Simulation of the CEV process and the local martingale property.

A. E. Lindsay

Mathematics Department, University of Arizona
617 N. Santa Rita Ave
Tucson AZ, USA 85721.

D. R. Brecher

FINCAD
Central City, Suite 1750, 13450 102nd Avenue
Surrey, B.C. V3T 5X3, Canada

Abstract

We consider the Constant Elasticity of Variance (CEV) process, reviewing the relationships between its transition density and that of the non-central chi-squared distribution. When the CEV parameter exceeds one, the forward price process is a strictly local martingale, and the price of a plain vanilla European call option reflects this fact. We develop techniques for Monte Carlo simulation of the CEV process, for all parameter regimes, and compare the results against the analytic expressions for plain vanilla European option prices. Using these techniques, we also verify the local martingale property.

Keywords: CEV process, Bessel process, local martingale.

1. Introduction

Pricing derivatives under the assumption of constant volatility, as in the classic Black-Scholes-Merton model [3, 19] of option pricing, is well-known to give results which cannot be reconciled with market observations, although these problems did not widely manifest themselves until the 1987 market crash. After this event, the so-called *volatility smile* or *volatility skew* became common place in equity markets.

The volatility smile is a market phenomenon whereby the Black-Scholes implied volatility of an option exhibits a dependence on the strike price. An alternative to the Black-Scholes model, which exhibits such a volatility skew, is the constant elasticity of variance (CEV) process, first proposed by Cox & Ross [5].

The CEV model is a continuous time diffusion process satisfying[1]

$$\mathrm{d}F = \sigma F^{\alpha}\,\mathrm{d}W, \qquad F(0) = F_0 > 0, \tag{1}$$

where $F(t)$ is the state variable representing the forward price of some underlying asset at time t and W is a standard Brownian motion. The parameter α is called the elasticity and

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Email addresses: alindsay@math.arizona.edu (A. E. Lindsay), d.brecher@fincad.com (D. R. Brecher)

[1]We ignore a possible drift term here, but its inclusion is straightforward, and has no substantial effects.

we take $\alpha \neq 1$ to distinguish (1) from the Black-Scholes model. To ensure σ has the correct dimensions, we take $\sigma = \sigma_{\mathrm{LN}} F_0^{1-\alpha}$, where σ_{LN} is the effective lognormal volatility.

Feller's work [10] on singular diffusion processes underpins much of our theoretical understanding of (1) and demonstrates that the CEV process admits three distinct types of solution according to the parameter regimes $\alpha < 1/2$, $1/2 \leq \alpha < 1$ and $\alpha > 1$. These distinct regimes allow the CEV model to capture qualitative features of several asset classes present in the financial industry. For example, in the case $\alpha < 1$, it is well known that $F = 0$ is an accessible case and therefore the CEV model allows for bankruptcy.

The pricing of derivatives under the CEV process was originally investigated by Cox & Ross [4, 5] who obtained closed-form European option prices when $\alpha < 1$ in terms of a sum of incomplete gamma functions. Schroder [20] established a connection between these pricing formulae and the non-central chi-squared distribution.

The case $\alpha > 1$ was studied by Emanuel & MacBeth [9] by constructing the relevant transition density function, which was in turn used to show that $\mathbb{E}[F_T | F_0] \neq F_0$. Therefore F_T is a strictly local martingale when $\alpha > 1$ (F_T can be shown to be martingale for $\alpha < 1$). An important consequence of this martingale property is that the widely-quoted European call price (see, *e.g.*, [14]; the origin of these results being [20]) does not represent an arbitrage free value unless augmented with a correction term. This feature of (1) has been noticed by Lewis [17], discussed by Atlan & Leblanc [2, 15] and is the main focus herein.

Of particular interest is the numerical treatment of (1) in the light of this local martingale property and the non-zero absorption probability at $F = 0$ when $\alpha < 1$. In the former case, paths of (1) are constructed from inversion of a cumulative distribution function and the results are consistent with the local martingale property. In the latter case, we show how to account both for the absorption probability and the unusual distribution while obtaining results consistent with the analytic properties. Such numerical techniques can thus be used to price arbitrary derivatives written on an underlying asset whose dynamics is described by (1).

