One-relator groups

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Abstract

We give a classically flavored introduction to the theory of one-relator groups. Topics include Magnus's Freiheitsatz, the solution of the word problem, the classification of torsion, Newman's Spelling Theorem together with the hyperbolicity (and thus solution to the conjugacy problem) for one-relator groups with torsion, and Lyndon's Identity Theorem together with the fact that the presentation 2-complex for a torsion-free one-relator group is aspherical.

1 Introduction

A one-relator group is a group with a presentation of the form $\langle S \mid r \rangle$, where r is a single element in the free group $F(S)$ on the generating set S. One fundamental example is a surface group $\langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$. These groups are basic examples in combinatorial/geometric group theory and possess an extensive literature. In these notes, we will discuss several important classical results about them. The outline is as follows.

- We begin in §2 by discussing Magnus's Freiheitsatz [Ma1], which he proved in his 1931 thesis. This theorem says that certain subgroups of one-relator groups are free. The techniques introduced in its proof ended up playing a decisive role in all subsequent work on the subject.
- In §3, we will show how to solve the word problem in one-relator groups. This was proved by Magnus [Ma2] soon after the Freiheitsatz.
- In §4, we will prove a theorem of Karrass, Magnus, and Solitar [KMaSo] that says that the only torsion in a one-relator group is the "obvious" torsion. In particular, if the relator r cannot be written as a proper power, then the group is torsion-free.
- In §5, we will prove Newman's Spelling Theorem [N], which implies that one-relator groups that contain torsion are hyperbolic. In particular, they have a solvable conjugacy problem. Whether or not torsion-free one-relator groups have a solvable conjugacy problem is a famous and difficult open question.
- In §6, we will prove a theorem of Cohen and Lyndon [CohL] that gives a basis for the relations in a one-relator group. A consequence is Lyndon's Identity Theorem [L], which says that the "relation module" of a one-relator group is cyclic.
- In §7, we will prove that the presentation 2-complex of a torsion-free one-relator group is aspherical. This is an important consequence of Lyndon's Identity Theorem and is implicit in Lyndon's work, though it was first noticed by Cockcroft [Coc].

Remark. Our exposition in §2–§5 is heavily influenced by the McCool and Schupp's elegant reworking of the theory in [McSchup].

Notation 1.1. Most of the time, we will confuse words in the generators of a group with the associated elements of the group. However, sometimes this will lead to confusion, so occasionally if w is a word in the generators of a group, then we will write $\overline{w}$ for the associated element of the group.

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2 Magnus's Freiheitsatz

If $G = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$ is a surface group and $H \subset G$ is the subgroup generated by a proper subset of $\{a_1, b_1, \ldots, a_g, b_g\}$, then H is the fundamental group of an infinite cover of a genus g surface. This infinite cover is a noncompact surface. Every non-compact smooth n-manifold deformation retracts onto an $(n-1)$-dimensional spine, so H is the fundamental group of a 1-dimensional CW complex, i.e. a graph. In other words H is a free group. The first important result about one-relator groups is the following theorem of Magnus [Ma1], which generalizes this.

Theorem 2.1 (Freiheitsatz). *Let $G = \langle S \mid r \rangle$ be a one-relator group and let $T \subset S$ be such that r cannot be written as a word in T. Then T is a basis for a free subgroup of G.*

This theorem appeared in Magnus's 1931 PhD thesis, which was supervised by Dehn. Though it might appear to be of specialized interest, the techniques introduced in its proof played a fundamental role in future work on one-relator groups and we will use them in all subsequent topics discussed in these notes. These proof techniques are algebraic and combinatorial, which displeased the geometer Dehn. In fact, Magnus relates the following anecdote in [Ma3]: "When told that the proof was purely algebraic, Dehn said: *Da sind Sie also blind gegangen* (So you proceeded with a blindfold over your eyes)."

