_n$ the image of $P(k)$ in $\mathcal{R}_i$ is a unit.*

3. *The map $\mathcal{R} \to \prod_{i\in I_n} \mathcal{R}_i$ is injective.*

Then for any closed subfield $E \subset \mathbb{C}_p$ and any continuous map $f : \mathcal{R} \to E$ there is an E-linear $G_{\mathbb{Q}_p}$-equivariant map

$$M^* \otimes_{\mathcal{R}} E \to (B^+_{\mathrm{cris}} \widehat{\otimes}_{\mathbb{Q}_p} E)^{\varphi = Y}$$

(4.3.3) We now get to the main theorem of this chapter. Let f be as in §4.2.4, i.e., a holomorphic Siegel modular form of level N, weight $\kappa^B = (k+1, 2)$ with $k \geq 1$, giving rise to an irreducible admissible cuspidal automorphic representation π of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$, whose infinite component π_∞ is a limit of discrete series with Harish-Chandra parameter $\kappa = (k, 0)$. Let $\rho_{\pi,p} : G_{\mathbb{Q}} \to \mathrm{GL}(4, \overline{\mathbb{Q}}_p)$ be the Galois representation associated to π. Let $\{\alpha_p, \beta_p, \gamma_p, \delta_p\}$ are the Satake parameters of the local representation π_p.

Theorem 4.3.4. *If $\rho_{\pi,p}$ is irreducible, then*

$$\dim_{\mathbb{Q}_p} D_{\mathrm{cris}} \left(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \right) \geq \#\{\alpha_p, \beta_p, \gamma_p, \delta_p\}$$

In particular, if the Satake parameters $\{\alpha_p, \beta_p, \gamma_p, \delta_p\}$ are all **distinct***, then $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is a crystalline representation, and the characteristic polynomial of the crystalline Frobenius φ is*

$$(x - \alpha_p)(x - \beta_p)(x - \gamma_p)(x - \delta_p)$$

Proof. Let α be one of the Satake parameters, let f_α be as in §4.2.4 and let z be the point on the eigenvariety coming from f_α. Under the assumption that $\rho_{\pi,p}$ is absolutely irreducibile, by Proposition 4.2.6 there exists a Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(4, \mathcal{R})$ where $\mathrm{Sp}\,\mathcal{R}$ is some affinoid neighborhood of z in $\mathcal{E}_\sigma$. For a point $c \in \mathcal{E}_\sigma(L)$ we write ρ_c for the specialization at c. Let M be the $\mathcal{R}$-dual of ρ.

Let E_t be the residue fields at the points z_t. The Galois representations ρ_{z_t} are crystalline at p by construction (since they are the Galois representations associated to regular holomorphic Siegel modular forms of level pN, which are old at p; cf. §4.2.4). Therefore, if α_t is the $U_{p,1}$-eigenvalue of z_t, there exists a nontrivial $G_{\mathbb{Q}_p}$-equivariant map

$$M^* \otimes_{\mathcal{R}} E_t \to (B^+_{\mathrm{cris}} \widehat{\otimes}_{\mathbb{Q}_p} E_t)^{\varphi=\alpha_t}$$

(In effect, we are exhibiting a weak refinement in the sense of [BCa, 4.2.7].)

We will apply Theorem 4.3.2 to the affinoids $\mathcal{R}_t = E_t$ and (for $n > 0$) for the sets I_n consisting of those points z_t such that $k_0(p-1)p^t > n$. In that case, since the weight of z_t is $w(z_t) = (k, 0) + k_0(p-1)(p^t, p^t)$, the Hodge-Tate weights of ρ_{z_t} are included in the set

$$\{0, k_0(p-1)p^t, k + k_0(p-1)p^t, k + 2k_0(p-1)p^t\}$$

and, since the points z_t come from *holomorphic* Siegel modular forms, the Hodge-Tate weights contain the set

$$\{0, k + 2k_0(p-1)p^t\}$$

([Urb05, 3.4]). By choice of t, we have $k_0(p-1)p^t > n$, so none of the roots of the Sen polynomial $P(T)$ specialized at z_t is equal to n, therefore $P(n)$ is invertible in E_t. This verifies condition (2) of Theorem 4.3.2. Finally, condition (1) of the theorem is verified by the above, and condition (3) is verified because the points z_t are dense in $\mathrm{Sp}\,\mathcal{R}$ (see the third paragraph of §4.2.4).

