The isoperimetric inequality in the plane

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Let \( \gamma \) be a simple closed curve in \( \mathbb{R}^2 \). Choose a parameterization \( f : [0, 1] \to \mathbb{R}^2 \) of \( \gamma \) and define

\[
\text{length}(\gamma) = \sup_{0=t_0<t_1<\cdots<t_k=1} \sum_{i=1}^{k} d_{\mathbb{R}^2}(f(t_{i-1}), f(t_i)) \in \mathbb{R} \cup \{\infty\}; \tag{0.1}
\]

here the supremum is taken over all partitions of \([0, 1]\). This does not depend on the choice of parameterization. By the Jordan curve theorem, the simple closed curve \( \gamma \) encloses a bounded region in \( \mathbb{R}^2 \); define \( \text{area}(\gamma) \) to be the Lebesgue measure of this bounded region. The classical isoperimetric inequality is as follows.

**Isoperimetric Inequality.** If \( \gamma \) is a simple closed curve in \( \mathbb{R}^2 \), then

\[
\text{area}(\gamma) \leq \frac{1}{4\pi} \text{length}(\gamma)^2 \text{ with equality if and only if } \gamma \text{ is a round circle.}
\]

This inequality was stated by Greeks but was first rigorously proved by Weierstrass in the 19th century. In this note, we will give a simple and elementary proof based on geometric ideas of Steiner. For a discussion of the history of the isoperimetric inequality and a sample of the enormous number of known proofs of it, see [1] and [3].

Our proof will require three lemmas. The first is a sort of “discrete” version of the isoperimetric inequality. A polygon in \( \mathbb{R}^2 \) is **cyclic** if it can be inscribed in a circle.

**Lemma 1.** Let \( P \) be a noncyclic polygon in \( \mathbb{R}^2 \). Then there exists a cyclic polygon \( P' \) in \( \mathbb{R}^2 \) with the same cyclically ordered side lengths as \( P \) satisfying \( \text{area}(P) < \text{area}(P') \).

**Proof.** All triangles are cyclic, so \( P \) has at least 4 sides. The set of all polygons in \( \mathbb{R}^2 \) with the same cyclically ordered side lengths as \( P \) and with one vertex at the origin is compact. It follows that there exists a polygon \( P' \) in \( \mathbb{R}^2 \) with the same cyclically ordered side lengths as \( P \) whose area is maximal among all such polygons. We will prove that \( P' \) is cyclic. It is clear that \( P' \) is convex. There are now two cases.

**Case 1.** The polygon \( P \) has 4 sides.

We remark that this case could be deduced immediately from Bretschneider’s formula for the area of a convex quadrilateral (see [2]), but we will give a self-contained proof.

Let \( a, b, c, \) and \( d \) be the side lengths of \( P \) (cyclically ordered). Consider a convex polygon \( Q \) with the same cyclically ordered side lengths as \( P \). Let
Figure 1: The quadrilateral $Q$ in Step 1 of the proof of Lemma 1.

$q_1, \ldots, q_4$ be the vertices and let $\theta_1$ and $\theta_2$ be the angles labeled in Figure 1. Since any three non-colinear points determine a circle, there are circles containing \{q_1, q_2, q_4\} and \{q_2, q_3, q_4\}. These circles will be the same (and hence $Q$ will be cyclic) exactly when $\theta_1 + \theta_2 = \pi$.

It is clear that the isometry class of $Q$ is determined by $\theta_1$ and $\theta_2$. However, not all pairs of angles are possible; indeed, computing the length of the diagonal from $q_2$ to $q_4$ using the law of cosines in two ways, we see that

$$a^2 + b^2 - 2ab \cos(\theta_1) = c^2 + d^2 - 2cd \cos(\theta_2). \quad (0.2)$$

Conversely, any angles $\theta_1$ and $\theta_2$ satisfying (0.2) and $0 \leq \theta_1, \theta_2 \leq \pi$ can be realized by some convex polygon as above. The area of $Q$ is $\frac{1}{2} ab \sin(\theta_1) + \frac{1}{2} cd \sin(\theta_2)$. Letting $f(\theta_1, \theta_2) = ab \sin(\theta_1) + cd \sin(\theta_2) \text{ and } g(\theta_1, \theta_2) = a^2 + b^2 - 2ab \cos(\theta_1) - c^2 - d^2 + 2cd \cos(\theta_2)$, our goal therefore is to show that among all angles satisfying $0 \leq \theta_1, \theta_2 \leq \pi$ and $g(\theta_1, \theta_2) = 0$, the function $f(\theta_1, \theta_2)$ is maximized when $\theta_1 + \theta_2 = \pi$.