Proof of Theorem 2.1. The proof will be by induction on the length of r. The base cases where r has length 0 or 1 are trivial, so assume that r has length at least 2 and that the theorem is true for all shorter relators. We now make several reductions:

- Without loss of generality, we can assume that r is cyclically reduced.
- Letting $S' \subset S$ be the set of letters that appear in r and $S'' = S \setminus S'$, we have
$$G = \langle S \mid r \rangle = \langle S' \mid r \rangle * F(S'').$$
Letting $T' = T \cap S'$ and $T'' = T \cap S''$, the subgroup of G generated by T is the free product of the subgroups generated by T' and T''. It follows that it is enough to prove the theorem for $T' \subset \langle S' \mid r \rangle$. In other words, we can assume without loss of generality that every element of S appears in r.
- If $|S| = 1$, then $T = \varnothing$ and the theorem is trivial. We thus can assume without loss of generality that $|S| > 1$. Since every element of S appears in r, this implies that r is not simply a power of a single generator.

For $t \in S$, define a homomorphism $\sigma_t \colon F(S) \to \mathbb{Z}$ via the formula
$$\sigma_t(s) = \begin{cases} 1 & \text{if } s = t, \\ 0 & \text{if } s \neq t \end{cases} \qquad (s \in S).$$

The proof divides into two cases.

Case 1. *There exists some $t \in S$ such that $\sigma_t(r) = 0$.*

If $t \in T$, then pick some $x \in S \smallsetminus T$. If $t \notin T$, then choose $x \in S \smallsetminus \{t\}$ arbitrarily. Cyclically conjugating r if necessary, we can assume that r begins with either x or x^{-1}.

We will show that G can be expressed as an HNN extension of a one-relator group $G' = \langle S' \mid r' \rangle$ whose relator r' is shorter than r. The letter t will be the stable letter of this HNN extension and the letter x will play an essential role to be described later. Define S'' to be the set of formal symbols $\{s_i \mid s \in S \smallsetminus \{t\},\ i \in \mathbb{Z}\}$. There is an isomorphism $\zeta\colon F(S'') \to \ker(\sigma_t)$ defined via the formula

$$\zeta(s_i) = t^i s t^{-i} \qquad (s \in S \smallsetminus \{t\}, i \in \mathbb{Z}).$$

Define $r' = \zeta^{-1}(r) \in F(S'')$. Since t appears in r, the word r' is shorter than r. Recall that we have chosen a special letter x above. Let $a \in \mathbb{Z}$ (resp. $b \in \mathbb{Z}$) be the smallest (resp. largest) integer such that x_a appears in r'. Since either x or x^{-1} is the first letter of r, we have $a \leq 0 \leq b$. Define

$$S' = \{s_i \mid s \in S \smallsetminus \{t\},\ i \in \mathbb{Z}, \text{ and } a \leq i \leq b \text{ if } s = x\} \subset S''$$

and $G' = \langle S' \mid r' \rangle$. The restriction of the isomorphism $\zeta\colon F(S'') \to \ker(\sigma_t)$ to $F(S')$ descends to a homomorphism $\eta\colon G' \to G$. Set

$$A = \{s_i \mid s \in S \smallsetminus \{t\},\ i \in \mathbb{Z}, \text{ and } a \leq i \leq b-1 \text{ if } s = x\}$$

and

$$B = \{s_i \mid s \in S \smallsetminus \{t\},\ i \in \mathbb{Z}, \text{ and } a+1 \leq i \leq b \text{ if } s = x\}.$$

Since r' is shorter than r, our inductive hypothesis says that the inclusion maps $F(A) \hookrightarrow F(S')$ and $F(B) \hookrightarrow F(S')$ descend to injective maps $F(A) \hookrightarrow G'$ and $F(B) \hookrightarrow G'$. There is an isomorphism $\phi\colon F(A) \to F(B)$ that takes $s_i \in A$ to $s_{i+1} \in B$. We can therefore form the HNN extension

$$\widehat{G} := G' *_{\phi}.$$

Letting t be the stable letter of $\widehat{G}$, we can define homomorphisms $\widehat{\eta}\colon \widehat{G} \to G$ and $\theta\colon G \to \widehat{G}$ as follows.

