# Two nonstandard constructions of free groups

Andrew Putman

**Abstract**

We give two nonstandard constructions of free groups, one using geometric topology and the other inspired by category theory.

## 1 Introduction

Let $S$ be a set. Recall that a free group on $S$ is a group $F(S)$ together with a set map $\eta\colon S \to F(S)$ satisfying the following universal property: for all groups $G$ and all set maps $\phi\colon S \to G$, there exists a unique homomorphism $\Phi\colon F(S) \to G$ such that the diagram

$$\begin{tikzcd} & F(S) \arrow[d, "\Phi"] \\ S \arrow[ur, "\eta"] \arrow[r, "\phi"] & G \end{tikzcd}$$

commutes. The usual argument shows that $F(S)$ is unique if it exists, but of course its existence is not obvious. The standard construction of $F(S)$ exhibits it as a set of equivalence classes of words in the formal symbols $\{s, s^{-1} \mid s \in S\}$, where two words are equivalent if one can be obtained from the other by a sequence of insertions and deletions of the trivial words $ss^{-1}$ or $s^{-1}s$ with $s \in S$. This is a mildly awkward construction; in particular, it requires a small trick to show that every word is equivalent to a **unique** reduced word (i.e. one that contains no trivial subwords $ss^{-1}$ or $s^{-1}s$ with $s \in S$).

In this note, we give two alternate constructions of $F(S)$ that work directly from the above universal property. One construction uses basic geometric topology, and the other is inspired by category theory.

## 2 Geometric topology

The key to our first construction is the follow lemma:

**Lemma 2.1.** *For all groups $G$, there exists a based space $(X, x_0)$ such that $\pi_1(X, x_0) \cong G$.*

One has to be careful here: the usual construction of a two-dimensional CW-complex whose fundamental group is a given group starts with a group presentation, a notion that depends on having first constructed a free group! The following argument avoids this circularity:

*Proof of Lemma 2.1.* The natural candidate for such a space is an Eilenberg–MacLane space for $G$. Here are two approaches to constructing this:

- The first uses Milnor's "infinite join" construction from [M]. Regard $G$ as a discrete space, and inductively define spaces $Z_n$ as follows:

$$Z_1 = G \quad \text{and} \quad Z_{n+1} = G * Z_n.$$

  Here $*$ denotes the join of these two spaces. We have

$$Z_1 \subset Z_2 \subset Z_3 \subset \cdots .$$

Define

$$Z = \bigcup_{n=1}^{\infty} Z_n.$$

The space $Z$ is contractible. Indeed, since $Z$ can be endowed with the structure of a CW-complex, it is enough to prove that $\pi_k(Z) = 0$ for $k \geq 0$. Let $f\colon S^k \to Z$ be any continuous map. Since the $k$-sphere $S^k$ is compact, there exists some $n \gg 0$ such that $\mathrm{Im}(f) \subset Z_n$. But the inclusion map $Z_n \hookrightarrow Z_{n+1}$ is clearly nullhomotopic, so $f$ is as well. The group $G$ acts freely on $Z$, and the quotient $Z/G$ is a space whose fundamental group is isomorphic to $G$.

- The second approach is a bit more abstract. Recall that a classifying space for $G$ is a based space $BG$ such that for all based connected CW-complexes $Y$, there is a bijection between the set of based principal $G$-bundles on $Y$ and the set $[Y, BG]$ of all homotopy classes of based maps from $Y$ to $BG$. Brown's representability theorem ([B]; see [P] for an expository account) shows that such a space $BG$ exists. Elements of $\pi_1(BG) = [S^1, BG]$ are the same as based principal $G$-bundles on $S^1$, which by the clutching construction are the same as elements of $\pi_0(G) = G$. We conclude that $\pi_1(BG) \cong G$. We remark that $BG$ is actually an Eilenberg–MacLane space: for $n \geq 2$ elements of $\pi_n(BG) = [S^n, BG]$ are the same as based principal $G$-bundles on $S^n$, which by the clutching construction are the same as elements of $\pi_{n-1}(G)$. This latter group is trivial since $G$ is a discrete set. □

We now prove that free groups exist.