The paper is structured as follows. In §2, we review the CEV process, the results of Feller concerning the transition density function, highlight the connections with the non-central chi-squared distribution, and establish a symmetry relationship between the regimes $\alpha < 1$ and $\alpha > 1$. In §3, we compute the expected value of F_T itself, and confirm the local martingale nature of the process in this case. Closed-form expressions for the prices of plain vanilla European options, derived directly from the transition density function, are also developed. In §4, we develop techniques for Monte Carlo simulation of the CEV process (1), in all parameter regimes, and test them against the analytic result for both the forward price of the underlying, and the prices of plain vanilla European options. We find very good agreement. Finally in §5 we discuss our results.

2. General properties of the CEV process

The CEV process applied to the forward price $F(t)$ of some underlying asset satisfies the stochastic differential equation (SDE) (1). We take the view that (1) applies only up to the stopping time

$$\tau \equiv \inf_{t>0} \{F(t) = 0\} . \tag{2}$$

The treatment of the process after the stopping time requires consideration of the underlying financial problem. For example, when F represents some equity asset price, the stopping time

(2) would indicate the time of bankruptcy. In other financial scenarios, however, F could represent a volatility in which case a return to $F > 0$ after the stopping time would be more sensible.

It is advantageous to work with the transformed variable

$$X = \frac{F^{2(1-\alpha)}}{\sigma^2 (1-\alpha)^2}, \tag{3}$$

which follows the square root process

$$\mathrm{d}X = \delta \, \mathrm{d}t + 2\sqrt{X} \, \mathrm{d}W, \qquad \delta \equiv \frac{1-2\alpha}{1-\alpha}. \tag{4}$$

Equation (4) is a squared Bessel process BESQ$^\delta$, with δ degrees of freedom. We again take the view that a path of (4) is defined only up to the stopping time.

For δ a positive integer, equation (4) is that governing the squared distance, X, from the origin of a Brownian particle in δ spatial dimensions. Accordingly, it has a non-central chi-squared distribution

$$p_{\chi'^2}(x; \delta, \lambda) = \frac{1}{2} \left(\frac{x}{\lambda} \right)^{\nu/2} \exp \left[-\frac{x+\lambda}{2} \right] I_\nu(\sqrt{x\lambda}), \qquad \nu = \frac{\delta}{2} - 1. \tag{5}$$

where $\chi'^2(x; k, \lambda)$ is the non-central chi-squared distribution with degrees of freedom k and non-centrality parameter λ, $I_\nu(x)$ is the modified Bessel function of the first kind, and ν is the *index* of the squared Bessel process. Of course, δ is generally not an integer and need not be positive. For $\delta \in \mathbb{R}^+$, however, the properties of (4) are well developed, but the case $\delta < 0 (\alpha > 1)$ has received less attention (see [21, 22, 12, 7, 8] and references therein).

To build a theory for the CEV process for all $\alpha \neq 1$, the classic analysis of Feller [10] is employed. From a direct solution of the Fokker-Planck equation, Feller showed that the square root process (4) has very different properties according to whether

- $\delta \leq 0$, or $\alpha \in [0.5, 1)$: the boundary $X = 0$ is attainable and absorbing.
- $0 < \delta < 2$, or $\alpha < 0.5$: the boundary $X = 0$ is attainable, and can be absorbing or reflecting.
- $\delta > 2$, or $\alpha > 1$: the boundary $X = 0$ is not attainable.

Since the boundary $X = 0$ may be accessible and absorbing, probability mass can be lost. Solutions, $p_\delta(X_T, T; X_0)$, of the Fokker-Planck equation can thus be norm-*decreasing*:

$$\int_0^\infty p_\delta(X, T; X_0) \, \mathrm{d}X < 1. \tag{6}$$

A norm-*preserving* solution, on the other hand, satisfies (6) but with an equality replacing the inequality.

