The action on homology of finite groups of automorphisms of surfaces and graphs

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Abstract

We prove that aside from trivial cases, finite-order homeomorphisms of surfaces and graphs must act nontrivially on homology. For surfaces, this classical theorem is usually deduced from the Lefschetz fixed point theorem, while for graphs it is usually deduced via combinatorial manipulations. Our proof is different and is in the same spirit as the original proof (due to Hurwitz) of this theorem for surfaces.

In this note, we prove that if S is a compact oriented surface whose genus is at least 2 and $f\colon S \to S$ is a periodic homeomorphism with $f \neq \mathrm{id}$, then the induced map $f_*\colon \mathrm{H}_1(S;\mathbb{Z}) \to \mathrm{H}_1(S;\mathbb{Z})$ must be nontrivial. This is a well-known theorem of Hurwitz [H]. The proof that appears in most modern textbooks (due to Serre) deduces it from the Lefschetz fixed point theorem; see, e.g., [FM]. Our goal is to give a proof that is more in the spirit of Hurwitz's original proof. While it is a little longer, I feel that this proof is quite instructive. Moreover, unlike the proof using the Lefschetz fixed point theorem, it can easily be adapted to prove the analogous result for graphs (I do not know who to attribute this analogous result to, though it is an easy consequence of work of Baumslag–Taylor [BT]). The key is a proposition concerning the homology groups of orbit spaces of finite group actions which we prove in §1. We then prove our main theorem for surfaces in §2 and for graphs in §3.

1 The homology of quotient spaces

If G is a group and M is a G-representation over a field $\mathbb{F}$ (that is, an $\mathbb{F}$-vector space on which G acts linearly; this is the same as a $\mathbb{F}[G]$-module), then the **invariants** of M are

$$M^G := \{m \in M \mid g(m) = m \text{ for all } g \in G\}$$

and the **coinvariants** of M are

$$M_G := M/I \quad \text{with} \quad I = \langle g(m) - m \mid g \in G,\, m \in M\rangle.$$

These are related by duality: letting $M^* = \mathrm{Hom}_{\mathbb{F}}(M,\mathbb{F})$, we have a natural isomorphism $(M^*)^G \cong (M_G)^*$. The goal of this section is to prove the following simple proposition. I do not know who to attribute it to, but it can be found in e.g. [Br, Theorem III.2.4].

Proposition 1.1. *Let X be a simplicial complex and G be a finite group acting simplicially on X. Then*

$$\mathrm{H}^k(X/G;\mathbb{Q}) \cong (\mathrm{H}^k(X;\mathbb{Q}))^G \quad \textit{and} \quad \mathrm{H}_k(X/G;\mathbb{Q}) \cong (\mathrm{H}_k(X;\mathbb{Q}))_G$$

for all $k \geq 0$.

Remark. If G acts freely, then Proposition 1.1 is a consequence of the standard transfer lemma. In fact, the proof in [Br, Theorem III.2.4] goes by constructing a generalized transfer map for branched covers, though our proof below is different.

Proof of Proposition 1.1. Since we are working over $\mathbb{Q}$, cohomology is dual to homology and the two assertions are equivalent. We will prove the first one. Subdividing X appropriately, we can assume that X/G is a simplicial complex whose p-simplices are in bijection with the G-orbits of p-simplices of X for all $p \geq 0$. The action of G on X turns the simplicial cochain complex $\mathcal{C}^*(X;\mathbb{Q})$ into a cochain complex of $\mathbb{Q}[G]$-modules. Our subdivision ensures that

$$\mathcal{C}^*(X/G;\mathbb{Q}) \cong (\mathcal{C}^*(X;\mathbb{Q}))^G.$$

The proposition now follows from Lemma 1.2 below. □

Lemma 1.2. *Let G be a finite group and let C^* be a cochain complex of $\mathbb{Q}[G]$-modules. Then*

$$(\mathrm{H}^*(C^*))^G = \mathrm{H}^*((C^*)^G)$$

Proof. This is an easy fact about representation theory. Recall that since G is a finite group, it has finitely many irreducible representations over $\mathbb{Q}$. If these irreducible representations are $V_1, \ldots, V_p$ and if V is a $\mathbb{Q}[G]$-module, then we can uniquely write