- The homomorphism $\eta\colon G' \to G$ extends to a homomorphism $\widehat{\eta}\colon \widehat{G} \to G$ with $\widehat{\eta}(t) = t$.
- The homomorphism θ is defined via the formula

$$\theta(s) = \begin{cases} t & \text{if } s = t, \\ s_0 & \text{if } s \neq t \end{cases} \qquad (s \in S).$$

It is clear that $\eta \circ \theta = \mathrm{id}$ and $\theta \circ \eta = \mathrm{id}$, so G is isomorphic to $\widehat{G}$, as claimed.

The proof of Case 1 now divides into two subcases.

Subcase 1.1. *We have $t \notin T$.*

This implies that $\theta(T)$ is a subset of the generating set S' for $G' \subset \widehat{G}$ and r' cannot be written as a word in $\theta(T)$. By our induction hypothesis, $\theta(T)$ is the basis for a free subgroup of G'. This completes the proof of Subcase 1.1.

Subcase 1.2. *We have $t \in T$.*

Consider $w \in F(T)$ with $w \neq 1$. Our goal is to show that $\overline{w} \neq 1$. If $\sigma_t(w) \neq 0$, then we are done; indeed, the fact that $\sigma_t(r) = 0$ implies that $\sigma_t\colon F(S) \to \mathbb{Z}$ descends to a homomorphism $G \to \mathbb{Z}$. We can therefore assume that $\sigma_t(w) = 0$. Recalling that ζ is the isomorphism between $F(S'')$ and $\ker(\sigma_t)$, define $w' = \zeta^{-1}(w)$. Since $x \in S \setminus T$, the element w' is a non-identity element of the free group on

$$C := \{s_i \mid s \in S \setminus \{x, t\},\ i \in \mathbb{Z}\} \subset S'.$$

Since either x or x^{-1} appears in r, the word r' does not lie in $F(C)$. Our inductive hypothesis therefore implies that C is a basis for a free subgroup of G'. In particular, the image of w' in G' is nontrivial. This implies that $\theta(\overline{w}) \neq 1$, so $\overline{w} \neq 1$, as desired. This completes the proof of Subcase 1.2 and thus also of Case 1.

Case 2. *We have $\sigma_t(r) \neq 0$ for all $t \in S$.*

Pick $t \in T$ and $x \in S \setminus T$. Set $\alpha = \sigma_t(r)$ and $\beta = \sigma_x(r)$, so $\alpha, \beta \neq 0$. Set $S_1 = S$ and define a homomorphism $\psi\colon F(S) \to F(S_1)$ via the formula

$$\psi(s) = \begin{cases} t^{\beta} & \text{if } s = t, \\ xt^{-\alpha} & \text{if } s = x, \\ s & \text{otherwise} \end{cases} \qquad (s \in S).$$

Let r_1 be the result of cyclically reducing $\psi(r)$ and $G_1 = \langle S_1 \mid r_1 \rangle$ and $T_1 = T$. The homomorphism ψ descends to a homomorphism $\overline{\psi}\colon G \to G'$. By construction, we have

$$\sigma_t(r_1) = \alpha\beta - \beta\alpha = 0.$$

The only problem is that r_1 is not shorter than r, but this only occurs because r_1 has some extra copies of the letter t. Apply the argument in Case 1 to $G_1 = \langle S_1 \mid r_1 \rangle$ and $T_1 \subset S_1$ (using our already-made choice of t and x, which are compatible with the requirements at the beginning of that case). The extra t-generators in r_1 disappear in the relator r_1' that appears in the HNN extension in that case, and thus r_1' is shorter than r and the induction goes through. We deduce that T_1 is a basis for a free subgroup of G_1. This implies that the set $T_2 := \{t^{\beta}, xt^{-\alpha}\} \cup (T_1 \setminus \{x, t\}$ is also a basis for a free subgroup of G_1. The homomorphism $\overline{\psi}\colon G \to G'$ takes T to T_2, so we conclude that T is a basis for a free subgroup of G, as desired. This completes the proof of Case 2 and thus also of Theorem 2.1. □

Throughout the rest of these notes, the homomorphisms σ_t will be defined like in the proof of Theorem 2.1. Almost all of our subsequent results will be proved following the outline of the proof above:

- First, will will prove them for relators r such that there exists some $t \in S$ such that $\sigma_t(r) = 0$. These arguments will use an HNN extension like in Case 1 above.
- Next, we will use a homomorphism like that used in Case 2 above to reduce the general case to the case where there exists some $t \in S$ such that $\sigma_t(r) = 0$.

