Harmonic functions on finite graphs

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Abstract

We give three proofs that functions on subgraphs of finite graphs can be uniquely extended to functions that are harmonic on the remainder of the graph.

1 Introduction

All graphs in this note are undirected and have no self-loops or multiple edges. Consider a finite graph G with vertex set $V(G)$ and edge set $E(G)$. For vertices $v, v' \in V(G)$, we will write $v \sim v'$ if v and v' are joined by an edge. A function $f\colon V(G) \to \mathbb{R}$ is said to be *harmonic* at a vertex $v \in V(G)$ if

$$f(v) = \frac{1}{\deg(v)} \sum_{v' \sim v} f(v').$$

The main theorem discussed in this note is as follows.

Theorem 1.1 (Dirichlet problem for finite graphs)**.** *Let G be a finite connected graph, let $W \subset V(G)$ be a nonempty set of vertices, and let $f\colon W \to \mathbb{R}$ be a function. Then there exists a unique extension $f\colon V(G) \to \mathbb{R}$ of f that is harmonic at all vertices in $V(G) \setminus W$.*

We will prove the uniqueness part in §2. We will then give three different proofs of existence: in §3 we will give a proof using linear algebra, in §4 will give a proof by minimizing an energy functional, and finally in §5 we will give a proof using random walks.

2 Uniqueness

We start by proving that the harmonic extension purported to exist by Theorem 1.1 is unique. Subtracting two such extensions, we see that it is enough to prove the following. Let G be a finite connected graph, let $W \subset V(G)$ be a nonempty set of vertices, and let $f\colon V(G) \to \mathbb{R}$ be a function such that $f|_W = 0$ and such that f is harmonic at all points of $V(G) \setminus W$. We must prove that $f = 0$. We will prove this via a maximum principle.

Assume that $f \neq 0$. Replacing f by $-f$ if necessary, we can assume that f achieves some positive value. Since G is connected and $f|_W = 0$, we can find some vertex $v \in V(G)$ such that $f(v)$ is maximal and such that there exists some $v_0 \in V(G)$ such that $v_0 \sim v$ and $f(v_0) < f(v)$. Since $f(v) > 0$, we must have $v \in V(G) \setminus W$, so f is harmonic at v, i.e.

$$f(v) = \frac{1}{\deg(v)} \sum_{v' \sim v} f(v').$$

This is an average of numbers that are at most $f(v)$, and at least one of them (namely $f(v_0)$) is less than $f(v)$, a contradiction.

3 Existence via linear algebra

We now give our first proof of existence. Let G be a finite connected graph, let $W \subset V(G)$ be a nonempty set of vertices, and let $f\colon W \to \mathbb{R}$ be a function. Our goal is to extend f to a function $f\colon V(G) \to \mathbb{R}$ that is harmonic on $V(G) \setminus W$.

For a set X, let $\mathrm{Fun}(X)$ be the vector space of real-valued functions on X. Let

$$\mathcal{U} = \{g \in \mathrm{Fun}(V(G)) \mid g|_W = f\}.$$

The set $\mathcal{U}$ is an affine subspace of the vector space $\mathrm{Fun}(V(G))$. Define

$$\Psi\colon \mathcal{U} \to \mathrm{Fun}(V(G) \setminus W)$$

via the formula

$$\Psi(g)(v) = g(v) - \frac{1}{\deg(v)} \sum_{v' \sim v} g(v') \qquad \text{for } v \in V(G) \setminus W.$$

The map Ψ is an affine-linear map.

We claim that Ψ is injective. Indeed, consider $g_1, g_2 \in \mathcal{U}$ with $\Psi(g_1) = \Psi(g_2)$. By assumption, we have

$$(g_1 - g_2)|_W = f - f = 0,$$

while for $v \in V(G) \setminus W$ we have

$$g_1(v) - \frac{1}{\deg(v)} \sum_{v' \sim v} g_1(v') = g_2(v) - \frac{1}{\deg(v)} \sum_{v' \sim v} g_2(v'),$$

so

$$(g_1 - g_2)(v) = \frac{1}{\deg(v)} \sum_{v' \sim v} (g_1 - g_2)(v').$$

In other words, $g_1 - g_2$ is a harmonic extension to $V(G)$ of the zero function on W, so by the already proved uniqueness of harmonic extensions we must have $g_1 - g_2 = 0$, as desired.

Since Ψ is an injective affine-linear map between affine spaces of the same dimension, it must be surjective. In particular, there must exist some $g \in \mathcal{U}$ with $\Psi(g) = 0$. This is our desired harmonic extension of f.

4 Existence via energy

We now give our second proof of existence. Let G be a finite connected graph, let $W \subset V(G)$ be a nonempty set of vertices, and let $f\colon W \to \mathbb{R}$ be a function. Our goal is to extend f to a function $f\colon V(G) \to \mathbb{R}$ that is harmonic on $V(G) \setminus W$.

