Abstract

We sketch the proof of the Brown representability theorem and give a few applications of it, the most important being the construction of the classifying space for principal $G$-bundles.

Let $G$ be a topological group and let $\text{CW}$ be the homotopy category of based connected CW-complexes. A classifying space for $G$ is a space $BG$ such that for all $X \in \text{CW}$, there exists a bijection between the set of based principal $G$-bundles on $X$ and the set $[X, BG]$ of all homotopy classes of basepoint-preserving maps from $X$ to $BG$. This determines the homotopy groups of $BG$ in the following way.

**Theorem 0.1.** For all $n \geq 1$, we have $\pi_n(BG) \cong \pi_{n-1}(G)$.

**Remark.** For instance, if $G$ is a discrete group, we deduce that $BG$ is a $K(G, 1)$.

**Proof of Theorem 0.1.** A based map $S^n \to BG$ is the same as a based principal $G$-bundle on $S^n$. Via the clutching construction, this is the same as a based map $S^{n-1} \to G$.

Theorem 0.1 might suggest to the reader that the following is true.

**Theorem 0.2.** The group $G$ is homotopy equivalent to $\Omega BG$.

**Proof.** Let us give two equivalent descriptions of the map $f : G \to \Omega BG$.

- A map $f : G \to \Omega BG$ is the same as a map $\Sigma G \to BG$, which is the same as a based principal $G$-bundle on $\Sigma G$. There is an obvious choice of such a bundle, again arising from the clutching construction.

- More directly, we define $f : G \to \Omega BG$ as follows. Consider $g \in G$. Define $f(g) \in \Omega BG$ to be the based loop $f(g) : S^1 \to BG$ that classifies the $G$-bundle on $S^1$ obtained by gluing the ends of the trivial $G$-bundle on $[0, 1]$ together using $g$.

To prove that $f$ is a homotopy equivalence, we show that it induces an isomorphism on all homotopy groups. In other words, for all $n \geq 1$ the map $[S^n, G] \to [S^n, \Omega BG]$ induced by $f$ is an isomorphism. Now, using the clutching construction an element of $[S^n, G]$ is the same as a principle $G$-bundle on $S^{n+1}$, i.e. an element of $[S^{n+1}, BG] = [\Sigma S^n, BG] = [S^n, \Omega BG]$. It is easy to see that this bijection is induced by $f$.

Of course, it is not obvious that a classifying space $BG$ for $G$ exists! There are several explicit constructions of $BG$, the first being due to Milnor [M]. Our next task is to give an abstract nonsense reason why $BG$ must exist. The key is the following theorem, which was first proved by Brown [Bro].

**Theorem 0.3** (Brown representability). Let $F : \text{CW} \to \text{Set}$ be a contravariant functor satisfying the following two properties.

1. Given any collection $\{X_\alpha\}$ of elements of $\text{CW}$, we have $F(\vee_\alpha X_\alpha) = \prod_\alpha F(X_\alpha)$. 


2. Let $X$ be an object of $CW$. Consider a cover $X = Y \cup Z$ by subcomplexes such that $Y, Z, Y \cap Z \in CW$. Then for all $y \in F(Y)$ and $z \in F(Z)$ that restrict to the same element of $F(Y \cap Z)$, there exists some $x \in F(X)$ that restricts to $y \in F(Y)$ and $z \in F(Z)$.

Then there exists some $C \in CW$ and some $c \in F(c)$ such that for all $X \in CW$, the map $[X, C] \to F(X)$ taking $f: X \to C$ to $f^*(c)$ is a bijection.

**Remark.** It is absolutely necessary for us to consider based connected CW-complexes. The theorem is false without these assumptions; see [Bra].

Before we prove Theorem 0.3, let us give several examples of how it can be used.

**Example.** If $G$ is a topological group, then we can apply Theorem 0.3 to the functor taking $X$ to the set of based principal $G$-bundles; the result is the classifying space $BG$ for $G$.

**Example.** For all $n \geq 1$, we can apply Theorem 0.3 to the cohomology functor $H^n(\cdot, A)$; the result is a $K(A, n)$ (as can be seen by plugging spheres into the statement).

**Example.** If $T$ is a based connected topological space, then we can apply Theorem 0.3 to the functor $[\cdot, T]$. The result is a CW-approximation for $T$, i.e. a based connected CW-complex $C$ such that $[X, C] = [X, T]$ for all $X \in CW$. We remark that the image in $[C, T]$ of the identity in $[C, C]$ is the usual map that arises in a CW-approximation theorem.