We conclude that if E is the residue field at z, there exists a nontrivial $G_{\mathbb{Q}_p}$-equivariant E-linear map $M^* \otimes_{\mathcal{R}} E \to (B^+_{\mathrm{cris}} \widehat{\otimes}_{\mathbb{Q}_p} E)^{\varphi=\alpha}$, in other words, the crystalline Frobenius φ acting on $D_{\mathrm{cris}}(\rho_z)$ has an eigenvector with eigenvalue α. Therefore, each distinct Satake parameter provides a crystalline eigenvalue, hence

$$\dim_{\mathbb{Q}_p} D_{\mathrm{cris}} \left(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \right) \geq \#\{\alpha_p, \beta_p, \gamma_p, \delta_p\}$$

If, moreover, the Satake parameters $\{\alpha_p, \beta_p, \gamma_p, \delta_p\}$ are all distinct, we conclude that $\dim_{\mathbb{Q}_p} D_{\mathrm{cris}}(\rho_z) \geq 4$, therefore ρ_z is a crystalline representation. The statement about the characteristic polynomial of φ results from the fact that $\{\alpha_p, \beta_p, \gamma_p, \delta_p\}$ are all the eigenvalues of φ acting on the four dimensional vector space $D_{\mathrm{cris}}(\rho_z)$. $\square$

Chapter 5

Modular Forms on $\mathrm{GL}(2)_{/K}$

5.1 Functorial Transfer to $\mathrm{GSp}(4)_{/\mathbb{Q}}$

(5.1.1) We start with a brief overview of the results of [HST93].

Let K be an imaginary quadratic field and let π be an irreducible cuspidal automorphic regular algebraic representation of $\mathrm{GL}(2, \mathbb{A}_K)$ whose central character χ_π satisfies $\chi_\pi \cong \chi_\pi^c$ (where c is the nontrivial element of $\mathrm{Gal}(K/\mathbb{Q})$) and such that the Langlands parameter associated to the infinity component π_∞ is

$$z \mapsto \begin{pmatrix} z^{1-k} & \\ & (z^c)^{1-k} \end{pmatrix}$$

where $k \geq 2$. For π as above let α_v and β_v be the Satake parameters of π_v if it is an unramified principal series.

(5.1.2) If $W \subset M_{2\times 2}(K)$ is the subspace of Hermitian matrices, then $-\det$ is a bilinear form on W of signature $(3,1)$. The map $\sigma : \mathrm{GL}(2)_{/K} \to \mathrm{GO}(W) = \mathrm{GO}(3,1)^\circ_{/\mathbb{Q}}$ given by $\sigma(g) : x \mapsto gxg^{ct}$ gives a bijection between cuspidal automorphic representations $\widetilde{\pi}$ of $\mathrm{GO}(3,1)^\circ_{/\mathbb{Q}}$ and pairs $(\pi, \widetilde{\chi})$ where π is a cuspidal automorphic representation of $\mathrm{GL}(2)_{/K}$ and $\widetilde{K} : \mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ is a character such that $\widetilde{\chi} \circ N_{K/\mathbb{Q}} = \chi_\pi$.

Using the fact that $\mathrm{GO}(3,1)^\circ_{/\mathbb{Q}}$ is the connected component of the identity in $\mathrm{GO}(3,1)_{/\mathbb{Q}}$ one obtains a bijection between cuspidal automorphic representations $\widehat{\pi}$ of $\mathrm{GO}(3,1)_{/\mathbb{Q}}$ and triples $(\pi, \widetilde{\chi}, \delta)$ where $(\pi, \widetilde{\chi})$ are as above, and δ is a map from the places of $\mathbb{Q}$ to $\{-1, 1\}$, which is 1 at almost all places, it is 1 at places v such that $\pi_v \cong \pi_v^c$ and, if $\pi \cong \pi^c$ then $\prod_v \delta(v) = 1$ (cf. [HST93, p. 383]).