2.1. The case $\delta \leq 0$

For $\delta \leq 0$, or $\alpha \in [0.5, 1)$, the Fokker-Planck equation has a unique norm-decreasing solution. Hitting the $X = 0$ boundary in (4) is equivalent to hitting the $F = 0$ boundary in the original CEV process (1), so that the process is also absorbed at the origin. The transition density[2] is

$$p_\delta(X_T, T; X_0) = \frac{1}{2T}\left(\frac{X_T}{X_0}\right)^{\nu/2} \exp\left[-\frac{X_T + X_0}{2T}\right] I_{-\nu}\left(\frac{\sqrt{X_T X_0}}{T}\right). \tag{7}$$

This expression is obtained after solving the Fokker-Planck equation via Laplace transforms, see [13] for detailed derivation. Direct integration of $p_\delta(X_T, T; X_0)$ gives:

$$\int_0^\infty p_\delta(X, T; X_0)\,\mathrm{d}X = \Gamma\left(-\nu; \frac{X_0}{2T}\right) < 1, \tag{8}$$

where $\Gamma(n; x)$ is the normalized incomplete gamma function

$$\Gamma(n; x) = \frac{1}{\Gamma(n)} \int_0^x t^{n-1} e^{-t}\,\mathrm{d}t. \tag{9}$$

Equation (8) gives the probability that the process has not become trapped at $X = 0$ by time T. As shown in Figure 1, in the limit as $T \to \infty$, the integral vanishes, indicating that *every* path will be trapped at $X = 0$ for $\delta \leq 0$.

The absorbing nature of the $X = 0$ ($F = 0$) boundary in this regime allows the CEV process to represent assets whose value may reach zero. The most natural example of this scenario would be the company, or indeed an interest rate, whose value reaches zero. On the other hand this parameter regime should not be used to model processes which should remain positive *i.e.* volatility processes.

The full norm-*preserving* transition density is thus given by the sum of the defective density (7) and a Dirac measure at zero, namely

$$p_\delta^{\text{full}}(X_T, T; X_0) = 2\left[1 - \Gamma\left(-\nu; \frac{X_0}{2T}\right)\right] \delta\left(X_T\right) + p_\delta(X_T, T; X_0), \tag{10}$$

The coupling of a defective density with a Dirac mass at the origin has been used to study the non-central chi-squared distribution with zero degrees of freedom [21] and the squared Bessel process (4) with $\delta = 0$ [12]. Indeed, taking $\delta = 0$ in expression (10) agrees with the result presented in [12].

Finally, the transition density is related to the non-central chi-squared distribution (5) since by inspection,

$$p_\delta(X, T; X_0) = p_{\chi'^2}\left(\frac{X_0}{T}; 4 - \delta, \frac{X}{T}\right) \frac{1}{T}, \tag{11}$$

where the right-hand side is well-defined since $4 - \delta > 0$, but note that it is a function of the non-centrality parameter. Schroder, however, proves[3] that [20]

$$\int_x^\infty p_{\chi'^2}\left(X_0; 4 - \delta, X\right)\,\mathrm{d}X = \chi'^2\left(X_0; 2 - \delta, x\right), \tag{12}$$

[2] This is a special case of the solution arrived at by Feller, although there is a minor typo in his work: the term $4b^2$ in equation (6.2) of [10] should be 1. After that one may take the limit as $b \to 0$ to arrive at (7). This error was also noticed by Lewis [17].

[3] There is also a claim in [20] that $\chi'^2(x; k, \lambda) + \chi'^2(\lambda; 2 - k, x) = 1$, but the proof of this relies on using the identity $I_{-n}(x) = I_n(x)$, which is true only for integer n, so in general the result will not hold.

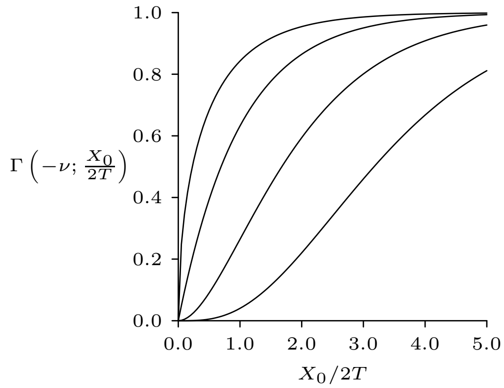

Figure 1: The right hand side of (8) is plotted for several values of $\delta < 2$. At the center line, we have from left to right curves for $\delta = 1, 0, -2, -5$, or $\alpha = 0, 1/2, 3/4, 6/7$, respectively. For fixed X_0, all paths will eventually be trapped as each curve tends to 0 for $T \to \infty$. For fixed T, the probability of a path having been trapped by time T increases (decreases) when X_0 decreases (increases), *i.e.* when $\alpha \to -\infty$ ($\alpha \to 1$).