To execute the second step, we will need the following lemma.

Lemma 2.2. *Let $G = \langle S \mid r\rangle$ be a one-relator group, let $t, x \in S$ be distinct elements, and let $\alpha, \beta \in \mathbb{Z}$ be nonzero. Set $S_1 = S$ and define a homomorphism $\psi: F(S) \to F(S_1)$ via the formula*

$$\psi(s) = \begin{cases} t^\beta & \text{if } s = t, \\ xt^{-\alpha} & \text{if } s = x, \\ s & \text{otherwise} \end{cases} \qquad (s \in S).$$

Finally, set $r_1 = \psi(r)$ and $G_1 = \langle S_1 \mid r_1\rangle$. Then the homomorphism $\overline{\psi}: G \to G_1$ induced by ψ is injective.

Proof. Set $S_2 = S$ and define $\psi_2: F(S) \to F(S_2)$ and $\psi_1: F(S_2) \to F(S_1)$ via the formulas

$$\psi_2(s) = \begin{cases} t^\beta & \text{if } s = t, \\ s & \text{otherwise} \end{cases} \qquad (s \in S)$$

and

$$\psi_1(s) = \begin{cases} xt^{-\alpha} & \text{if } s = x, \\ s & \text{otherwise} \end{cases} \qquad (s \in S_2).$$

We thus have $\psi = \psi_1 \circ \psi_2$. Define $r_2 = \psi_2(r)$ and $G_2 = \langle S_2 \mid r\rangle$, so ψ_2 and ψ_1 induce homomorphisms $\overline{\psi}_2: G \to G_2$ and $\overline{\psi}_1: G_2 \to G_1$ such that $\overline{\psi} = \overline{\psi}_1 \circ \overline{\psi}_2$. Since ψ_1 is an isomorphism, it follows that $\overline{\psi}_1$ is an isomorphism, so it is enough to prove that $\overline{\psi}_2$ is injective. The Freiheitsatz (Theorem 2.1) implies that t is an infinite-order element of G and G_2. Using this, it is clear that G_2 equals the free product of G and $\mathbb{Z}$ amalgamated along the subgroups $\langle t\rangle \subset G$ and $\beta \cdot \mathbb{Z} \subset \mathbb{Z}$. The homomorphism $\overline{\psi}_2$ is the inclusion of G into this amalgamated free product, which is necessarily injective. □

3 The word problem

Our next goal is to prove the following theorem of Magnus [Ma2].

Theorem 3.1. *The word problem is solvable in all one-relator groups.*

For surface groups, this was originally proved by Dehn. The modern understanding of Dehn's proof is that it exploits the fact that surface groups are hyperbolic. No such proof is available for general one-relator groups since they can be very far from hyperbolic. For example, $\mathbb{Z}^2 = \langle x, y \mid [x, y]\rangle$ is not hyperbolic. Even worse things than this can happen; indeed, hyperbolic groups can be characterized as those having linear isoperimetric inequalities, and it is known that one-relator groups can have isoperimetric inequalities that are even worse than exponential; see [P]. However, in §5 we will prove Newman's Spelling Theorem, which implies that one-relator groups with torsion are hyperbolic and possess a "Dehn-type" algorithm for solving the word problem.

Proof of Theorem 3.1. Let $G = \langle S \mid r\rangle$ be a one-relator group. We will prove that the word problem for G is solvable by induction on the length of r. Actually, to make our induction work we will have to prove a stronger theorem, namely that for all recursive subsets $T \subset S$ the generalized word problem for G with respect to T is solvable, i.e. there exists an algorithm to determine whether or not a word w lies in the subgroup generated by T. The ordinary word problem corresponds to the case where $T = \varnothing$.