We now sketch the proof of Theorem 0.3.

Proof sketch of Theorem 0.3. We begin by observing that it is enough to construct $C \in CW$ and $c \in F(C)$ that satisfy the conclusion of the theorem for all spheres $S^n$ with $n \geq 1$. Indeed, if $X \in CW$ is arbitrary, then for $x \in F(X)$ we can construct $f: X \to C$ satisfying $f^*(c) = x$ by the usual “cell by cell” procedure, and similarly if $f, f': X \to C$ satisfy $f^*(c) = (f')^*(c) = x$, then we can construct a homotopy from $f$ to $f'$ cell by cell.

We will construct $C$ as follows. Start with $C_0 = \{*\}$ and $c_0$ the unique element of $F(C_0)$. Assume that $C_{n-1}$ and $c_{n-1} \in F(C_{n-1})$ has been constructed such that for all $1 \leq k \leq n-1$, we have $[S^k, C_{n-1}] = F(S^k)$ via pullback of $c_{n-1}$. We will construct a CW-complex $C_n$ containing $C_{n-1}$ as a subcomplex together with $c_n \in F(C_n)$ that restricts to $c_{n-1} \in F(C_{n-1})$. The complex $C_n$ will be obtained from $C_{n-1}$ by attaching $n$-cells and $(n+1)$-cells, and from this it is easy to see that we still have $[S^k, C_n] = F(S^k)$ via pullback of $c_n$ for all $1 \leq k \leq n-1$. We just have to find the right cells to attach to make this true for $S^n$ as well. There are two parts to this (generators and relations):

- First, we wedge on an $n$-sphere for each element of $F(S^n)$ to get a complex $C'_n$. Using the first condition in the theorem, we can extend $c_{n-1}$ to $c'_n \in F(C'_n)$ such that the map $f_x: S^n \to C'_n$ taking $S^n$ to the sphere representing $x \in F(S^n)$ satisfies $(f_x)^*(c'_n) = x$. This implies that the map $[S^n, C'_n] \to F(S^n)$ is surjective.

- We now want to make it injective. Observe that the cogroup structure on $S^n$ (the same one that makes homotopy groups into groups) makes $F(S^n)$ into a group. The map $[S^n, C'_n] \to F(S^n)$ is then a group homomorphism. To construct $C_n$, we attach cells to $C'_n$ to kill off the kernel. Extending $c'_n$ over $C_n$ requires the second condition in the theorem.
Repeating this procedure, we get an increasing sequence

\[ C_0 \subset C_1 \subset C_2 \subset \cdots \]

of based connected CW-complexes. Define

\[ C = \bigcup_{n=0}^{\infty} C_n. \]

We now come to the final subtle point of the proof, namely constructing an element \( c \in F(C) \) that restricts to \( c_n \in F(C_n) \) for all \( n \). The issue here is that we have not assumed any kind of “continuity” for our functor \( F \). Indeed, there is a map

\[ F(C) \to \lim_{\leftarrow} F(C_n), \]

but this map need not be bijective. However, it is surjective, which is good enough for us. To prove that it is surjective, replace \( C \) by the telescoping collection of mapping cylinders \( M(C_n \to C_{n+1}) \) (with the basepoints all collapsed to points so that everything is based). This does not change the homotopy type of \( C \); however, we can now decompose \( C \) as \( X \cup Y \), where \( X \) is the union of the even mapping cylinders \( M(C_{2n} \to C_{2n+1}) \) and \( Y \) is the union of the odd mapping cylinders \( M(C_{2n+1} \to C_{2n+2}) \). Since we have collapsed basepoints, \( X \) is actually the wedge of the spaces \( M(C_{2n} \to C_{2n+1}) \), and similarly for \( Y \). The space \( M(C_n \to C_{n+1}) \) is homotopy equivalent to \( C_{n+1} \), so we can view \( c_{n+1} \) as an element of \( F(M(C_n \to C_{n+1})) \). Using the first condition in the theorem, we can then construct elements \( c_x \in F(X) \) and \( c_y \in F(Y) \) restricting to the various \( c_n \). It is clear that \( c_x \) and \( c_y \) restrict to the same element of \( F(X \cap Y) \) (here \( X \cap Y \) is another wedge!), so the second condition in the theorem allows us to glue \( c_x \) and \( c_y \) together to an element \( c \in F(C) \).

References


Andrew Putman
Department of Mathematics
University of Notre Dame
255 Hurley Hall
Notre Dame, IN 46556
andyp@nd.edu