Lie groups and automatic continuity

Andrew Putman

Abstract

We prove a theorem of Pettis that says that all measurable homomorphisms between Lie groups are continuous.

Throughout this note, all Lie groups are equipped with their left-invariant Haar measure. Our goal is to prove the following.

Theorem 0.1 (Pettis, [P]). *Every measurable homomorphism between Lie groups is continuous.*

Remark 0.2. Pettis's paper [P] proves this theorem for a much wider class of topological groups. We restrict our attention to Lie groups to avoid spelling out all the technical assumptions that must be made (but for those that care, the proof below only uses that the groups are locally compact and second countable). □

Remark 0.3. The requirement that the homomorphisms be measurable is needed. Indeed, noncontinuous field automorphisms $\mathbb{C} \to \mathbb{C}$ can be constructed by permuting a transcendence basis for $\mathbb{C}$ over the algebraic closure of $\mathbb{Q}$. These exotic field automorphisms give noncontinuous automorphisms of both the additive and multiplicative groups of $\mathbb{C}$. □

Proof of Theorem 0.1. Let $f\colon G \to H$ be a measurable homomorphism between Lie groups. Let $U \subset H$ be an open neighborhood of 1. It is enough to prove that $f^{-1}(U)$ contains an open neighborhood of the identity.[1] Pick an open neighborhood $V \subset H$ of 1 such that $VV^{-1} \subset U$. The set $f^{-1}(V)$ cannot have zero measure.[2] By Lemma 0.4 below, the set $f^{-1}(V) \cdot (f^{-1}(V))^{-1}$ contains an open neighborhood of the identity. Since

$$f^{-1}(V) \cdot \left(f^{-1}(V)\right)^{-1} = f^{-1}(V) \cdot f^{-1}\left(V^{-1}\right) \subset f^{-1}\left(V \cdot V^{-1}\right) \subset f^{-1}(U),$$

the theorem follows. □

Lemma 0.4 (Steinhaus). *Let G be a Lie group and let $X \subset G$ be a set that does not have measure zero. Then $X \cdot X^{-1}$ contains an open neighborhood of 1.*

Proof. Let μ be the Haar measure on G. Shrinking X, we can assume that $0 < \mu(X) < \infty$. We can then approximate X by compact and open sets in the sense that we can find compact sets $K \subset X$ (resp. open sets $U \supset X$) such that $\mu(K)$ (resp. $\mu(U)$) is as close to $\mu(X)$ as we like. In particular, we can find a compact set K and an open set U such that $K \subset X \subset U$

[1] Using the group action, this implies that for all $x \in G$, if U is an open neighborhood of $f(x)$, then $f^{-1}(U)$ contains an open neighborhood of x. Using this, we deduce that for all open sets V of H the preimage $f^{-1}(V)$ is open, so f is continuous.

[2] Assume it does. Since H is second countable, there exists a sequence $\{x_i\}_{i=1}^{\infty}$ of elements of G such that $\{f(x_i) \cdot V\}_{i=1}^{\infty}$ is an open cover of $f(G)$. We have $f^{-1}(f(x_i) \cdot V) = x_i \cdot f^{-1}(V)$, so we deduce that $G = \cup_{i=1}^{\infty} x_i \cdot f^{-1}(V)$. Each of the sets $x_i \cdot f^{-1}(V)$ has measure zero, so we conclude that G has measure zero, a contradiction.

and such that $2\mu(K) > \mu(U)$. Below we will prove that there exists an open neighborhood V of 1 such that $V \cdot K \subset U$. Assuming this for the moment, we claim that $V \subset XX^{-1}$. Indeed, consider $v \in V$. We have $K \cup v \cdot K \subset U$, so K and $v \cdot K$ cannot be disjoint since then $\mu(U) \geq \mu(K) + \mu(v \cdot K) = 2\mu(K)$, contradicting our choice of K and U. We can thus find $k_1, k_2 \in K$ such that $k_1 = vk_2$, so $v = k_1 k_2^{-1} \in K \cdot K^{-1} \subset X \cdot X^{-1}$, as desired.

It remains to find an open neighborhood V of 1 such that $V \cdot K \subset U$. For all $k \in K$, the set $U \cdot k^{-1}$ is an open neighborhood of 1, so we can find an open neighborhood V_k of 1 such that $V_k \cdot V_k \subset U \cdot k^{-1}$. The sets $\{V_k \cdot k\}_{k \in K}$ cover the compact set K, so we can find $k_1, \ldots, k_n \in K$ such that $\{V_{k_i} \cdot k_i\}_{i=1}^n$ covers K. Define $V = V_{k_1} \cap \cdots \cap V_{k_n}$. The set V is an open neighborhood of 1, and

$$\begin{aligned}
V \cdot K &\subset V \cdot (V_{k_1} \cdot k_1 \cup \cdots \cup V_{k_n} \cdot k_n) \\
&\subset (V_{k_1} \cdot V_{k_1}) \cdot k_1 \cup \cdots \cup (V_{k_n} \cdot V_{k_n}) \cdot k_n \\
&\subset \left(U \cdot k_1^{-1}\right) \cdot k_1 \cup \cdots \cup \left(U \cdot k_n^{-1}\right) \cdot k_n \\
&= U,
\end{aligned}$$

as desired. □

References

[P] B.J. Pettis, On continuity and openness of homomorphisms in topological groups, Ann. of Math. (2) 52, (1950). 293-308.

Andrew Putman
Department of Mathematics
University of Notre Dame
255 Hurley Hall
Notre Dame, IN 46556
andyp@nd.edu