The base cases where r has length 0 and 1 are trivial, so assume that r has length at least 2 and that the theorem is true for all shorter relators. We now make several reductions:

- Without loss of generality, we can assume that r is cyclically reduced.
- Letting $S' \subset S$ be the set of letters that appear in r and $S'' = S \smallsetminus S'$, we have

$$G = \langle S \mid r \rangle = \langle S' \mid r \rangle * F(S'').$$

  Letting $T' = T \cap S'$, it is enough to prove the theorem for $T' \subset \langle S' \mid r \rangle$. We can thus assume without loss of generality that every element of S appears in r.
- If $|S| = 1$, then r must be a nontrivial power of that generator and thus G must be a finite cyclic group. The theorem is trivial in this case, so we can assume without loss of generality that $|S| > 1$. Since every element of S appears in r, this implies that r is not simply a power of a single generator.

As in the proof of the Freiheitsatz (Theorem 2.1), the proof divides into two cases.

Case 1. *There exists some $t \in S$ such that $\sigma_t(r) = 0$.*

Just as in the proof of the Freiheitsatz (Theorem 2.1), this implies that the following exist:

- a one-relator group $G' = \langle S' \mid r' \rangle$ with r' shorter than r, and
- sets $A, B \subset S'$ that form bases for free subgroups $F(A), F(B) \subset G'$, and
- an isomorphism $\phi\colon F(A) \to F(B)$, and
- an isomorphism $\theta\colon G \to \widehat{G}$, where $\widehat{G} = G' *_{\phi}$.

The image $\theta(T) \subset \widehat{G}$ consists of a recursive subset of S' together with possibly the stable letter. Our inductive hypothesis implies that the generalized word problem for G' with respect to A and B is solvable. Using the usual normal form for elements of an HNN extension, this implies that the generalized word problem for $\widehat{G}$ with respect to $\theta(T)$ is solvable, and thus that the generalized word problem for G with respect to T is solvable, as desired. This completes the proof of Case 1.

Case 2. *We have $\sigma_t(r) \neq 0$ for all $t \in S$.*

The case where $T = S$ is trivial, so we can assume that $T \neq S$. We can thus choose $x \in S \smallsetminus T$ and $t \in S \smallsetminus \{x\}$. Set $\alpha = \sigma_t(r)$ and $\beta = \sigma_x(r)$, so $\alpha, \beta \neq 0$. As in the proof of Case 2 of the Freiheitsatz (Theorem 2.1), set $S_1 = S$ and define a homomorphism $\psi\colon F(S) \to F(S_1)$ via the formula

$$\psi(s) = \begin{cases} t^{\beta} & \text{if } s = t, \\ xt^{-\alpha} & \text{if } s = x, \\ s & \text{otherwise} \end{cases} \qquad (s \in S).$$

Also, set $r_1 = \psi(r)$ and $G_1 = \langle S_1 \mid r_1 \rangle$. We thus have $\sigma_t(r) = 0$, and by an argument similar to that in the proof of Case 2 of the Freiheitsatz (Theorem 2.1) we can use our inductive hypothesis to show that the theorem holds for G_1. The homomorphism ψ induces a homomorphism $\overline{\psi}\colon G \to G_1$ which Lemma 2.2 says is injective.

The proof of Case 2 now divides into two subcases.

Subcase 2.1. *We have $t \notin T$.*

This implies that $\overline{\psi}(T)$ is a recursive subset of S_1, so we can solve the generalized word problem for G_1 with respect to $\overline{\psi}(T)$. Since $\overline{\psi}$ is injective, this implies that we can solve the generalized word problem for G with respect to T, as desired. This completes the proof of Subcase 2.1.