(5.1.3) Assume π is a cuspidal automorphic representation of $\mathrm{GL}(2)_{/K}$ as above and suppose that $\widehat{\pi} = (\pi, \widetilde{\chi}, \delta)$ is a cuspidal representation of $\mathrm{GO}(3,1,\mathbb{A}_\mathbb{Q})$ such that

- $\delta(\infty) = -1$,
- there exists a character $\phi : K^\times\backslash\mathbb{A}_K^\times \to \mathbb{C}^\times$ with $\phi|_{\mathbb{A}_\mathbb{Q}^\times} = \widetilde{\chi}$ such that if $\pi_v \cong \pi_v^c$ then $\delta(v) = \widetilde{\chi}_v(-1)\varepsilon\left(\pi_v \otimes \phi_v^{-1}, \dfrac{1}{2}\right)$ and
- $L\left(\pi \otimes \phi^{-1}, \dfrac{1}{2}\right) \neq 0$.

The main result of [HST93] (contained in Proposition 3 loc. cit.) is that under the assumptions enumerated above, the theta lift $\Theta(\widehat{\pi})$ from $\mathrm{GO}(3,1)_{/\mathbb{Q}}$ to $\mathrm{GSp}(4)_{/\mathbb{Q}}$ is a *nonzero cuspidal* automorphic representation of $\mathrm{GSp}(4,\mathbb{A}_\mathbb{Q})$. Moreover, if Π is an irreducible quotient of $\Theta(\widehat{\pi})$, then Π_∞ is a *holomorphic* discrete series representation.

5.2 The Galois Representations

This paragraph is based on [Tay91], [Tay93] and [BH07].

(5.2.1) For an automorphic representation π of $\mathrm{GL}(2,\mathbb{A}_K)$ as in §5.1.1, let $S(\pi)$ be the set of places of $\mathbb{Q}$ containing:

- the infinite place,
- places p such that $v \mid p$ ramifies over $\mathbb{Q}$,

- inert primes $p = v$ such that π_v is not an unramified principal series, and

- split primes $p = v \cdot v^c$ such that π_v or π_{v^c} is not an unramified principal series.

A last bit of notation: a set $\mathcal{M}$ of quadratic characters $\mu : \mathbb{A}_K^\times / K^\times \to \mathbb{C}^\times$ is *dense* if for any quadratic character $\widetilde{\mu} : \mathbb{A}_K^\times / K^\times \to \mathbb{C}^\times$ and any finite set S of places of K there exists a character $\mu \in \mathcal{M}$ such that $\widetilde{\mu}_v = \mu_v$ for $v \in S$.

(5.2.2) We have seen that, associated to the cuspidal representation π, together with choices of $\widetilde{\chi}$ and δ satisfying certain conditions there exists a nonzero cuspidal automorphic representation Π of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$. In order to ensure that Π does not degenerate (the essential problem being the nonvanishing of the L-function at $\frac{1}{2}$), one must twist the original representation π by a quadratic character. Using a dense set of such characters allows one to recover the Galois representation associated to π. The following theorem is [BH07, Theorem 2.3]:

Theorem 5.2.3. *For π as above there exists a dense set $\mathcal{M}$ of quadratic characters such that for each $\mu \in \mathcal{M}$ there exists a cuspidal automorphic representation Π^μ of* $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ *with the following properties:*

- *the component at infinity Π^μ_∞ is a limit of discrete series representation with Harish-Chandra parameter $(k-1, 0)$, and*

- *if $p = v \notin S(\pi \otimes \mu)$ then Π^μ_p is an unramified principal series with Satake parameters*

$$\pm\sqrt{\alpha_v \mu_v(\varpi_v)}, \pm\sqrt{\beta_v \mu_v(\varpi_v)}$$

- *if $p = v \cdot v^c \notin S(\pi \otimes \mu)$ then Π^μ_p is an unramified principal series with Satake parameters*

$$\alpha_v \mu_v(\varpi_v), \alpha_{v^c} \mu_{v^c}(\varpi_{v^c}), \beta_v \mu_v(\varpi_v), \beta_{v^c} \mu_{v^c}(\varpi_{v^c})$$

where for each finite place v, we denote by ϖ_v a uniformizer for K_v.

