The Borel density theorem

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Abstract

We discuss the Borel density theorem and prove it for $\mathrm{SL}_n(\mathbb{Z})$.

This short note is devoted to the Borel density theorem.

Lattices. If G is a Lie group, then a subgroup $\Gamma < G$ is a *lattice* if Γ is discrete and the quotient G/Γ supports a G-invariant Riemannian metric of finite volume. Here we are regarding G/Γ as the space of left cosets $g\Gamma$, so G acts on the left.

Example. Let $G = \mathbb{R}^n$ and $\Gamma = \mathbb{Z}^n$. Then Γ is clearly discrete and G/Γ is an n-dimensional torus. The G-invariant Riemannian metric on G/Γ is the usual flat one.

Example. Let $G = \mathrm{SL}_n(\mathbb{R})$ and $\Gamma = \mathrm{SL}_n(\mathbb{Z})$. While Γ is clearly discrete in G, the quotient G/Γ is noncompact, so the fact that it supports a finite-volume G-invariant Riemannian metric is nontrivial.

Borel density theorem. The following is one of the fundamental properties of lattices.

Theorem 1 (Borel density theorem)**.** *Let G be a connected semisimple $\mathbb{R}$-algebraic group without compact factors and let $\Gamma < G$ be a lattice. Then Γ is Zariski dense in G.*

The statement that Γ is Zariski dense in G can be rephrased as saying that any polynomial function on G that vanishes on Γ must vanish identically (the notion of a polynomial function makes sense in this context because G is endowed with the structure of a $\mathbb{R}$-algebraic group). For example, Theorem 1 applies to $\mathrm{SL}_n(\mathbb{Z}) \subset \mathrm{SL}_n(\mathbb{R})$. Letting $x_{ij} : \mathrm{SL}_n(\mathbb{R}) \to \mathbb{R}$ be the function that returns the matrix entry at position (i, j), it asserts that any polynomial in the x_{ij} with real coefficients that vanishes on $\mathrm{SL}_n(\mathbb{Z}) \subset \mathrm{SL}_n(\mathbb{R})$ must vanish on all of $\mathrm{SL}_n(\mathbb{R})$ (warning: it need not literally be the zero polynomial; for instance, the function $\det(M) - 1$ is a polynomial in the x_{ij} which vanishes on $\mathrm{SL}_n(\mathbb{R})$). We will prove this special case of Theorem 1 below.

History and references. Theorem 1 was originally proved in [1]. See [4, Chapter V] for a textbook treatment of Borel's original proof. See also [6, §3.2] and [5, §4F] for textbook treatments of alternate proofs.

Application to representation theory. Lie groups act on many things via polynomials, and Theorem 1 says that the restrictions of these actions to lattices faithfully reflect the properties of the original actions. This is quite useful in numerous contexts. One sample application is as follows.

Definition 2. Let G be an $\mathbb{R}$-algebraic group. A *polynomial representation* of G is a homomorphism $\phi : G \to \mathrm{GL}_m(\mathbb{R})$ whose matrix entries are polynomial functions on G. □

Remark. For semisimple $\mathbb{R}$-algebraic groups G, almost all representations you are likely to meet in nature are polynomial. For example, it is proved in [2, p. 236] that measurable representations are automatically continuous, and most (but not all) continuous representations are actually polynomial (see [3]). In particular, for $\mathrm{SL}_n(\mathbb{R})$ all continuous representations are polynomial.

Theorem 3. *Let G be a connected semisimple $\mathbb{R}$-algebraic group and let $\Gamma < G$ be a lattice. Let $\phi : G \to \mathrm{GL}_m(\mathbb{R})$ be an irreducible polynomial representation of G. Then the restriction of ϕ to Γ is irreducible.*

Proof. Assume that the restriction of ϕ to Γ is not irreducible, and let $V \subset \mathbb{R}^m$ be a proper nonzero Γ-subrepresentation. Changing bases, we can assume that $V = \langle \vec{e}_1, \ldots, \vec{e}_k \rangle$, where $\{\vec{e}_1, \ldots, \vec{e}_m\}$ is the standard basis for $\mathbb{R}^m$ and $1 \leq k < m$. In other words, we can assume that in block form we have

$$\phi(g) = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

for all $g \in \Gamma$. Here A is a $k \times k$ matrix, B is a $k \times (m-k)$ matrix, and C is an $(m-k) \times (m-k)$ matrix. For $k < i \leq m$ and $1 \leq j \leq k$, the (i,j) matrix entry of $\phi(g)$ is thus 0 for all $g \in \Gamma$, so Theorem 1 implies that it is 0 for all $g \in G$. We conclude that V is G-invariant, a contradiction. □

Remark. Though we deduced Theorem 3 from the Borel density theorem, Theorem 3 actually forms one of the key steps in the the original proof of the Borel density theorem in [1]. The other proofs referenced above do not make use of Theorem 3.