which gives

$$\int_0^{X_T} p_\delta(X, T; X_0)\, \mathrm{d}X = \Gamma\left(-\nu; \frac{X_0}{2T}\right) - {\chi'}^2\left(\frac{X_0}{T}; 2-\delta, \frac{X_T}{T}\right). \tag{13}$$

Including the Dirac mass in the full transition density (10), X is thus distributed according to

$$\Pr\left(X \leq X_T \middle|\, X_0\right) = \int_0^{X_T} p_\delta^{\text{full}}(X, T; X_0)\, \mathrm{d}X = 1 - {\chi'}^2\left(\frac{X_0}{T}; 2-\delta, \frac{X_T}{T}\right). \tag{14}$$

2.2. The case $0 < \delta < 2$

For $0 < \delta < 2$, or $\alpha < 0.5$, the boundary $X = 0$ is accessible, just as when $\delta \leq 0$. When such a path hits $X = 0$, we may either impose an absorbing boundary and terminate the process, or impose a reflecting boundary and return to $X > 0$. Hitting the $X = 0$ boundary in (4) is again equivalent to hitting the $F = 0$ boundary in the original CEV process (1), so appropriate boundary conditions must also be applied at the origin for that process in the regime $\alpha < 0.5$.

This freedom to return the process to $X > 0$ after hitting the origin suggests that asset classes which must remain strictly positive are best described by the CEV process in this regime. Examples of such processes include interest rates and volatilities.

The unique fundamental solution assuming an *absorbing* boundary condition is given by (7). Accordingly, all paths will eventually be trapped at the origin as in the $\delta \leq 0$ case. As for that case, probability mass is present at the origin, and the full norm-preserving transition density is given by (10). This equation is thus valid for $\delta < 2$, with an absorbing boundary at $X = 0$.

Feller does not derive the corresponding solution given a *reflecting* boundary condition, but by imposing a zero flux at $X = 0$ it is straight forward to show[4] that the transition

[4]Take equation (3.9) of [10], set $f(t) = 0$, and invert the resulting reduced Laplace transform.

density in this case is

$$p_\delta(X_T, T; X_0) = \frac{1}{2T}\left(\frac{X_T}{X_0}\right)^{\nu/2} \exp\left[-\frac{(X_T + X_0)}{2T}\right] I_\nu\left(\frac{\sqrt{X_T X_0}}{T}\right), \tag{15}$$

and that it is norm-preserving. We thus have[5]

$$\Pr\left(X \le X_T \middle| X_0\right) = \int_0^{X_T} p_\delta(X, T; X_0)\,\mathrm{d}X = \chi'^2\left(\frac{X_T}{T};\, \delta,\, \frac{X_0}{T}\right). \tag{16}$$

2.3. The case $\delta > 2$

Finally when $\delta > 2$, or $\alpha > 1$, a unique norm-preserving solution exists, the process never hits $X = 0$, and boundary conditions cannot be imposed. The transition density for $\delta > 2$ is given by (15), but in this case $p_\delta \to 0$ as $X_T \to 0$, paths being pushed away from the origin. From the point of view of the original CEV process, however, this is a statement about the $F = \infty$ boundary and indicates that solutions of (1) remain finite for all time[6]. On the other hand, the Feller test for explosions (see, *e.g.*, [16, 18]) shows that X will never become unbounded in finite time if $\delta > 2$. The origin $F = 0$ in the CEV process is thus also not accessible (as in the limiting $\alpha = 1$ lognormal case).

2.4. Symmetry of the transition density

If we choose *absorbing* boundary conditions at $X = 0$ when appropriate, then the norm-decreasing part of the transition density is given by (7) for all $\delta < 2$. On the other hand, for $\delta > 2$, the density is given by (15). It is straightforward to verify the following symmetry between these two expressions.

If $\delta < 2$ ($\delta > 2$) then, for $\delta > 2$ ($\delta < 2$), we have

$$p_\delta(X_T, T; X_0) = p_{4-\delta}(X_0, T; X_T). \tag{17}$$

The boundary case at $\delta = 2$ corresponds to $\alpha = 1$, and so there is a similar symmetry relation over $\alpha \to 2 - \alpha$. This feature can be used to generate the density for $\alpha > 1$ ($\alpha < 1$) from that for $\alpha < 1$ ($\alpha > 1$), although one must be careful to (re-)include the Dirac mass in (10) when appropriate.