For each $\mu \in \mathcal{M}$, Taylor associates Galois representations $\rho_{\Pi^\mu,p} : G_K \to \mathrm{GL}(4, \overline{\mathbb{Q}}_p)$ to Π_μ using congruences (given by the specialization of the Galois representation ρ from Proposition 3.4.3 at the system of Hecke eigenvalues coming from Π^μ). In [BH07] it is shown that there exists $\rho_{\pi,p} : G_K \to \mathrm{GL}(2, \overline{\mathbb{Q}}_p)$ such that

$$\mathrm{Ind}_K^{\mathbb{Q}} (\rho_{\pi,p} \otimes \mu) \cong \rho_{\Pi^\mu,p}$$

We cite the theorem below ([BH07, Theorem 1.1]):

Theorem 5.2.4. *There exists a continuous irreducible representation $\rho_{\pi,p} : G_K \to \mathrm{GL}(2, \overline{\mathbb{Q}}_p)$ such that if $v \notin S(\pi) \cup \{v \mid p\}$ is a place of K, then $\rho_{\pi,p}|_{G_{K_v}}$ is unramified and*

$$L(\pi_v, s) = \det(1 - \rho_{\pi,p}(\mathrm{Frob}_v) q_v^{-s} | \rho_{\pi,p}|_{G_{K_v}})^{-1}$$

Remark 5.2.5. *In the original paper due to Taylor, the existence of the dense set of characters $\mathcal{M}$ was proven with some restrictions on the weight. In [BH07] this restriction is removed, by using a result of Friedberg and Hoffstein on nonvanishing of L-functions.*

We end this section with a lemma.

Lemma 5.2.6. *Let $\rho : G_K \to \mathrm{GL}(2, \overline{\mathbb{Q}}_p)$ be an irreducible Galois representation. Then $\mathrm{Ind}_K^{\mathbb{Q}} \rho : G_{\mathbb{Q}} \to \mathrm{GL}(4, \overline{\mathbb{Q}}_p)$ is irreducible if and only if $\rho \ncong \rho^c$.*

Proof. Assume $\mathrm{Ind}_K^{\mathbb{Q}}$ is reducible and let $f : \tau \hookrightarrow \mathrm{Ind}_K^{\mathbb{Q}} \rho$ be a nontrivial subrepresentation. By Frobenius reciprocity $g : \tau|_{G_K} \hookrightarrow \rho$ is also a subrepresentation. Since ρ is irreducible, it follows that $g : \tau|_{G_K} \cong \rho$, where the map g can be computed as $g(v) = f(v)(1)$. Then $v \mapsto f(v)(c)$ is an isomorphism between ρ and ρ^c.

Reciprocally, if $\rho \cong \rho^c$ and e_1, e_2, e_3, e_4 is the basis of $\mathrm{Ind}_K^{\mathbb{Q}} \rho$ with respect to which for $h \in G_K$ we have

$$(\operatorname{Ind}_K^{\mathbb{Q}} \rho)(h) = \begin{pmatrix} \rho(h) & \\ & \rho(h) \end{pmatrix}$$

and

$$(\operatorname{Ind}_K^{\mathbb{Q}} \rho)(ch) = \begin{pmatrix} & \rho(h) \\ \rho(h) & \end{pmatrix}$$

then the subspace generated by $e_1 + e_3$ and $e_2 + e_4$ is a subrepresentation isomorphic to ρ. $\square$

5.3 Behavior at p

The purpose of this thesis is to answer the question of what happens in Theorem 5.2.4 when $v \mid p$. Our main result is the following:

Theorem 5.3.1. *Let π be a cuspidal automorphic representation of* $\mathrm{GL}(2, \mathbb{A}_K)$ *with infinite component π_∞ having Langlands parameter $z \mapsto \begin{pmatrix} z^{1-k} & \\ & \overline{z}^{1-k} \end{pmatrix}$ where $k \geq 2$, and such that the central character χ_π satisfies $\chi_\pi \cong \chi_\pi^c$. Let p be a prime number such that $K/\mathbb{Q}$ is not ramified at p.*

- *If $p = v$ is inert in K assume π_v is an unramified principal series with distinct Satake parameters α_v, β_v;*
- *If $p = v \cdot v^c$ splits assume that π_v and π_{v^c} are unramified principal series with Satake parameters α_v, β_v and $\alpha_{v^c}, \beta_{v^c}$ respectively, such that $\{\alpha_v, \beta_v, \alpha_{v^c}, \beta_{v^c}\}$ are all distinct.*

Then $\rho_{\pi,p}|_{G_{K_v}}$ (as well as $\rho_{\pi,p}|_{G_{K_{v^c}}}$ in the split case) is a crystalline representation.

