The Jacobson density theorem

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Abstract

We prove the Jacobson density theorem concerning simple modules over rings. This implies, for instance, that if G is a group and V is a finite-dimensional irreducible complex representation of G, then the natural map $\mathbb{C}[G] \to \mathrm{End}_{\mathbb{C}}(V)$ is surjective.

Let R be a ring, not necessarily commutative. For a simple left R-module M, Schur's Lemma implies that $D = \mathrm{End}_R(M)$ is a division ring. In this note, we prove the following theorem, which is known as the Jacobson Density Theorem [1].

Theorem A. *Let R be a ring and let M be a simple left R-module. Set $D = \mathrm{End}_R(M)$. Let $x_1, \ldots, x_n \in M$ be linearly independent over D. Then for all $y_1, \ldots, y_n \in M$, there exists some $r \in R$ such that $r \cdot x_i = y_i$ for all $1 \leq i \leq n$.*

Here is an instructive special case. Assume that G is a group and that V is an n-dimensional irreducible representation of G over $\mathbb{C}$. Thus V is a simple left $\mathbb{C}[G]$-module, and Schur's Lemma implies that $\mathrm{End}_{\mathbb{C}[G]}(V) = \mathbb{C}$. Fixing a basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ for V as a complex vector space, the elements $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent over $\mathrm{End}_{\mathbb{C}[G]}(V)$, so Theorem A says that for all $\vec{w}_1, \ldots, \vec{w}_n \in V$, there exists some $\omega \in \mathbb{C}[G]$ such that $\omega \cdot \vec{v}_i = \vec{w}_i$ for all $1 \leq i \leq n$. In other words, the natural map

$$\mathbb{C}[G] \longrightarrow \mathrm{End}_{\mathbb{C}}(V) \cong \mathrm{Mat}_n(\mathbb{C})$$

is surjective.

Remark. For G finite, this follows from the Wedderburn Structure Theorem, which says that in that case $\mathbb{C}[G]$ is a product of matrix rings over $\mathbb{C}$. For G infinite, however, no such structure theorem is available.

Proof of Theorem A. The proof will be by induction on n. For the base case $n = 1$, we have $x_1 \neq 0$ since x_1 itself is linearly independent over D. Since M is simple, the R-orbit of x_1 must therefore be M, and in particular we can find some $r \in R$ such that $r \cdot x_1 = y_1$.

Assume now that $n > 1$ and that the theorem is true for all smaller n. Below we will prove the following:

(†) There exist $\lambda_1, \ldots, \lambda_n \in R$ such that $\lambda_i \cdot x_i \neq 0$ for all $1 \leq i \leq n$ and $\lambda_i \cdot x_j = 0$ for all distinct $1 \leq i, j \leq n$.

Before we do this, we explain why it implies the theorem. By the base case $n = 1$, for $1 \leq i \leq n$ we can find some $r_i \in R$ such that $r_i \lambda_i \cdot x_i = y_i$. Setting

$$r = r_1 \lambda_1 + \cdots + r_n \lambda_n \in R,$$

for $1 \leq i \leq n$ we then have

$$r \cdot x_i = (r_1 \lambda_1 + \cdots + r_n \lambda_n) \cdot x_i = r_i \lambda_i \cdot x_i = y_i,$$

as desired.

It remains to prove (†). To keep the notation from getting out of hand, we will show how to construct λ_n. Assume to the contrary that the desired λ_n does not exist. What this means is that

(††) if $\lambda \in R$ satisfies $\lambda \cdot x_i = 0$ for $1 \leq i \leq n-1$, then $\lambda \cdot x_n = 0$.
We now define an R-linear map $\phi\colon M^{n-1} \to M$ as follows. Consider $(z_1, \ldots, z_{n-1}) \in M^{n-1}$. By our inductive hypothesis, there exists some $a \in R$ such that $a \cdot x_i = z_i$ for $1 \leq i \leq n-1$. We then define

$$\phi(z_1, \ldots, z_{n-1}) = a \cdot x_n.$$

Of course, this depends a priori on the choice of a, but if $a' \in R$ also satisfies $a' \cdot x_i = z_i$ for $1 \leq i \leq n-1$, then $(a - a') \cdot x_i = 0$ for $1 \leq i \leq n-1$, so (††) implies that $(a - a') \cdot x_n = 0$, and thus $a \cdot x_n = a' \cdot x_n$. It follows that ϕ is well-defined.

For $1 \leq i \leq n-1$, define $\zeta_i \in \mathrm{End}_R(M)$ to be the composition

$$M \hookrightarrow M^{n-1} \xrightarrow{\phi} M,$$

where the first inclusion is the inclusion into the i^{th} factor. For $z_1, \ldots, z_{n-1} \in M$, we thus have

$$\phi(z_1, \ldots, z_{n-1}) = \zeta_1 \cdot z_1 + \cdots + \zeta_{n-1} \cdot z_{n-1}.$$

In particular, we have

$$x_n = 1 \cdot x_n = \phi(x_1, \ldots, x_{n-1}) = \zeta_1 \cdot x_1 + \cdots + \zeta_{n-1} \cdot z_{n-1}.$$

This contradicts the fact that the x_i are linearly independent over $D = \mathrm{End}_R(M)$. It follows that our assumption that λ_n does not exist is false, so it exists. $\square$

References

[1] N. Jacobson, Structure theory of simple rings without finiteness assumptions, Trans. Amer. Math. Soc. **57** (1945), 228–245.

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