A categorical construction of free groups

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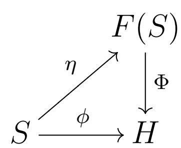
Abstract

This note contains a proof (from Lang's book on algebra, though it does not seem to be widely known) that free groups exist. In contrast to the usual proof, words in the generators do not appear. Even worse, uncountable groups appear even when constructing the free group on two generators. This proof is essentially a specialization to the situation at hand of the usual proof of the adjoint functor theorem.

Let S be a set. Recall that a free group on S is a group $F(S)$ together with a set map $\eta\colon S \to F(S)$ satisfying the following universal property: for all groups H and all set maps $\phi\colon S \to H$, there exists a unique homomorphism $\Phi\colon F(S) \to H$ such that the diagram



commutes. The usual argument shows that $F(S)$ is unique if it exists, but of course its existence is not obvious. The standard construction of $F(S)$ exhibits it as a set of equivalence classes of words in the formal symbols $\{s, s^{-1} \mid s \in S\}$, where two words are equivalent if one can be obtained from the other by a sequence of insertions and deletions of the trivial words ss^{-1} or $s^{-1}s$ with $s \in S$. This is a mildly awkward construction; in particular, it requires a small trick to show that every word is equivalent to a **unique** reduced word (i.e. one that contains no trivial subwords ss^{-1} or $s^{-1}s$ with $s \in S$).

The purpose of this note is to advertise a beautiful alternate construction of the free group that is contained in Lang's book on algebra; see [L, §I.12]. From a categorical point of view, the construction of a free group is a functor $\mathcal{F}\colon \texttt{Sets} \to \texttt{Groups}$ which is a left adjoint to the forgetful functor $\mathcal{U}\colon \texttt{Groups} \to \texttt{Sets}$, i.e. such that there is a natural bijection

$$\mathrm{Hom}_{\texttt{Sets}}(S, \mathcal{U}(\Gamma)) \longleftrightarrow \mathrm{Hom}_{\texttt{Groups}}(\mathcal{F}(S), \Gamma).$$

In category theory, there is a very general theorem (due to Freyd) known as the adjoint functor theorem that constructs such left adjoints. Lang's proof is essentially a specialization of the usual proof of the adjoint functor theorem to the setting of groups. As a geometric group theorist, the thing that surprises me the most about this proof is that perhaps the most subtle issues in it are set-theoretic; one constructs an enormous collection of groups, and it requires care to make sure that this collection is a set and not just a class.

The proof is as follows.

Theorem 1. *For any set S, there exists a free group on S.*

Proof. We begin with a sequence of definitions. Let $\mathfrak{C}$ be the set of isomorphism classes of groups that can be generated by a set of cardinality at most $|S|$ (subtle point: there is such a set; the reader should carefully convince themself of this). Next, define

$$\mathfrak{D} = \{(G, \psi) \mid G \in \mathfrak{C} \text{ and } \psi\colon S \to G \text{ is a set map whose image generates } G\}.$$

Let

$$\widehat{\Gamma} = \prod_{(G,\psi)\in\mathfrak{D}} G$$

and let $\widehat{\eta}\colon S \to \widehat{\Gamma}$ be the set map whose (G, ψ)-coordinate function is ψ. Finally, define Γ to be the subgroup of $\widehat{\Gamma}$ generated by the image of $\widehat{\eta}$ and let $\eta\colon S \to \Gamma$ be the set map obtained by restricting the target of $\widehat{\eta}$.

We claim that Γ is a free group on S (with respect to the set map $\eta\colon S \to \Gamma$). Indeed, let H be any group and let $\phi\colon S \to H$ be a set map. Letting $G \subset H$ be the subgroup generated by the image of ϕ and $\psi\colon S \to G$ be the set map obtained by restricting the target of ϕ, we obtain an element $(G, \psi) \in \mathfrak{D}$. The projection of $\widehat{\Gamma}$ onto its (G, ψ)-factor is a homomorphism $\widehat{\Phi}\colon \widehat{\Gamma} \to G$ such that the diagram

$$\begin{array}{ccc} & & \widehat{\Gamma} \\ & \overset{\widehat{\eta}}{\nearrow} & \downarrow{\scriptstyle \widehat{\Phi}} \\ S & \xrightarrow{\psi} & G \end{array}$$

commutes. The desired homomorphism $\Phi\colon \Gamma \to H$ is then obtained by restricting $\widehat{\Phi}$ to $\Gamma \subset \widehat{\Gamma}$ and then including G into H. The uniqueness of Φ is obvious (the key point being that Γ is generated by the image of η). $\square$

References

[L] S. Lang, *Algebra*, revised third edition, Graduate Texts in Mathematics, 211, Springer, New York, 2002.

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