Proof. Let $v \notin S(\pi)$. Choose $\mu \in \mathcal{M}$ such that

1. if $p = v$ is inert then μ is unramified at p and $\mu_p(p) = 1$,

2. if $p = v \cdot v^c$ is split then μ is unramified at v and v^c, and $\mu_v(\varpi_v) = \mu_{v^c}(\varpi_{v^c}) = 1$,

3. $(\rho_{\pi,p} \otimes \mu) \ncong (\rho_{\pi,p} \otimes \mu)^c$.

If for some place $w \notin S(\pi) \cup \{v\}$ we have $\rho_{\pi,p}|_{G_{K_w}} \ncong \rho_{\pi,p}|^c_{G_{K_w}}$ then condition 3 above is satisfied by requiring that μ be unramified at w and $\mu_w(\varpi_w) = \mu_{w^c}(\varpi_{w^c})$; otherwise, it is satisfied by requiring that μ be unramified at w and $\mu_w(\varpi_w) \neq \mu_{w^c}(\varpi_{w^c})$. Finally, there exists a quadratic character $\widetilde{\mu}$ satisfying the above conditions, and therefore there exists a quadratic character $\mu \in \mathcal{M}$ satisfying them, since $\mathcal{M}$ is dense. Note that conditions 1 and 2 imply that $S(\pi \otimes \mu)$ does not contain p.

By choice of μ and Lemma 5.2.6 it follows that $\rho_{\Pi^\mu,p} \cong \mathrm{Ind}_K^{\mathbb{Q}} \rho_\pi^\mu$ is irreducible. If we assume that $p = v \cdot v^c$ splits completely, then Π_p^μ is an unramified principal series with distinct Satake parameters $\alpha_v, \alpha_{v^c}, \beta_v, \beta_{v^c}$. Therefore, by Theorem 4.3.4, the Galois representation $\rho_{\Pi^\mu,p}|_{G_{\mathbb{Q}_p}}$ is crystalline. If we assume that $p = v$ is inert, then Π_p^μ is an unramified principal series with distinct Satake parameters $\pm\sqrt{\alpha_v}, \pm\sqrt{\beta_v}$. Again, by Theorem 4.3.4, the Galois representation $\rho_{\Pi^\mu,p}|_{G_{\mathbb{Q}_p}}$ is crystalline.

We conclude that $\rho_{\Pi^\mu,p}|_{G_{\mathbb{Q}_p}} \cong \left(\mathrm{Ind}_K^{\mathbb{Q}} \rho_{\pi,p}^\mu\right)_{G_{\mathbb{Q}_p}}$ is crystalline. We would like to deduce that $\rho_p|_{G_{K_v}}$ is crystalline when $v \mid p$.

First, start with the case when $p = v \cdot v^c$ splits completely in K. Then

$$\left(\mathrm{Ind}_K^{\mathbb{Q}} \rho_{\pi,p}^\mu\right)|_{G_{\mathbb{Q}_p}} = \rho_{\pi,p}^\mu|_{G_{K_v}} \oplus \rho_{\pi,p}^\mu|_{G_{K_{v^c}}}$$

Therefore

$$D^*_{\mathrm{cris}}\left(\mathrm{Ind}_K^{\mathbb{Q}} \rho_{\pi,p}^\mu\right)|_{G_{\mathbb{Q}_p}} = D^*_{\mathrm{cris}}\left(\rho_{\pi,p}^\mu|_{G_{K_v}}\right) \oplus D^*_{\mathrm{cris}}\left(\rho_{\pi,p}^\mu|_{G_{K_{v^c}}}\right)$$

Since $\left(\mathrm{Ind}_K^{\mathbb{Q}} \rho_{\pi,p}^\mu\right)|_{G_{\mathbb{Q}_p}}$ is crystalline, it follows that $\left(\rho_{\pi,p}^\mu\right)|_{G_{K_v}}$ and $\left(\rho_{\pi,p}^\mu\right)|_{G_{K_{v^c}}}$ are crystalline. Since we have chosen μ such that μ_v and μ_{v^c} are trivial, it follows that $(\rho_{\pi,p})|_{G_{K_v}}$ and $(\rho_{\pi,p})|_{G_{K_{v^c}}}$ are crystalline.

