The Noetherianity of group rings

Andrew Putman

Abstract

We discuss conditions under which the group ring of a group is and isn’t Noetherian. There are two main results. The first is a folklore theorem that says that if a group contains a non-finitely-generated subgroup, then its group ring is not Noetherian. The second is a theorem of Phillip Hall that says that group rings of virtually polycyclic groups are Noetherian.

Let $G$ be a group. One basic question about $G$ is whether or not its group ring $\mathbb{Z}[G]$ is (left) Noetherian, i.e. whether or not all submodules of finitely generated left $\mathbb{Z}[G]$-modules are themselves finitely generated. This is trivially true for finite $G$, but is quite subtle for infinite $G$. The first main result about this is the following folklore result.

**Theorem 0.1.** Let $G$ be a group that contains a non-finitely-generated subgroup. Then $\mathbb{Z}[G]$ is not Noetherian.

Most infinite groups that one encounters satisfy the hypotheses of Theorem 0.1. For instance, any group that contains a nonabelian free subgroup satisfies the hypotheses of this theorem. This includes nonelementary hyperbolic groups, mapping class groups, automorphism groups of free groups, lattices in semisimple Lie groups, etc.

**Remark 0.2.** A group all of whose subgroups are finitely generated is called a Noetherian group. Theorem 0.1 implies that if the group ring $\mathbb{Z}[G]$ of $G$ is Noetherian, then $G$ is Noetherian. As far as I know, the converse is open. Interesting examples here are the Tarski monster groups constructed by Olshanskii [O] all of whose proper subgroups are cyclic. Though these groups are trivially Noetherian, I believe it is still open whether or not their group rings are Noetherian.

**Proof of Theorem 0.1.** Since $G$ contains a non-finitely-generated subgroup, there exists a strictly increasing chain

$$G_1 \subsetneq G_2 \subsetneq G_3 \subsetneq \cdots$$

(0.1)

of subgroups of $G$. For $k \geq 1$, define $I_k$ to be the kernel of the natural map $\mathbb{Z}[G] \to \mathbb{Z}[G/G_k]$ of left $\mathbb{Z}[G]$-modules. From (0.1), we see that the increasing sequence

$$I_1 \subset I_2 \subset I_3 \subset \cdots$$

of $\mathbb{Z}[G]$-submodules of $\mathbb{Z}[G]$ is strictly increasing, so $\mathbb{Z}[G]$ is not Noetherian.

The second main result is a theorem of Phillip Hall. Recall that a group $G$ is polycyclic if there exists a subnormal sequence

$$G = G_1 \triangleright G_2 \triangleright \cdots \triangleright G_n = 1$$

such that $G_k/G_{k+1}$ is cyclic for all $1 \leq k < n$. The minimal length of such a subnormal sequence is the Hirsch length of $G$. A group is virtually polycyclic if it has a polycyclic subgroup of finite index. Hall’s theorem is as follows.
Theorem 0.3 (Hall, [H]). Let $G$ be a virtually polycyclic group. Then the group ring $\mathbb{Z}[G]$ is Noetherian.

The proof of Theorem 0.3 uses ideas that are reminiscent of the theory of Gröbner bases.

Proof of Theorem 0.3. Since having a Noetherian group ring is preserved when you pass to a finite-index subgroup, we can assume that $G$ is polycyclic. The proof will be by induction on the Hirsch length of $G$. The base case is where the Hirsch length is 0, so $G=1$ and the result is trivial. Assume, therefore, that the Hirsch length of $G$ is positive and that the theorem is true whenever it is smaller. Let $M$ be a finitely generated $\mathbb{Z}[G]$-module and let $N\subset M$ be a submodule. Our goal is to prove that $N$ is finitely generated. Let $S$ be a finite generating set for $M$.

Let $H$ be a normal subgroup of $G$ such that $G/H$ is cyclic and the Hirsch length of $H$ is smaller than $G$. Our inductive hypothesis thus says that $\mathbb{Z}[H]$ is Noetherian. Let $x \in G$ be an element that projects to a generator for $G/H \cong \mathbb{Z}$. Define $M_H$ to be the $\mathbb{Z}[H]$-submodule of $M$ spanned by $S$ (which we recall generates $M$ as a $\mathbb{Z}[G]$-module). Every element $m \in M$ can be written as

$$m = \sum_{i=0}^{\infty} x^i c_i$$

with $c_i \in M_H$ and only finitely many $c_i$ nonzero. Of course, this expression is far from unique. If $c_i = 0$ for all $i < 0$, then we will say that this is a polynomial expression. For such a polynomial expression, the value $d = \max\{i \mid c_i \neq 0\}$ is the degree of the expression and $c_d$ is the expression’s leading coefficient.

Let $L_{N,k} \subset M_H$ be the union of $\{0\}$ and the set of leading coefficients of elements of $N$ that can be expressed as polynomials of degree $k$. I claim that $L_{N,k}$ is a $\mathbb{Z}[H]$-submodule of $M_H$. We must check two things:

- It is closed under sums. This is obvious.
- It is closed under multiplication by elements of $\mathbb{Z}[H]$. Consider some $h \in \mathbb{Z}[H]$ together with an element

$$n = \sum_{i=0}^{k} x^i c_i$$

of $N$ satisfying $c_k \neq 0$, so $c_k$ is an arbitrary element of $L_{N,k}$. We want to show that $hc_k \in L_{N,k}$. This is trivial if $hc_k = 0$, so assume that it is nonzero. We have

$$x^{k}hx^{-k} n = \sum_{i=0}^{k} x^i \left(x^{k-i}hx^{-k+i} c_i\right).$$

Since $H$ is a normal subgroup of $G$, we have $x^{k-i}hx^{-k+i} \in \mathbb{Z}[H]$, so $x^{k-i}hx^{-k+i}c_i \in M_H$. This is thus a polynomial expression. Its leading term is $x^{k-k}hx^{-k+k} c_k = hc_k \neq 0$, so this polynomial expression has degree $k$. It follows that $hc_k \in L_{N,k}$, as desired. It now follows from our induction hypothesis that each $L_{N,k}$ is a finitely generated $\mathbb{Z}[H]$-module. Moreover, since a polynomial expression of degree $k$ can be multiplied by $x$ to get a polynomial expression of degree $k + 1$, we see that

$$L_{N,0} \subset L_{N,1} \subset L_{N,2} \subset \cdots \subset M_H.$$ 

Again using our induction hypothesis, this increasing sequence of $\mathbb{Z}[H]$-submodules of $M_H$ must stabilize. We can thus choose a finite set $\{f_1, f_2, \ldots, f_r\}$ of elements of $N$ that for
all $i \geq 0$ contains a subset that can be expressed as polynomials of degree at most $i$ whose
leading terms generate $L_{N,i}$. Let $N'$ be the $\mathbb{Z}[G]$-submodule of $N$ generated by the $f_i$. We
claim that $N' = N$. Indeed, consider $n \in N$. For some $\ell \in \mathbb{Z}$, we can write

$$x^\ell n = \sum_{i=0}^{k} x^i c_i$$

with each $c_i \in M_H$. By subtracting appropriate multiples of the $f_i$ to first kill off the terms
of degree $k$, then the terms of degree $k - 1$, etc., we can reduce this to 0. It follows that
$x^\ell n \in N'$, and thus that $n \in N'$, as desired. \qed

References


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Andrew Putman
Department of Mathematics
University of Notre Dame
255 Hurley Hall
Notre Dame, IN 46556
andyp@nd.edu