Subcase 2.2. *We have $t \in T$.*

This implies that $\overline{\psi}(T) = T_1 \cup \{t^\beta\}$, where T_1 is a recursive subset of S_1. We can solve the generalized word problem for G_1 with respect to $T_1 \cup \{t\}$. Also, by the Freiheitsatz (Theorem 2.1) the set $T_1 \cup \{t\}$ is the basis for a free subgroup of G_1. We can solve the generalized word problem for $F(T_1 \cup \{t\})$ with respect to $T_1 \cup \{t^\beta\}$. We deduce that we can solve the generalized word problem for G_1 with respect to $\overline{\psi}(T) = T_1 \cup \{t^\beta\}$. Since $\overline{\psi}$ is injective, this implies that we can solve the generalized word problem for G with respect to T, as desired. This completes the proof of Subcase 2.2 and thus the proof of Case 2, which itself completes the proof of Theorem 3.1. □

4 Torsion

Our next goal is prove the following theorem of Karrass, Magnus, and Solitar [KMaSo] which characterizes the torsion in a one-relator group.

Theorem 4.1. *For some $k \geq 1$, let $G = \langle S \mid r^k \rangle$ be a one-relator group such that $r \in F(S)$ is not a proper power. Then $\overline{r} \in G$ has order k and every torsion element of G is conjugate to a power of $\overline{r}$.*

The following is an immediate corollary.

Corollary 4.2. *Let $G = \langle S \mid r \rangle$ be a one-relator group such that $r \in F(S)$ is not a proper power. Then G is torsion free.*

This implies in particular that surface groups are torsion-free. Probably the easiest way to see that surface groups are torsion-free is to observe that surfaces of positive genus are aspherical, so surface groups have finite cohomological dimension and hence are torsion-free. Generalizing the fact that surfaces of positive genus are aspherical, we will prove in §7 below that the presentation 2-complexes of one-relator groups whose relators are not proper powers are aspherical. This will provide an alternate proof of Corollary 4.2.

Proof of Theorem 4.1. We will prove this by induction on the length of r. The base cases where r has length 0 and 1 are trivial, so assume that r has length at least 2 and that the theorem is true for all shorter r. We now make several reductions:

- Without loss of generality, we can assume that r is cyclically reduced.
- Letting $S' \subset S$ be the set of letters that appear in r and $S'' = S \setminus S'$, we have

$$G = \langle S \mid r \rangle = \langle S' \mid r \rangle * F(S'').$$

  All torsion elements of G must be conjugate to torsion elements of $\langle S' \mid r \rangle$. It is thus enough to prove the theorem for $\langle S' \mid r \rangle$. In other words, we can assume without loss of generality that every element of S appears in r.

- If $|S| = 1$, then r must be a nontrivial power of that generator and thus G must be a finite cyclic group. The theorem is trivial in this case, so we can assume without loss of generality that $|S| > 1$. Since every element of S appears in r, this implies that r is not simply a power of a single generator.

As in the proof of the Freiheitsatz (Theorem 2.1), the proof divides into two cases.

Case 1. *There exists some $t \in S$ such that $\sigma_t(r) = 0$.*

Just as in the proof of the Freiheitsatz (Theorem 2.1), this implies that the following exist:

- a one-relator group $G' = \langle S' \mid (r')^k \rangle$ with r' shorter than r and r' not a proper power, and
- sets $A, B \subset S'$ that form bases for free subgroups $F(A), F(B) \subset G'$, and
- an isomorphism $\phi: F(A) \to F(B)$, and
- an isomorphism $\theta: G \to \widehat{G}$, where $\widehat{G} = G' *_{\phi}$. This isomorphism takes $\overline{r}$ to $\overline{r}'$.

By induction, $\overline{r}' \in G'$ has order k and every torsion element of G' is conjugate to a power of $\overline{r}$. Standard results about HNN extensions (proved, for instance, with Bass-Serre theory) show that every torsion element of $\widehat{G}$ is conjugate to a torsion element of G'. Case 1 follows.

Case 2. *We have $\sigma_t(r) \neq 0$ for all $t \in S$.*

Choose distinct $x, t \in S$ and set $\alpha = \sigma_t(r)$ and $\beta = \sigma_x(r)$, so $\alpha, \beta \neq 0$. As in the proof of Case 2 of the Freiheitsatz (Theorem 2.1), set $S_1 = S$ and define a homomorphism $\psi: F(S) \to F(S_1)$ via the formula

$$\psi(s) = \begin{cases} t^{\beta} & \text{if } s = t, \\ xt^{-\alpha} & \text{if } s = x, \\ s & \text{otherwise} \end{cases} \qquad (s \in S).$$

Also, set $r_1 = \psi(r)$ and $G_1 = \langle S_1 \mid r_1 \rangle$. We thus have $\sigma_t(r) = 0$, and by an argument similar to that in the proof of Case 2 of the Freiheitsatz (Theorem 2.1) we can use our inductive hypothesis to show that the theorem holds for G_1. The homomorphism ψ induces a homomorphism $\overline{\psi}: G \to G_1$ which Lemma 2.2 says is injective.

