# The action of the deck group on the homology of finite covers of surfaces

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**Abstract**

We give two proofs of a theorem of Chevalley–Weil that describes the homology of a cover of a surface as a representation of the deck group.

Our goal is to prove the following beautiful theorem of Chevalley–Weil.

**Theorem 0.1** (Chevalley–Weil, [CW])**.** *Let $\Sigma_g$ be a genus $g$ surface and let $\widetilde{\Sigma}$ be a finite cover of $\Sigma_g$ with deck group $G$. Then as a $\mathbb{Q}[G]$-module we have*

$$\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q}) \cong (\mathbb{Q}[G])^{2g-2} \oplus \mathbb{Q}^2.$$

I learned the first proof from Thomas Church.

*Proof 1 (character theory).* Let $\chi$ be the character of the $\mathbb{Q}[G]$-representation $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$. We want to show that $\chi$ equals the character of $(\mathbb{Q}[G])^{2g-2} \oplus \mathbb{Q}^2$. Since the left-action of $G$ on itself freely permutes the elements of $G$, the character of $\mathbb{Q}[G]$ takes the value 0 on all non-identity elements of $G$. We deduce that our goal is to prove that

$$\chi(g) = \begin{cases} (2g-2)|G|+2 & \text{if } g = 1 \\ 2 & \text{if } g \neq 1 \end{cases} \qquad (g \in G).$$

We divide this into two cases.

- Let $g \in G$ be a nonidentity element. The action of $g$ on $\widetilde{\Sigma}$ has no fixed points, so the Lefschetz fixed point theorem says that

$$\begin{aligned} 0 &= \mathrm{trace}(\mathrm{H}_0(\widetilde{\Sigma};\mathbb{Q}) \xrightarrow{g} \mathrm{H}_0(\widetilde{\Sigma};\mathbb{Q})) - \mathrm{trace}(\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q}) \xrightarrow{g} \mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})) \\ &\qquad + \mathrm{trace}(\mathrm{H}_2(\widetilde{\Sigma};\mathbb{Q}) \xrightarrow{g} \mathrm{H}_2(\widetilde{\Sigma};\mathbb{Q})) \\ &= \mathrm{trace}(\mathbb{Q} \xrightarrow{\mathrm{id}} \mathbb{Q}) - \chi(g) + \mathrm{trace}(\mathbb{Q} \xrightarrow{\mathrm{id}} \mathbb{Q}) \\ &= 2 - \chi(g). \end{aligned}$$

  We deduce that $\chi(g) = 2$, as desired.
- We now deal with the identity. The surface $\widetilde{\Sigma}$ has Euler characteristic $|G|(2-2g)$. Letting $\widetilde{g}$ be the genus of $\widetilde{\Sigma}$, we thus see that $|G|(2-2g) = 2 - 2\widetilde{g}$, so

$$\widetilde{g} = \frac{1}{2}\left(2 - |G|\,(2-2g)\right) = (g-1)|G| + 1$$

  and

$$\chi(1) = \dim_{\mathbb{Q}} \mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q}) = (2g-2)|G| + 2,$$

  as desired. □

I learned the second proof from [GLLM].

*Proof 2 (topology).* Endow $\Sigma_g$ with the usual CW-complex structure consisting of a single vertex $*$ and $2g$ edges $e_1, \ldots, e_{2g}$ and a single face $f$. Lift this to a CW-complex structure on $\widetilde{\Sigma}$. The cellular chain complex

$$0 \to C_2(\widetilde{\Sigma}; \mathbb{Q}) \to C_1(\widetilde{\Sigma}; \mathbb{Q}) \to C_0(\widetilde{\Sigma}; \mathbb{Q}) \to 0$$

is a chain complex of $\mathbb{Q}[G]$-representations. We can identify these representations as follows. Let $\widetilde{*}$ be an arbitrary lift of $*$, let $\widetilde{e}_1, \ldots, \widetilde{e}_{2g}$ be arbitrary lifts of $e_1, \ldots, e_{2g}$, and let $\widetilde{f}$ be an arbitrary lift of $f$. The group $G$ freely permutes the cells of $\widetilde{\Sigma}$, so the 0-cells of $\widetilde{\Sigma}$ are precisely $G \cdot *$, the 1-cells are precisely $G \cdot \widetilde{e}_1, \ldots, G \cdot \widetilde{e}_{2g}$, and the 2-cells are precisely $G \cdot \widetilde{f}$. We conclude that the cellular chain complex of $\widetilde{\Sigma}$ takes the form

$$0 \to \mathbb{Q}[G] \to (\mathbb{Q}[G])^{2g} \to \mathbb{Q}[G] \to 0.$$

When we take the homology of this chain complex, the $0^{\text{th}}$ and $2^{\text{nd}}$ homology groups should be $\mathbb{Q}$. We deduce that when we form $\mathrm{H}_1(\widetilde{\Sigma}; \mathbb{Q})$, we eliminate all but the trivial representation from two copies of $\mathbb{Q}[G]$ (though of course these copies are not embedded in the indicated product in a standard way!). It follows that

$$\mathrm{H}_1(\widetilde{\Sigma}; \mathbb{Q}) \cong (\mathbb{Q}[G])^{2g-2} \oplus \mathbb{Q}^2,$$

as desired. □

## References


[CW] C. Chevalley and A. Weil. Über das Verhalten der Integrale 1. Gattung bei Automorphismen des Funktionenkörpers. Abh. Math. Sem. Univ. Hamburg, 10 (1934), 358–361

[GLLM] F. Grunewald, M. Larsen, A. Lubotzky, and J. Malestein, Arithmetic quotients of the mapping class group, Geom. Funct. Anal. **25** (2015), no. 5, 1493–1542.



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