3. Local Martingale Property and Closed-form European option prices

In this section, we derive expressions for the expected value of the process F_T based on the transition densities developed in sections §2.1-§2.4. These quantities are then used to establish closed form expressions for European options prices which take into account the local martingale property of F_T when $\alpha > 1$.

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[5] We also note that $p_\delta(X_T, T; X_0) = \mathcal{O}(X_T^\nu)$ as $X_T \to 0$. Since $-1 < \nu < 0$, the transition density (15) is not finite at $X_T = 0$ although its expectation *is*, since $\mathbb{E}[X_T] = \mathcal{O}(X_T^{1+\nu})$ as $X_T \to 0$. In terms of the squared Bessel process (4), this indicates that paths have a propensity towards the vicinity of the origin.

[6] This is in contrast to the deterministic case $\mathrm{d}x = x^\alpha \mathrm{d}t$ which is guaranteed to blow-up in finite time whenever $x(0) > 0$ and $\alpha > 1$.

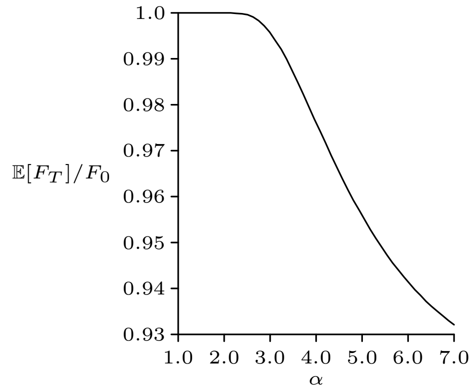

Figure 2: Plot of $\mathbb{E}[F_T]/F_0$ against $\alpha > 1$ for parameter values $F_0 = 100$, $T = 1$, $\sigma_{\mathrm{LN}} = 0.2$.

3.1. Expected value of F, and the local martingale property

For any δ, the expected value of F_T is

$$\mathbb{E}\left[F_T \middle| F_0\right] \equiv \mu_F = \sigma^{-2\nu}(1-\alpha)^{-2\nu} \int_0^\infty X^{-\nu}\, p_\delta(X, T; X_0)\, \mathrm{d}X,$$

where, for $\delta < 2$, the Dirac mass vanishes from the integral of p_δ^{full}. Substitution of (7) or (15), and using the symmetry (17), gives

$$X^{-\nu} p_\delta(X, T; X_0) = X_0^{-\nu} p_\delta(X_0, T; X) = X_0^{-\nu} p_{4-\delta}(X, T; X_0),$$

so that

$$\mu_F = F_0 \int_0^\infty p_{4-\delta}(X, T; X_0)\, \mathrm{d}X.$$

For $\delta < 2$ ($\alpha < 1$), this integral is norm-preserving, making F a martingale as expected. However, for $\delta > 2$ ($\alpha > 1$), the integral is norm-decreasing, so that F is a *strictly local martingale*[7]:

$$\mu_F = F_0\, \Gamma\left(-\nu; \frac{X_0}{2T}\right) < F_0. \tag{18}$$

As discussed by Lewis [17], the CEV process (1) with $\alpha > 1$ is such that the forward price is only a local martingale. The discrepancy from a true martingale is shown in Figure 2.

3.2. Closed-form European option prices.

Here we establish the prices of plain vanilla European options, based on the results developed in sections §2.1-§2.4. The forward, or at-expiry, price of a plain vanilla European call option is

$$C = \mathbb{E}\left[\max\left(F_T - K, 0\right) \middle| F_0\right] = \int_K^\infty (F - K)\; p_\delta(F, T; F_0)\, \mathrm{d}F,$$

where K is the strike price of the option, with maturity T. The forward price of the corresponding put option can be found through put-call parity:

$$P = C - \mathbb{E}\left[F_T - K \middle| F_0\right] = C - \mathbb{E}\left[F_T \middle| F_0\right] - K, \tag{19}$$

[7]A rigorous definition of a strictly local martingale is given in, *e.g.*, [16]. For our purposes, however, if will suffice to take a local martingale to be a process $\mathrm{d}F = \sigma(F, t)\, \mathrm{d}W$ with $\mathbb{E}[F_T | F_0] < F_0$.

where, if F_T is only a local martingale, the right-hand side is *not* equal to $C - F_0 - K$.