Second, assume that $p = v$ is inert in K. Then

$$\left(\mathrm{Ind}_K^{\mathbb{Q}} \rho_{\pi,p}^{\mu}\right)|_{G_{\mathbb{Q}_p}} = \mathrm{Ind}_{K_v}^{\mathbb{Q}_p} \left(\rho_{\pi,p}^{\mu}|_{G_{K_v}}\right)$$

Since $K_v/\mathbb{Q}_p$ is a finite extension, $\mathrm{Ind}_{K_v}^{\mathbb{Q}_p}$ is also a left adjoint to restriction, so as K_v-vector spaces (remember that $v \notin S(\pi \otimes \mu)$ implies that $K_v/\mathbb{Q}_p$ is unramified, therefore D_{cris} is in fact a K_v-vector space) we have

$$\begin{aligned} D_{\mathrm{cris}}^{*}\left(\rho_{\pi,p}^{\mu}|_{G_{K_v}}\right) &\cong \mathrm{Hom}_{G_{K_v}}\left(\rho_{\pi,p}^{\mu}|_{G_{K_v}}, B_{\mathrm{cris}}\right) \\ &\cong \mathrm{Hom}_{G_{\mathbb{Q}_p}}\left(\mathrm{Ind}_{K_v}^{\mathbb{Q}_p}\left(\rho_{\pi,p}^{\mu}|_{G_{K_v}}\right), B_{\mathrm{cris}}\right) \\ &\cong D_{\mathrm{cris}}^{*}\left(\mathrm{Ind}_{K_v}^{\mathbb{Q}_p}\left(\rho_{\pi,p}^{\mu}|_{G_{K_v}}\right)\right) \end{aligned}$$

Since $\left(\mathrm{Ind}_K^{\mathbb{Q}} \rho_{\pi,p}^{\mu}\right)|_{G_{\mathbb{Q}_p}}$ is crystalline, we get

$$\dim_{\mathbb{Q}_p} D_{\mathrm{cris}}^{*}\left(\mathrm{Ind}_{K_v}^{\mathbb{Q}_p}\left(\rho_{\pi,p}^{\mu}|_{G_{K_v}}\right)\right) = 4$$

therefore

$$\dim_{K_v} D_{\mathrm{cris}}^{*}\left(\rho_{\pi,p}^{\mu}|_{G_{K_v}}\right) = 2$$

which shows that $\left(\rho_{\pi,p}^{\mu}\right)_{G_{K_v}}$ is crystalline. As before, we get $(\rho_{\pi,p})_{G_{K_v}}$ is crystalline. $\square$

Chapter 6

Concluding Remarks

(6.1) The main hypothesis of Theorem 5.3.1, that of distinct Satake parameters, is essential for our method of proof. If one could construct congruences between holomorphic Siegel modular forms with equal Satake parameters at p and bounded weight Siegel modular forms, potentially increasing the level away from p, then one could use integral p-adic Hodge theory to extend the result to the case of equal parameters. However, constructing congruences between nonregular Siegel modular forms is beyond our current means.

(6.2) We mentioned in Theorem 4.3.4, that if the Satake parameters at p are distinct, then the Galois representation crystalline and the characteristic polynomial of the crystalline Frobenius φ is $(x-\alpha_p)(x-\beta_p)(x-\gamma_p)(x-\delta_p)$. However, we cannot readily conclude that the characteristic polynomial of φ on $D_{\mathrm{cris}}(\rho_{\pi,p}|_{G_{K_v}})$ is $(x-\alpha_v)(x-\beta_v)$ since we deduced crystallinity of the latter by a dimension count.

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