Just like in the previous section, let

$$\mathcal{U} = \{g \in \text{Fun}(V(G)) \mid g|_W = f\}.$$

Define the *energy functional* to be the function $\mathfrak{E}\colon \mathcal{U} \to \mathbb{R}$ given by the formula

$$\mathfrak{E}(g) = \sum_{\substack{e \in E(G) \text{ edge w/} \\ \text{endpts } v, v' \in V(G)}} (g(v) - g(v'))^2.$$

We thus have $\mathfrak{E}(g) \geq 0$ for all $g \in \mathcal{U}$. We claim that $\mathfrak{E}$ must achieve a global minimum. To see this, it is enough to show that $\mathfrak{E}(g) \mapsto \infty$ as $g \mapsto \infty$ in $\mathcal{U}$. But this is clear: if $|g(v)|$ is large for any single vertex $v \in V(G)$, then since $g|_W$ is our fixed function f and $V(G)$ is connected there must be an edge connecting vertices $v', v'' \in V(G)$ such that $(g(v') - g(v''))^2$ is large, forcing $\mathfrak{E}(g)$ to be large.

Let $g \in \mathcal{U}$ be a global minimum for $\mathfrak{E}$. We claim that g is harmonic on $V(G) \setminus W$. Indeed, consider a vertex $v \in V(G) \setminus W$. View $\mathcal{U}$ as Euclidean space with coordinates the values the functions $g \in \mathcal{U}$ on vertices in $V(G) \setminus W$. We can then differentiate $\mathfrak{E}$ with respect to any of these coordinates. Since $g \in \mathcal{U}$ is a global minimum, we must have

$$0 = \frac{\partial \mathfrak{E}}{\partial v}(g) = 2 \sum_{v' \sim v} \left(g\left(v\right) - g\left(v'\right)\right) = 2\left(\deg\left(v\right) g\left(v\right) - \sum_{v' \sim v} g\left(v'\right)\right).$$

Rearranging, we see that

$$g(v) = \frac{1}{\deg(g)} \sum_{v' \sim v} g(v'),$$

so g is harmonic at v, as desired.

5 Existence via random walks

We now give our third proof of existence. Let G be a finite connected graph, let $W \subset V(G)$ be a nonempty set of vertices, and let $f\colon W \to \mathbb{R}$ be a function. Our goal is to extend f to a function $f\colon V(G) \to \mathbb{R}$ that is harmonic on $V(G) \setminus W$.

Consider a vertex $v \in V(G)$. Consider a random walk

$$v = v_1, v_2, v_3, \ldots$$

starting at v. Thus at each vertex v_i of our random walk, we randomly choose an edge coming out of v_i (each such edge getting equal probability $\frac{1}{\deg(v_i)}$) and move along it to the next vertex v_{i+1}. It is a fun exercise to see that with probability 1, such a random walk will eventually visit every vertex of G infinitely many times (hint: prove it by induction on the number of vertices, and argue that the set of random walks that pass through a fixed vertex at most N times but pass through an adjacent vertex infinitely many times must have measure 0). It will thus eventually pass through a vertex of W. For each $w \in W$, let $P(v, w)$ be the probability that w is the first vertex of W encountered in a random walk starting at v.

We thus have

$$\sum_{w \in W} P(v, w) = 1 \qquad \text{for } v \in V(G)$$

and

$$P(w_1, w_2) = \begin{cases} 1 & \text{if } w_1 = w_2, \\ 0 & \text{if } w_1 \neq w_2 \end{cases} \qquad \text{for } w_1, w_2 \in W. \tag{5.1}$$

Define $g \colon V(G) \to \mathbb{R}$ via the formula

$$g(v) = \sum_{w \in W} f(w) \cdot P(v, w) \quad \text{for } v \in V(G).$$

By (5.1), we have $g|_W = f$.

We claim that g is harmonic on $V(G) \setminus W$. Indeed, consider some $v \in V$. A random walk starting at v first goes to one of the adjacent vertices of G, and the probability of going to each such vertex is $\frac{1}{\deg(v)}$. It follows that for $w \in W$, we have

$$P(v, w) = \frac{1}{\deg(v)} \sum_{v' \sim v} P(v', w).$$

In other words, $P(-, w)$ is harmonic at v. Plugging this into our formula for g, we see that

$$\begin{aligned} g(v) &= \sum_{w \in W} f(w) \cdot P(v, w) \\ &= \sum_{w \in W} \left(f(w) \cdot \frac{1}{\deg(v)} \sum_{v' \sim v} P(v', w) \right) \\ &= \frac{1}{\deg(v)} \sum_{v' \sim v} \sum_{w \in W} f(w) P(v', w) \\ &= \frac{1}{\deg(v)} \sum_{v' \sim v} g(v'), \end{aligned}$$

as desired.

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