Once we transform to the X coordinate, we need to differentiate between the $\delta < 2$ and $\delta > 2$ regimes. In the former, one can derive expressions which agree with [20]:

$$C_{\delta<2} = F_0 \left[1 - \chi'^2\left(\frac{\tilde{K}}{T}; 4-\delta, \frac{X_0}{T}\right)\right] - K\chi'^2\left(\frac{X_0}{T}; 2-\delta, \frac{\tilde{K}}{T}\right), \tag{20}$$

However, for $\delta > 2$, or $\alpha > 1$, we have

$$C_{\delta>2} = \int_0^{\tilde{K}} \left[\frac{X^{-\nu}}{(\sigma(1-\alpha))^{2\nu}} - K\right] p_\delta(X, T; X_0)\,\mathrm{d}X,$$

where the relevant (norm-preserving) transition density is given by (15). The second integral is a cumulative non-central chi-squared distribution, but the first integral becomes

$$F_0 \int_0^{\tilde{K}} p_{4-\delta}(X, T; X_0)\,\mathrm{d}X,$$

upon use of the symmetry (17). Since $\delta > 2$, the transition density $p_{4-\delta}$ is the norm-decreasing density given in (7) and, from (13), we have

$$C_{\delta>2} = F_0 \left[\Gamma\left(\nu; \frac{X_0}{2T}\right) - \chi'^2\left(\frac{X_0}{T}; \delta-2, \frac{X_T}{T}\right)\right] - K\chi'^2\left(\frac{\tilde{K}}{T}; \delta, \frac{X_0}{T}\right). \tag{21}$$

The expression (21) for $C_{\delta>2}$ agrees with Lewis [17], and as he has pointed out, it is *not* the same as that widely reported in the literature. On the other hand, the put price does agree with that in the literature, but only by a cancellation of errors: the replacement of $F_0\Gamma(\nu; X_0/(2T))$ with F_0 in (21) is canceled out by the mistaken assumption that F_T is a martingale.

4. Monte Carlo simulation

In this section, we perform an exact simulation of the squared Bessel and CEV processes, and compute both the forward prices $\mathbb{E}[X_T]$ and $\mathbb{E}[F_T]$, as well as plain vanilla European prices. We find good agreement with the analytic results obtained above. We actually perform *quasi*-Monte Carlo simulations, using the Sobol sequence of numbers (see, *e.g.*, [11] for an overview of these techniques). We always sample X_T, and obtain F_T through inversion of (3). We assume an absorbing boundary where appropriate, and always take $N = 2^{20} - 1$ in our simulations[8].

As X_t and X_T are arbitrary, this method can be also be used to generate detailed paths required for the pricing of more exotic derivatives. By calculating paths using this method, one automatically takes care of the local martingale property of the CEV process with $\alpha > 1$. Additionally, discretization is not required with the effect that good accuracy is guaranteed and extraneous work is minimized by only sampling at points required for derivative pricing. Detracting from this method is the relatively large amount of computational effort required for each sample obtained.

[8]For quasi-Monte Carlo simulations with the Sobol sequence of numbers, one should use $2^n - 1$ paths, with n an integer, so that the mean of the set of numbers used is precisely equal to 1/2.

4.1. The case $\delta < 2$

For $\delta < 2$, or $\alpha < 1$, the full transition density (10) consists of the norm-decreasing density (15) and a Dirac mass at the origin.

To simulate X_T, one might think to sample directly from the distribution given in (14). Drawing numbers $U = \Pr\left(X \leq X_T \middle| X_0\right) \in (0,1)$ from a uniform distribution, one would have

$$\chi'^2\left(\frac{X_0}{T}; 2-\delta, \frac{X_T}{T}\right) = U, \tag{22}$$

so X_T/T would be the formal inverse of the non-central chi-squared distribution as a function of the non-centrality parameter: with $\mathcal{F}(x) \equiv \chi'^2(X_0/T; 2-\delta, x)$, we would have

$$\frac{X_T}{T} = \mathcal{F}^{-1}(U).$$

However, accounting for the absorption at $X = 0$ that this distribution captures is difficult numerically; there are many values of U for which $X_T = 0$ is the correct solution.