$$V = V_1^{k_1} \oplus \cdots \oplus V_p^{k_p} \qquad (k_1, \ldots, k_p \in \mathbb{Z}_{\geq 0});$$

the subspace $V_i^{k_i}$ of V is called the V_i**-isotypic component** of V, and the decomposition itself is called the **isotypic decomposition** of V. This works even if V is infinite-dimensional. Each term of C^* has an isotypic decomposition, and this isotypic decomposition is preserved by the coboundary map. This implies that for all $1 \leq i \leq p$ and $k \in \mathbb{Z}$, the V_i-isotypic component of $\mathrm{H}^k(C^*)$ is the same as the k^{th} cohomology group of the cochain complex composed of the V_i-isotypic components of C^*. This is in particular true for the trivial representation, which is the assertion of the lemma. □

2 Finite-order automorphisms of surfaces

To prove our theorem for surfaces, we need the following classical fact.

Theorem 2.1 (Riemann-Hurwitz formula)**.** *Let $\pi\colon S \to S'$ be an orientation-preserving degree d branched cover between closed oriented surfaces. Assume that the orders of the branch points are $\lambda_1, \ldots, \lambda_k \in \mathbb{Z}_{\geq 2}$. Then*

$$\chi(S) = d\chi(S') - \sum_{i=1}^{k}(\lambda_i - 1).$$

Proof. Choosing a triangulation of S' that contains the images of all the branch points and lifting this triangulation to S, we can assume that S and S' are simplicial complexes and π is a simplicial map. Assume that S' has v vertices, e edges, and t triangles. Our assumptions then imply that S has de edges and dt triangles. As for vertices, we need a correction term to account for the branch points: S has $dv - \sum_{i=1}^{k}(\lambda_i - 1)$ vertices. We conclude that

$$\chi(S) = (dv - \sum_{i=1}^{k}(\lambda_i - 1)) - de + dt = d\chi(S') - \sum_{i=1}^{k}(\lambda_i - 1),$$

as desired. □

Our theorem then is as follows.

Theorem 2.2. *Let S be a compact oriented surface with $\chi(S) < 0$ and let $f\colon S \to S$ be a periodic homeomorphism with $f \neq id$. Then f acts nontrivially on $\mathrm{H}_1(S;\mathbb{Z})$.*

Remark. Theorem 2.2 is clearly false for $g = 0$ and $g = 1$.

Proof of Theorem 2.2. If f is not orientation-preserving, then f acts nontrivially on $\mathrm{H}^2(S;\mathbb{Z})$. Considering cup products, we see that f must therefore act nontrivially on $\mathrm{H}^1(S;\mathbb{Z})$, and thus on $\mathrm{H}_1(S;\mathbb{Z})$. We can therefore assume that f is orientation-preserving. Assume for the sake of contradiction that f acts trivially on homology. Let $G \subset \mathrm{Homeo}^+(S)$ be the subgroup generated by f and let $\pi\colon S \to S/G$ be the projection. Since G preserves orientation, the map π is a branched covering. Letting $d = |G| \geq 2$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{Z}_{\geq 2}$ be the orders of the branch points, we can apply Theorem 2.1 and get that

$$\chi(S) = d\chi(S/G) - \sum_{i=1}^{k}(\lambda_i - 1).$$

Proposition 1.1 together with our assumption that f acts trivially on homology implies that $\chi(S) = \chi(S')$. Combining this with the above sum, we get that

$$\sum_{i=1}^{k}(\lambda_i - 1) = (d-1)\chi(S).$$

The left hand side of this is positive. However, since $d \geq 2$ and $\chi(S) < 0$, the right hand side is negative, a contradiction. □

3 Finite-order automorphisms of graphs

Let Λ be a graph and let $f\colon \Lambda \to \Lambda$ be a periodic homeomorphism. It is possible for f to not be the identity but to act trivially on $\mathrm{H}_1(\Lambda;\mathbb{Z})$. Here are two examples.

- Let Λ be a p-vertex triangulation of the circle and let $f\colon \Lambda \to \Lambda$ be an order p rotation.
- Let Λ be an arbitrary graph such that there exists some vertex $v \in \Lambda$ and two edges e_1 and e_2 containing v and ending at valence 1 vertices. Then we can let $f\colon \Lambda \to \Lambda$ flip e_1 and e_2 while fixing the remainder of Λ (including v).