It is clear that this maximum will occur when $0 < \theta_1, \theta_2 < \pi$, so using Lagrange multipliers we see that at this maximum, there will exist some $\lambda \in \mathbb{R}$ such that $\nabla f = \lambda \nabla g$, i.e. such that

$$ab \cos(\theta_1) = 2ab\lambda \sin(\theta_1) \text{ and } cd \cos(\theta_2) = -2cd\lambda \sin(\theta_2).$$

Since $0 < \theta_1, \theta_2 < \pi$, we have $\sin(\theta_1) \neq 0$ and $\sin(\theta_2) \neq 0$, so we can manipulate the above formulas and see that $\cot(\theta_1) = -\cot(\theta_2)$. This implies that $\theta_1 + \theta_2 = \pi$, as desired.

Case 2. The polygon $P$ has more than 4 sides.

Assume that $P'$ is not cyclic. This implies that there exist four vertices $q_1, \ldots, q_4$ of $P'$ that do not lie on a circle. Let $Q$ be the quadrilateral with these four vertices. Using Case 1, there exist a cyclic quadrilateral $Q'$ with the same side lengths as $Q$ but with area($Q$) < area($Q'$). Let $X_1, \ldots, X_4$ be the components of $P' \setminus Q$ adjacent to the four sides of $Q$ (possibly some of the $X_i$ are empty), so area($P'$) = area($Q$) + area($X_1$) + \cdots + area($X_4$). As is shown in Figure 2, we can attach the $X_i$ to $Q'$ to form a polygon $P''$ whose cyclically ordered side lengths are the same as those of $P'$ but whose area equals area($Q'$) + area($X_1$) + \cdots + area($X_4$). But this implies that area($P''$) > area($P'$), contradicting the maximality of the area of $P'$. \qed
Figure 2: Changing the quadrilateral $Q$ in $P$ to $Q'$ (without changing the side lengths of $P$) increases the area since $\text{area}(Q) < \text{area}(Q')$ but the four pieces $X_1, \ldots, X_4$ making up the rest of $P$ just are rotated without their area changing.

For the second lemma, say that a simple closed curve in $\mathbb{R}^2$ is convex if it encloses a convex region.

**Lemma 2.** Let $\gamma$ be a simple closed curve in $\mathbb{R}^2$ and let $\gamma'$ be the boundary of the convex hull of the closed region enclosed by $\gamma$. Then $\gamma'$ is a convex simple closed curve satisfying $\text{length}(\gamma') \leq \text{length}(\gamma)$.

**Proof.** Parameterize $\gamma$ as $f : [0, 1] \to \mathbb{R}^2$ and define $\Lambda = f^{-1}(\gamma \cap \gamma')$. Choose $f$ such that $f(0) \in \gamma'$, and hence $0, 1 \in \Lambda$. The set $\Lambda$ is nonempty and closed, so its complement consists of at most countably many disjoint open intervals $\{I_\alpha\}_{\alpha \in A}$. For $\alpha \in A$, write $\partial I_\alpha = \{x_\alpha, y_\alpha\} \subset \Lambda$ with $x_\alpha < y_\alpha$. Define $f' : [0, 1] \to \mathbb{R}^2$ to equal $f$ on $\Lambda$ and to parameterize a straight line from $f(x_\alpha)$ to $f(y_\alpha)$ on $I_\alpha$ for all $\alpha \in A$. The function $f$ is then a parameterization of $\gamma'$. Let $\mathcal{P}$ be the set of all partitions of $[0, 1]$. For $P \in \mathcal{P}$ written as $0 = t_0 < t_1 < \cdots < t_k = 1$, define

$$\ell(f, P) = \sum_{i=1}^{k} d_{\mathbb{R}^2}(f(t_{i-1}), f(t_i))$$

and

$$\ell(f', P) = \sum_{i=1}^{k} d_{\mathbb{R}^2}(f'(t_{i-1}), f'(t_i)).$$

Our goal is to show that $\sup_{P \in \mathcal{P}} \ell(f', P) \leq \sup_{P \in \mathcal{P}} \ell(f, P)$.

Define $\mathcal{P}_1$ to be the set of partitions $P$ of $[0, 1]$ such that if a point of $I_\alpha$ appears in $P$ for some $\alpha \in A$, then both $x_\alpha$ and $y_\alpha$ appear in $P$. Since every partition can be refined to a partition in $\mathcal{P}_1$, we have

$$\sup_{P \in \mathcal{P}_1} \ell(f', P) = \sup_{P \in \mathcal{P}} \ell(f', P). \quad (0.3)$$

Next, define $\mathcal{P}_2$ to be the set of partitions $P$ of $[0, 1]$ that contain no points of $I_\alpha$ for any $\alpha \in A$. For $P \in \mathcal{P}_1$, define $\hat{P} \in \mathcal{P}_2$ to be the result of deleting all points that lie in $I_\alpha$ for some $\alpha \in A$. The key observation is that

$$\ell(f', P) = \ell(f', \hat{P}) = \ell(f, \hat{P}) \quad (P \in \mathcal{P}_1).$$
This implies that
\[
\sup_{P \in \mathcal{P}_1} \ell(f', P) = \sup_{P \in \mathcal{P}_2} \ell(f', P) = \sup_{P \in \mathcal{P}_2} \ell(f, P) \leq \sup_{P \in \mathcal{P}} \ell(f, P). \tag{0.4}
\]
Combining (0.3) and (0.4), the lemma follows. \hfill \Box

**Lemma 3.** Let \( \gamma \) be a convex simple closed curve in \( \mathbb{R}^2 \). Then for all \( \epsilon > 0 \), there exists a polygon \( P \) inscribed in \( \gamma \) satisfying \( \text{area}(P) > \text{area}(\gamma) - \epsilon \).