It is more straightforward to account for the absorption "by hand" as it were. We draw numbers $U \in (0,1)$ from a uniform distribution. If

$$U > U_{\max} \equiv \Gamma\left(-\nu; \frac{X_0}{2T}\right),$$

we simply set $X_T = 0$. On the other hand, if $U \leq U_{\max}$, then we sample from the norm-decreasing density (7) by inverting the integral (13) to obtain

$$\frac{X_T}{T} = \mathcal{F}^{-1}(U_{\max} - U).$$

To compute this numerically, we perform a root search over values of X_T. For each such value, we construct a *new* cumulative non-central chi-squared distribution, which is nevertheless always evaluated at the point X_0/T.

The cumulative non-central chi-squared distribution is easily computed numerically along the lines of [6]. This method uses the fact that

$$\chi'^2(x; k, \lambda) = \sum_{i=1}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^i}{i!} \chi^2(x; k+2i).$$

To avoid numerical difficulties, the sum should be performed starting from the $i = \lfloor \lambda/2 \rfloor$ term for which the coefficient of $\chi^2(x; k+2i)$ is a maximum, and should then proceed for increasing and decreasing i until convergence is reached in both directions.

The simulated values of European call and put prices are compared with (20). We again find good agreement, as shown for selected values of α in Table 1, in which we have again taken $F_0 = 100$, $\sigma_{\mathrm{LN}} = 50\%$, $T = 4$ and have calculated option prices for $K = 100$. Similarly good agreement is observed for other values of K.

4.2. The case $\delta > 2$

For $\delta > 2$, or $\alpha > 1$, the variable X_T can be simulated by sampling from a non-central chi-squared distribution directly, as in (16). This can be done in various ways (see, *e.g.* [11]). The

	Call $K = 100$		Put $K = 100$	
α	Simulated	Analytic	Simulated	Analytic
-2	34.42926 ± 0.03259	34.42928	34.42925 ± 0.04602	34.42928
-1	37.38754 ± 0.04146	37.38750	37.38744 ± 0.04630	37.38750
0	39.04504 ± 0.05709	39.04516	39.04513 ± 0.04396	39.04516
0.1	39.00895 ± 0.05943	39.00891	39.00885 ± 0.04327	39.00891
0.2	38.93068 ± 0.06202	38.93077	38.93073 ± 0.04242	38.93077
0.3	38.82058 ± 0.06492	38.82088	38.82086 ± 0.04139	38.82088
0.4	38.69621 ± 0.06822	38.69635	38.69630 ± 0.04019	38.69635
0.5	38.57511 ± 0.07203	38.57528	38.57523 ± 0.03886	38.57528
0.6	38.47236 ± 0.07653	38.47251	38.47246 ± 0.03746	38.47251
0.7	38.39167 ± 0.08199	38.39215	38.39212 ± 0.03608	38.39215
0.8	38.33770 ± 0.08895	38.33554	38.33544 ± 0.03476	38.33554
0.9	38.30141 ± 0.09812	38.30366	38.30365 ± 0.03350	38.30366

Table 1: Simulated and analytic values of call and put prices for $\alpha < 1$, $F_0 = 100$, $\sigma_{\text{LN}} = 50\%$, $T = 4$ and $K = 100$. Similarly good agreement is observed for a variety of strikes K.

simplest method, and the one we choose here, is to draw numbers $U = \Pr\left(X \leq X_T \middle| X_0\right) \in (0,1)$ from a uniform distribution, and invert (16) directly to give

$$\frac{X_T}{T} = \chi'^{-1}\left(U; \delta, \frac{X_0}{T}\right),$$

where $\chi'^{-1}(x; k, \lambda)$ denotes the inverse cumulative non-central chi-squared distribution, with k degrees of freedom and non-centrality parameter λ.

The inversion of the non-central chi-squared distribution is again performed using a root search over the cumulative non-central chi-squared distribution. Although computationally expensive, this method will suffice for our purposes. An alternative would be to use the quadratic-exponential method described by Andersen [1].

We compare the simulated values of $\mathbb{E}[X_T]$ and $\mathbb{E}[F_T]$ with the standard result for the non-central chi-squared distribution and (18) respectively, and find good agreement. In particular, the latter comparison confirms that F_T is a strictly local martingale when $\alpha > 1$: a plot of the simulated value of $\mathbb{E}[F_T]$ would show no difference to the analytic result in Figure 2.

