The Cauchy–Binet formula

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Abstract

We give a proof of the Cauchy–Binet formula for the determinant of the product of two matrices that mostly avoids explicit matrix manipulations.

Let $\mathbf{k}$ be a field. All matrices in this note have entries in $\mathbf{k}$. Let A be an $n \times m$ matrix and let B be an $m \times n$ matrix. The product AB is thus an $n \times n$ matrix. The Cauchy–Binet formula shows how to express the determinant of AB in terms of A and B. When $n = m$, it reduces to the familiar fact that $\det(AB) = \det(A)\det(B)$.

Stating it requires introducing some notation. Let $[m] = \{1, \ldots, m\}$. For $I \subset [m]$, let A_I be the $n \times |I|$ submatrix of A consisting of the rows of A lying in I. Similarly, let ${}_IB$ be the $|I| \times n$ submatrix of B consisting of the columns of B lying in I.

Theorem 0.1 (Cauchy–Binet formula)**.** *Let A be an $n \times m$ matrix and let B be an $m \times n$ matrix. Then*

$$\det(AB) = \sum_{\substack{I \subset [m] \\ |I| = n}} \det(A_I)\det({}_IB).$$

Proof. For an $r \times s$ matrix C, let $\phi_C \colon \mathbf{k}^s \to \mathbf{k}^r$ be the associated linear map. Thus ϕ_{AB} equals the composition

$$\mathbf{k}^n \xrightarrow{\phi_B} \mathbf{k}^m \xrightarrow{\phi_A} \mathbf{k}^n.$$

Letting $\{\vec{e}_1, \ldots, \vec{e}_n\}$ be the standard basis for $\mathbf{k}^n$, we thus have that

$$\phi_A \circ \phi_B(\vec{e}_1 \wedge \cdots \wedge \vec{e}_n) = \det(AB)\vec{e}_1 \wedge \cdots \wedge \vec{e}_n.$$

To express this in terms of A and B, we will have to first understand $\phi_B \colon \wedge^n \mathbf{k}^n \to \wedge^n \mathbf{k}^m$.

Let $\{\vec{f}_1, \ldots, \vec{f}_n\}$ be the standard basis $\mathbf{k}^m$. The vector space $\wedge^n \mathbf{k}^m$ thus has a basis

$$\{\vec{f}_{i_1} \wedge \cdots \wedge \vec{f}_{i_n} \mid \{i_1 < \cdots < i_n\} \subset [m]\}.$$

We claim that

$$\phi_B(\vec{e}_1 \wedge \cdots \wedge \vec{e}_n) = \sum_{I = \{i_1 < \cdots < i_n\} \subset [m]} \det({}_IB)\vec{f}_{i_1} \wedge \cdots \wedge \vec{f}_{i_n}. \tag{0.1}$$

To see this, fix some $I = \{i_1 < \cdots < i_n\} \subset [m]$. Let $V_I = \langle \vec{f}_{i_1}, \ldots, \vec{f}_{i_n} \rangle \subset \mathbf{k}^m$ and let $\pi_I \colon \mathbf{k}^m \to V_I$ be the projection whose kernel is generated by the $\vec{f}_j$ with $j \notin I$. Identifying V_I with $\mathbf{k}^n$ via its natural basis, the composition

$$\mathbf{k}^n \xrightarrow{\phi_B} \mathbf{k}^m \xrightarrow{\pi_I} V_I$$

equals the linear map associated to ${}_I B$. We thus have

$$(\pi_I \circ \phi_B)_*(\vec{e}_1 \wedge \cdots \wedge \vec{e}_n) = \det({}_I B) \vec{f}_{i_1} \wedge \cdots \wedge \vec{f}_{i_n}.$$

The equation (0.1) follows.

Fixing some $I = \{i_1 < \cdots < i_n\} \subset [m]$ again, the next step is to observe that if we again identify V_I with $\mathbf{k}^n$ via its natural basis, the composition

$$V_I \hookrightarrow \mathbf{k}^m \xrightarrow{\phi_A} \mathbf{k}^n$$

equals the linear map associated to A_I. It follows that

$$\phi_A(\vec{f}_{i_1} \wedge \cdots \wedge \vec{f}_{i_n}) = \det(A_I) \vec{e}_1 \wedge \cdots \wedge \vec{e}_n.$$

Combining this with (0.1), we see that

$$\begin{aligned} \phi_A \circ \phi_B(\vec{e}_1 \wedge \cdots \wedge \vec{e}_n) &= \sum_{I=\{i_1<\cdots<i_n\}\subset[m]} \det({}_I B) \phi_A(\vec{f}_{i_1} \wedge \cdots \wedge \vec{f}_{i_n}) \\ &= \sum_{I=\{i_1<\cdots<i_n\}\subset[m]} \det({}_I B) \det(A_I) \vec{e}_1 \wedge \cdots \wedge \vec{e}_n. \end{aligned}$$

The theorem follows. □

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