The Noetherianity of group rings

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Abstract

We discuss conditions under with the group ring of a group is and isn't Noetherian. There are two main results. The first is a folklore theorem that says that if a group contains a non-finitely-generated subgroup, then its group ring is not Noetherian. The second is a theorem of Phillip Hall that says that group rings of virtually polycyclic groups are Noetherian.

Let G be a group. One basic question about G is whether or not its group ring $\mathbb{Z}[G]$ is (left) Noetherian, i.e. whether or not all submodules of finitely generated left $\mathbb{Z}[G]$-modules are themselves finitely generated. This is trivially true for finite G, but is quite subtle for infinite G. The first main result about this is the following folklore result.

Theorem 0.1. *Let G be a group that contains a non-finitely-generated subgroup. Then $\mathbb{Z}[G]$ is not Noetherian.*

Most infinite groups that one encounters satisfy the hypotheses of Theorem 0.1. For instance, any group that contains a nonabelian free subgroup satisfies the hypotheses of this theorem. This includes nonelementary hyperbolic groups, mapping class groups, automorphism groups of free groups, lattices in semisimple Lie groups, etc.

Remark 0.2. A group all of whose subgroups are finitely generated is called a *Noetherian group*. Theorem 0.1 implies that if the group ring $\mathbb{Z}[G]$ of G is Noetherian, then G is Noetherian. As far as I know, the converse is open. Interesting examples here are the Tarski monster groups constructed by Olshanskii [O] all of whose proper subgroups are cyclic. Though these groups are trivially Noetherian, I believe it is still open whether or not their group rings are Noetherian. □

Proof of Theorem 0.1. Since G contains a non-finitely-generated subgroup, there exists a strictly increasing chain

$$G_1 \subsetneq G_2 \subsetneq G_3 \subsetneq \cdots \tag{0.1}$$

of subgroups of G. For $k \geq 1$, define I_k to be the kernel of the natural map $\mathbb{Z}[G] \to \mathbb{Z}[G/G_k]$ of left $\mathbb{Z}[G]$-modules. From (0.1), we see that the increasing sequence

$$I_1 \subset I_2 \subset I_3 \subset \cdots$$

of $\mathbb{Z}[G]$-submodules of $\mathbb{Z}[G]$ is strictly increasing, so $\mathbb{Z}[G]$ is not Noetherian. □

The second main result is a theorem of Phillip Hall. Recall that a group G is *polycyclic* if there exists a subnormal sequence

$$G = G_1 \rhd G_2 \rhd \cdots \rhd G_n = 1$$

such that G_k/G_{k+1} is cyclic for all $1 \leq k < n$. The minimal length of such a subnormal sequence is the *Hirsch length* of G. A group is *virtually polycyclic* if it has a polycyclic subgroup of finite index. Hall's theorem is as follows.

Theorem 0.3 (Hall, [H])**.** *Let G be a virtually polycyclic group. Then the group ring $\mathbb{Z}[G]$ is Noetherian.*

The proof of Theorem 0.3 uses ideas that are reminiscent of the theory of Gröbner bases.

Proof of Theorem 0.3. Since having a Noetherian group ring is preserved when you pass to a finite-index subgroup, we can assume that G is polycyclic. The proof will be by induction on the Hirsch length of G. The base case is where the Hirsch length is 0, so $G = 1$ and the result is trivial. Assume, therefore, that the Hirsch length of G is positive and that the theorem is true whenever it is smaller. Let M be a finitely generated $\mathbb{Z}[G]$-module and let $N \subset M$ be a submodule. Our goal is to prove that N is finitely generated. Let S be a finite generating set for M.

Let H be a normal subgroup of G such that G/H is cyclic and the Hirsch length of H is smaller than G. Our inductive hypothesis thus says that $\mathbb{Z}[H]$ is Noetherian. Let $x \in G$ be an element that projects to a generator for $G/H \cong \mathbb{Z}$. Define M_H to be the $\mathbb{Z}[H]$-submodule of M spanned by S (which we recall generates M as a $\mathbb{Z}[G]$-module). Every element $m \in M$ can be written as

$$m = \sum_{i=-\infty}^{\infty} x^i c_i$$

with $c_i \in M_H$ and only finitely many c_i nonzero. Of course, this expression is far from unique. If $c_i = 0$ for all $i < 0$, then we will say that this is a *polynomial expression*. For such a polynomial expression, the value $d = \max\{i \mid c_i \neq 0\}$ is the *degree* of the expression and c_d is the expression's *leading coefficient*.

Let $L_{N,k} \subset M_H$ be the union of $\{0\}$ and the set of leading coefficients of elements of N that can be expressed as polynomials of degree k. I claim that $L_{N,k}$ is a $\mathbb{Z}[H]$-submodule of M_H. We must check two things:

- It is closed under sums. This is obvious.
- It is closed under multiplication by elements of $\mathbb{Z}[H]$. Consider some $h \in \mathbb{Z}[H]$ together with an element
$$n = \sum_{i=0}^{k} x^i c_i$$
of N satisfying $c_k \neq 0$, so c_k is an arbitrary element of $L_{N,k}$. We want to show that $hc_k \in L_{N,k}$. This is trivial if $hc_k = 0$, so assume that it is nonzero. We have
$$x^k h x^{-k} n = \sum_{i=0}^{k} x^i \left(x^{k-i} h x^{-k+i} c_i \right).$$
Since H is a normal subgroup of G, we have $x^{k-i} h x^{-k+i} \in \mathbb{Z}[H]$, so $x^{k-i} h x^{-k+i} c_i \in M_H$. This is thus a polynomial expression. Its leading term is $x^{k-k} h x^{-k+k} c_k = hc_k \neq 0$, so this polynomial expression has degree k. It follows that $hc_k \in L_{N,k}$, as desired.

It now follows from our induction hypothesis that each $L_{N,k}$ is a finitely generated $\mathbb{Z}[H]$-module. Moreover, since a polynomial expression of degree k can be multiplied by x to get a polynomial expression of degree $k+1$, we see that

$$L_{N,0} \subset L_{N,1} \subset L_{N,2} \subset \cdots \subset M_H.$$

Again using our induction hypothesis, this increasing sequence of $\mathbb{Z}[H]$-submodules of M_H must stabilize. We can thus choose a finite set $\{f_1, f_2, \ldots, f_r\}$ of elements of N that for all $i \geq 0$ contains a subset that can be expressed as polynomials of degree at most i whose leading terms generate $L_{N,i}$. Let N' be the $\mathbb{Z}[G]$-submodule of N generated by the f_i. We claim that $N' = N$. Indeed, consider $n \in N$. For some $\ell \in \mathbb{Z}$, we can write

$$x^\ell n = \sum_{i=0}^{k} x^i c_i$$

with each $c_i \in M_H$. By subtracting appropriate multiples of the f_i to first kill off the terms of degree k, then the terms of degree $k-1$, etc., we can reduce this to 0. It follows that $x^\ell n \in N'$, and thus that $n \in N'$, as desired. □

References

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