The simulated values of European call prices are compared with (21) and good agreement, as shown for selected values of α in Table 2, in which we have again taken $F_0 = 100$, $\sigma_{\text{LN}} = 20\%$, $T = 1$ and have calculated option prices for $K = 100$. We also include the standard call option prices, based on the Cox pricing formula, which clearly deviate from the corrected value, particularly for large values of α.

5. Conclusion

We have reviewed properties of the CEV process (1), discussing the Feller classification of boundary conditions and associated probability transition functions, according to the value of the CEV exponent α. Since the transition density of the squared Bessel process is norm-decreasing in the $\delta \leq 0$ regime (and also, given an absorbing boundary condition, in the $0 < \delta < 2$ regime), we have argued that it should be amended with a Dirac mass at the origin, with strength such that the resulting full transition density is norm-preserving. The

	Call $K = 100$			Put $K = 100$	
α	Simulated	Analytic	Standard	Simulated	Analytic
1.5	7.96872 ± 0.01385	7.96887	7.96885	7.96883 ± 0.00967	7.96885
2	7.96872 ± 0.01528	7.97879	7.97885	7.97883 ± 0.00924	7.97885
2.5	7.95335 ± 0.01742	7.95453	7.99598	7.99597 ± 0.00885	7.99598
3	7.95335 ± 0.01773	7.58979	8.02115	8.02113 ± 0.00849	8.02115
3.5	6.75691 ± 0.01597	6.75739	8.05605	8.05604 ± 0.00815	8.05605
4	5.71528 ± 0.01366	5.71516	8.10331	8.10330 ± 0.00783	8.10331
4.5	4.70077 ± 0.01151	4.70092	8.16394	8.16393 ± 0.00753	8.16394
5	3.82032 ± 0.00970	3.82051	8.23453	8.23453 ± 0.00723	8.23453
5.5	3.09733 ± 0.00822	3.09681	8.30843	8.30842 ± 0.00695	8.30843
6	2.51872 ± 0.00702	2.51885	8.37860	8.37860 ± 0.00667	8.37860
6.5	2.06033 ± 0.00605	2.06009	8.43965	8.43964 ± 0.00641	8.43965
7	1.69776 ± 0.00525	1.69846	8.48825	8.48824 ± 0.00616	8.48825

Table 2: Values of call and put prices for $\alpha > 1$, $F_0 = 100$, $\sigma_{\text{LN}} = 20\%$, $T = 1$ and $K = 100$. The column of standard results is obtained from the Cox pricing formula [4, 5]. Similarly good agreement is observed for a variety of strikes K.

cumulative distribution in this case is related to the non-central chi-squared distribution, but as a function of non-centrality parameter.

The forward price F_T is a strictly local martingale when $\alpha > 1$, and this gives rise to a correction to the standard European call price for $\alpha > 1$. This is included naturally in a direct calculation of the expectation value of the option payoff.

We have developed techniques for Monte Carlo simulation of the CEV process, in particular a new scheme has been given to simulate the forward price when $\alpha < 1$, which accounts for the probability of absorption at the origin. We used these techniques to simulate the expectation values, both of X and F, and of European option prices, and compared them to the analytic results. The agreement is extremely good.

For $\alpha > 1$, the simulated results show that the forward price in the CEV model is indeed a local martingale, and agrees with the corrected call price. The $\alpha > 1$ regime is not often discussed in the literature and at any rate, fairly large values of α must be considered to see these results.

The methods developed herein can be used to generate detailed paths required for the pricing of more exotic derivatives . By calculating paths using this method, one automatically takes care of the local martingale property of the CEV process with $\alpha > 1$. Additionally, discretization is not required with the effect that good accuracy is guaranteed and extraneous work is minimized by only sampling at points required for derivative pricing.

Detracting from this method is the relatively large amount of computational effort required for each sample obtained. One would expect discretization methods to become more competitive as the number of desired time points per path increases. There are several possible avenues for speeding up this *exact* method based on reduction of the underlying density. The quadratic exponential method, or QE [1] proposes a simplified form of the non-central chi squared density and has been successfully used in simulations of the Heston model.

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