**Theorem 2.2.** *For any set $S$, there exists a free group on $S$.*

*Proof.* Let $Y$ be the wedge of $|S|$ circles labeled by elements of $S$ with wedge point $y_0$. Define $\Gamma = \pi_1(Y, y_0)$ and let $\eta\colon S \to \Gamma$ take $s \in S$ to the element corresponding to the circle labeled by $S$. We claim that $\Gamma$ is a free group on $S$. To prove this, we verify the universal property. Let $G$ be a group and let $\phi\colon S \to G$ be a set map. Using Lemma 2.1, we can find a based space $(X, x_0)$ such that $\pi_1(X, x_0) \cong G$. Define $\Phi\colon \Gamma \to G$ to be the map on fundamental groups induced by the map $(Y, y_0) \to (X, x_0)$ that takes the circle labeled by $s \in S$ to a loop representing $\eta(s)$. It is clear that the diagram

$$\begin{tikzcd} & \Gamma \arrow[d, "\Phi"] \\ S \arrow[ur, "\eta"] \arrow[r, "\phi"'] & G \end{tikzcd}$$

commutes and that the resulting $\Phi$ is unique. □

# 3 Category theory

We now give a categorical construction of a free group that we learned about from Lang's book on algebra; see [L, §I.12]. From a categorical point of view, the construction of a free group is a functor $\mathcal{F}\colon \texttt{Sets} \to \texttt{Groups}$ which is a left adjoint to the forgetful functor $\mathcal{U}\colon \texttt{Groups} \to \texttt{Sets}$, i.e. such that there is a natural bijection

$$\mathrm{Hom}_{\texttt{Sets}}(S, \mathcal{U}(\Gamma)) \longleftrightarrow \mathrm{Hom}_{\texttt{Groups}}(\mathcal{F}(S), \Gamma).$$

In category theory, there is a very general theorem (due to Freyd) known as the adjoint functor theorem that constructs such left adjoints. Lang's proof is essentially a specialization of the usual proof of the adjoint functor theorem to the setting of groups. As a geometric group theorist, the thing that surprises me the most about this proof is that perhaps the most subtle issues in it are set-theoretic; one constructs an enormous collection of groups, and it requires care to make sure that this collection is a set and not just a class.

The proof is as follows.

**Theorem 3.1.** *For any set $S$, there exists a free group on $S$.*

*Proof.* We begin with a sequence of definitions. Let $\mathfrak{C}$ be the set of isomorphism classes of groups that can be generated by a set of cardinality at most $|S|$ (subtle point: there is such a set; the reader should carefully convince themself of this). Next, define

$$\mathfrak{D} = \{(G, \psi) \mid G \in \mathfrak{C} \text{ and } \psi\colon S \to G \text{ is a set map whose image generates } G\}.$$

Let

$$\widehat{\Gamma} = \prod_{(G,\psi) \in \mathfrak{D}} G$$

and let $\widehat{\eta}\colon S \to \widehat{\Gamma}$ be the set map whose $(G, \psi)$-coordinate function is $\psi$. Finally, define $\Gamma$ to be the subgroup of $\widehat{\Gamma}$ generated by the image of $\widehat{\eta}$ and let $\eta\colon S \to \Gamma$ be the set map obtained by restricting the target of $\widehat{\eta}$.

We claim that $\Gamma$ is a free group on $S$ (with respect to the set map $\eta\colon S \to \Gamma$). Indeed, let $H$ be any group and let $\phi\colon S \to H$ be a set map. Letting $G \subset H$ be the subgroup generated by the image of $\phi$ and $\psi\colon S \to G$ be the set map obtained by restricting the target of $\phi$, we obtain an element $(G, \psi) \in \mathfrak{D}$. The projection of $\widehat{\Gamma}$ onto its $(G, \psi)$-factor is a homomorphism $\widehat{\Phi}\colon \widehat{\Gamma} \to G$ such that the diagram

$$\begin{tikzcd} & \widehat{\Gamma} \arrow[d, "\widehat{\Phi}"] \\ S \arrow[ur, "\widehat{\eta}"] \arrow[r, "\psi"] & G \end{tikzcd}$$

commutes. The desired homomorphism $\Phi\colon \Gamma \to H$ is then obtained by restricting $\widehat{\Phi}$ to $\Gamma \subset \widehat{\Gamma}$ and then including $G$ into $H$. The uniqueness of $\Phi$ is obvious (the key point being that $\Gamma$ is generated by the image of $\eta$). $\square$

# References


[B] E. H. Brown, Jr., Cohomology theories, Ann. of Math. (2) 75 (1962), 467–484.

[L] S. Lang, *Algebra*, revised third edition, Graduate Texts in Mathematics, 211, Springer, New York, 2002.

[M] J. Milnor, Construction of universal bundles. II, Ann. of Math. (2) 63 (1956), 430–436.

[P] A. Putman, Classifying spaces and Brown representability, informal note, `http://www.nd.edu/~andyp/notes/BrownRepresentability.pdf`.



Andrew Putman
Department of Mathematics
University of Notre Dame
255 Hurley Hall
Notre Dame, IN 46556
`andyp@nd.edu`