The following theorem says that in some sense these are the only things that can go wrong.

Theorem 3.1. *Let Λ be a compact connected graph and let $f\colon \Lambda \to \Lambda$ be a periodic homeomorphism. Assume that the following hold.*

- *The homeomorphism f acts trivially on $\mathrm{H}_1(\Lambda;\mathbb{Z})$.*
- *The homeomorphism f fixes all valence 1 vertices of Λ.*
- *If $\mathrm{H}_1(\Lambda;\mathbb{Z}) \cong \mathbb{Z}$, then f fixes at least one point of Λ.*

Then $f = id$.

Proof. Assume for the sake of contradiction that $f \neq \mathrm{id}$. By subdividing Λ, we can assume that f is a simplicial map that does not flip any edges. Also, by replacing f with a power we can assume that the order $p \geq 2$ of f is prime. Let $G \subset \mathrm{Homeo}(\Lambda)$ be the subgroup generated by f, so $|G| = p$.

We will begin by dealing with the case that is closest to that of surfaces (and whose proof closely tracks that of Theorem 2.2), namely when f does not fix any edges of Λ (and hence has isolated fixed points). Since f acts trivially on $\mathrm{H}_1(\Lambda;\mathbb{Z})$, Proposition 1.1 implies that $\chi(\Lambda) = \chi(\Lambda/G)$. Let v be the number of vertices of Λ/G and e be the number of edges of Λ/G. Also let v_0 be the number of vertices of Λ that are fixed by f. Since $|G| = p$ is prime, all vertex and edge orbits have size either 1 or p. Since we are assuming that f does not fix any edges, we deduce that Λ has $p(v - v_0) + v_0$ vertices and pe edges. We therefore get that

$$v - e = \chi(\Lambda/G) = \chi(\Lambda) = p(v - v_0) + v_0 - pe.$$

Manipulating this, we see that $(p-1)(v-e) = (p-1)v_0$, so

$$v - e = v_0.$$

If $\mathrm{H}_1(\Lambda;\mathbb{Z})$ has rank at least 2, then $v - e < 0$ but $v_0 \geq 0$, so we get a contradiction. If $\mathrm{H}_1(\Lambda;\mathbb{Z})$ has rank 1, then $v - e = 0$, but by our assumptions we have $v_0 \geq 1$, so we also get a contradiction. Finally, if $\mathrm{H}_1(\Lambda;\mathbb{Z})$ has rank 0, then Λ is a tree and $v - e = 1$, so $v_0 = 1$. By assumption, f fixes all valence 1 vertices, so Λ has a single valence 1 vertex. The only compact tree with a single valence 1 vertex is the trivial one-point tree, so we conclude that Λ is a single point and $f = \mathrm{id}$, a contradiction.

We now reduce the general case to that of the previous paragraph. Let $\Lambda_{\text{fix}} = \{x \in \Lambda \mid f(x) = x\}$. Since we have assumed that $f \neq \mathrm{id}$, the graph Λ_{fix} must be a proper subgraph of Λ. We remark that Λ_{fix} need not be connected. Let Λ' be the graph that results from taking Λ and collapsing a maximal forest in Λ_{fix}. The projection map $\Lambda \to \Lambda'$ is a homotopy equivalence and f induces a simplicial homeomorphism $f' \colon \Lambda' \to \Lambda'$ that does not flip any edges and has order p. The key property of Λ' is that all its edges that are fixed by f' are loops; let $\Lambda'' \subset \Lambda'$ be the result of removing those fixed edges and let $f'' \colon \Lambda'' \to \Lambda''$ be the restriction of f'. Since Λ_{fix} is a proper subgraph of Λ, the graph Λ'' must contain edges and $f'' \neq \mathrm{id}$. If the rank of $\mathrm{H}_1(\Lambda''; \mathbb{Z})$ is one, then either the rank of $\mathrm{H}_1(\Lambda; \mathbb{Z})$ is one and by assumption f (and hence f'') must fix a point or we have have removed at least one loop when we formed Λ'', so again f'' must fix a point. We can thus apply the previous paragraph to the action of f'' on Λ'' to obtain a contradiction. $\square$

References

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