**Proof.** Translating \( \gamma \), we can assume that 0 lies in its interior. For \( 0 < \delta < 1 \), define \( \gamma_\delta = \{ \delta \cdot x \mid x \in \gamma \} \). Then \( \gamma_\delta \) is a convex simple closed curve contained in the interior of the region bounded by \( \gamma \) satisfying
\[
\text{area}(\gamma_\delta) = \delta^2 \cdot \text{area}(\gamma).
\]
Choose \( \delta \) sufficiently close to 1 such that \( \text{area}(\gamma_\delta) > \text{area}(\gamma) - \epsilon \). We can then find a polygon \( P \) inscribed in \( \gamma \) such that \( \gamma_\delta \) lies in the interior of \( P \), and hence \( \text{area}(P) > \text{area}(\gamma_\delta) > \text{area}(\gamma) - \epsilon \). \hfill \Box

**Proof of the isoperimetric inequality.** The theorem is trivial if \( \text{length}(\gamma) = \infty \), so assume without loss of generality that \( \text{length}(\gamma) < \infty \). Assume first that \( \gamma \) is not convex. Let \( \gamma' \) be the boundary of the convex hull of the region bounded by \( \gamma \). Lemma 2 says that \( \text{length}(\gamma') \leq \text{length}(\gamma) \), and it is clear that \( \text{area}(\gamma') > \text{area}(\gamma) \). It is therefore enough to prove the theorem for \( \gamma' \). Replacing \( \gamma \) with \( \gamma' \), we can therefore assume that \( \gamma \) is convex.

Fix some \( \epsilon > 0 \). Use Lemma 3 to find a polygon \( P \) inscribed in \( \gamma \) such that \( \text{area}(P) > \text{area}(\gamma) - \epsilon \). Since \( P \) is inscribed in \( \gamma \), we have \( \text{length}(P) \leq \text{length}(\gamma) \). Lemma 1 ensures that there exists a cyclic polygon \( P' \) with the same cyclically ordered side lengths as \( P \) satisfying \( \text{area}(P') \geq \text{area}(P) \). Let \( C \) be the circle in which \( P' \) is inscribed. Since \( P' \) is inscribed in \( C \), we have \( \text{area}(P') < \text{area}(C) \). Adding more vertices to \( P \), we can ensure that \( \text{length}(P') > \text{length}(C) - \epsilon \). We now combine all of the our estimates to deduce that
\[
\text{area}(\gamma) < \text{area}(P) + \epsilon \leq \text{area}(P') + \epsilon < \text{area}(C) + \epsilon
\]
\[
= \frac{1}{4\pi} \text{length}(C)^2 + \epsilon < \frac{1}{4\pi} \left( \text{length}(P') + \epsilon \right)^2 + \epsilon
\]
\[
= \frac{1}{4\pi} \left( \text{length}(P) + \epsilon \right)^2 + \epsilon \leq \frac{1}{4\pi} \left( \text{length}(\gamma) + \epsilon \right)^2 + \epsilon.
\]
Since \( \text{area}(\gamma) < \frac{1}{4\pi} \left( \text{length}(\gamma) + \epsilon \right)^2 + \epsilon \) for all \( \epsilon > 0 \), we conclude that \( \text{area}(\gamma) \leq \frac{1}{4\pi} \text{length}(\gamma) \), as desired.

To finish the proof, we must show that \( \text{area}(\gamma) < \frac{1}{4\pi} \text{length}(\gamma) \) when \( \gamma \) (still assumed to be convex) is not a round circle. Since \( \gamma \) is not a round circle, we can find four points \( q_1, \ldots, q_4 \in \gamma \) that do not lie on a circle. Let \( Q \) be the quadrilateral inscribed in \( \gamma \) with the vertices \( q_1, \ldots, q_4 \). By Lemma 1, we can find a cyclic quadrilateral \( Q' \) with the same side lengths as \( Q \) but with
Figure 3: Just like in the second step of the proof of Lemma 1, we change $Q$ to $Q'$ without changing the length of $\gamma$; each of the four shaded regions is merely rotated and glued onto $Q'$.

area($Q'$) > area($Q$). Just like in Case 2 of the proof of Lemma 1, we can use $Q'$ to find a simple closed curve $\gamma'$ with $\text{length}(\gamma') = \text{length}(\gamma)$ but with $\text{area}(\gamma') > \text{area}(\gamma)$ (see Figure 3). This implies that

$$\text{area}(\gamma) < \text{area}(\gamma') \leq \frac{1}{4\pi} \text{length}(\gamma')^2 = \frac{1}{4} \text{length}(\gamma)^2,$$

as desired. \qed

References


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