Proofs for special linear group. To give some intuition for Theorem 1, we will prove it for $\mathrm{SL}_n(\mathbb{Z})$. As a warm-up, we begin with the following two lemmas. Strictly speaking these, these lemmas are not special cases of the Borel density theorem since the groups in question are not semisimple, but the conclusion of the theorem holds for them.

Lemma 4. *The group $\mathbb{Z}$ is Zariski dense in $\mathbb{R}$.*

Proof. Any nonzero polynomial $f(x)$ has only finitely many roots, so any polynomial $f(x)$ such that $f(k) = 0$ for all $k \in \mathbb{Z}$ must vanish identically. □

Lemma 5. *The group $\mathbb{Z}^n$ is Zariski dense in $\mathbb{R}^n$.*

Proof. Let $f(x_1, \ldots, x_n)$ be any polynomial such that $f(\vec{v}) = 0$ for all $\vec{v} \in \mathbb{Z}^n$. Our goal is to show that $f = 0$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ (with the usual topology), it is enough to show that $f(q_1, \ldots, q_n) = 0$ for all nonzero $q_1, \ldots, q_n \in \mathbb{Q}$. Fixing some nonzero $q_1, \ldots, q_n \in \mathbb{Q}$, define $\phi : \mathbb{R} \to \mathbb{R}^n$ to be the function

$$\phi(t) = (tq_1, \ldots, tq_n).$$

Then the function $f \circ \phi : \mathbb{R} \to \mathbb{R}$ is a polynomial vanishing on all points of the form $t = nd$, where $n \in \mathbb{Z}$ and d is the least common multiple of the denominators of the q_i. There are infinitely many numbers of this form, so we deduce that $f \circ \phi = 0$. Plugging in $t = 1$, we obtain that $f(q_1, \ldots, q_n) = 0$, as desired. □

Proof of Theorem 1 for special linear group. We wish to prove that $\mathrm{SL}_n(\mathbb{Z})$ is Zariski dense in $\mathrm{SL}_n(\mathbb{R})$. Consider a polynomial f in the matrix entries x_{ij} of $\mathrm{SL}_n(\mathbb{R})$ such that $f(M) = 0$ for all $M \in \mathrm{SL}_n(\mathbb{Z})$. Our goal is to prove that $f(M) = 0$ for all $M \in \mathrm{SL}_n(\mathbb{R})$. Fix such an $M \in \mathrm{SL}_n(\mathbb{R})$. For $r \in \mathbb{R}$ and distinct $1 \leq i, j \leq n$, let $e_{ij}(r)$ be the elementary matrix obtained from the $n \times n$ identity matrix by inserting r at position (i, j). For any field $\mathbb{F}$, the group $\mathrm{SL}_n(\mathbb{F})$ is generated by elementary matrices. This hold in particular for $\mathbb{F} = \mathbb{R}$, so there exists an expression

$$M = e_{i_1 j_1}(r_1) \cdot e_{i_2 j_2}(r_2) \cdots e_{i_p j_p}(r_p).$$

Here

$$r_1, \ldots, r_p \in \mathbb{R} \quad \text{and} \quad 1 \leq i_1, j_1, \ldots, i_p, j_p \leq n.$$

Also, $i_\ell \neq j_\ell$ for all $1 \leq \ell \leq p$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and f is a continuous function, it is enough to prove that

$$f\left(e_{i_1 j_1}(q_1) \cdot e_{i_2 j_2}(q_2) \cdots e_{i_p j_p}(q_p)\right) = 0 \tag{0.1}$$

for all nonzero $q_1, \ldots, q_p \in \mathbb{Q}$. Fixing some nonzero $q_1, \ldots, q_p \in \mathbb{Q}$, define a function $\phi : \mathbb{R} \to \mathrm{SL}_n(\mathbb{R})$ via the formula

$$\phi(t) = e_{i_1 j_1}(tq_1) \cdot e_{i_2 j_2}(tq_2) \cdots e_{i_p j_p}(tq_p).$$

The function $f \circ \phi : \mathbb{R} \to \mathbb{R}$ is a polynomial which vanishes for all t of the form $t = nd$, where $n \in \mathbb{Z}$ and d is the least common multiple of the denominators of the q_i. There are infinitely many numbers of this form, so we deduce that $f \circ \phi = 0$. Plugging in $t = 1$, we obtain (0.1). □

References

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