This almost implies the theorem. The only thing we must verify is that if $g_1, g_2 \in G$ are such that $\overline{\psi}(g_1)$ is G_1-conjugate to $\overline{\psi}(g_2)$, then g_1 is G-conjugate to g_2. As in the proof of Lemma 2.2, we can find a group G_2 together with homomorphisms $\overline{\psi}_2: G \to G_2$ and $\overline{\psi}_1: G_2 \to G_1$ such that the following hold.

- $\overline{\psi} = \overline{\psi}_1 \circ \overline{\psi}_2$.
- $\overline{\psi}_1$ is an isomorphism.
- G_2 equals the free product of G and $\mathbb{Z}$ amalgamated along the subgroups $\langle t \rangle \subset G$ and $\beta \cdot \mathbb{Z} \subset \mathbb{Z}$, and the homomorphism $\overline{\psi}_2$ is the inclusion of G into this amalgamated free product.

Since $\overline{\psi}(g_1)$ is conjugate to $\overline{\psi}(g_2)$ and $\overline{\psi}_1$ is an isomorphism, it follows that $\overline{\psi}_2(g_1)$ is conjugate to $\overline{\psi}(g_2)$. The usual structure theorems for HNN extensions then imply that g_1 is conjugate to g_2, as desired. This completes the proof of Case 2 and thus of Theorem 4.1. □

5 Newman's Spelling Theorem

Our next goal is to prove the following theorem of Newman [N].

Theorem 5.1 (Spelling Theorem)**.** *For some $k \geq 1$, let $G = \langle S \mid r^k \rangle$ be a one-relator group. Assume that r is cyclically reduced. Let $w \in F(S)$ be a freely reduced word such that $\overline{w} = 1$. Then w contains a subword u such that either u or u^{-1} is a subword of r^k and such that the length of u is strictly more than $\frac{k-1}{k}$ times the length of r^k.*

Of course, this only has content when $k > 1$, i.e. when G has torsion. In that case, Theorem 5.1 leads to a simple and elegant algorithm to solve the word problem. Consider a freely reduced word $w \in F(S)$.

Step 1. Check to see if w contains a subword u such that either u or u^{-1} is a subword of r^k and the length of u is more than $\frac{k-1}{k}$ times the length of r^k. If it does not, then $\overline{w} \neq 1$.

Step 2. If such a u is found, then we can rewrite the relation $r^k = 1$ in the form $u = u'$. Replace the subword u of w with u' and freely reduce the result. If this results in the trivial word, then $\overline{w} = 1$. Otherwise, go back to Step 1.

This terminates since $\frac{k-1}{k} \geq \frac{1}{2}$, so in Step 2 the word u' is strictly shorter than u and replacing the subword u of w by u' shortens w. This is exactly the kind of algorithm that Dehn discovered for surface groups. Its importance goes far beyond showing that the word problem can be solved quickly: one of the fundamental theorems concerning hyperbolic groups says that a group is hyperbolic if and only if it has an algorithm for solving the word problem that is similar to the one above. We therefore deduce the following.

Corollary 5.2. *All one-relator groups with torsion are hyperbolic.*

The conjugacy problem is solvable in hyperbolic groups, so we deduce the following.

Corollary 5.3. *All one-relator groups with torsion have solvable conjugacy problem.*

Remark. It is unknown whether or not general one-relator groups have solvable conjugacy problem.

Proof of Theorem 5.1. For the sake of our induction, we will prove that the conclusion of the theorem holds more generally for words w such that $\overline{w} = \overline{v}$ for some word $v \in F(S)$ that omits a letter appearing in w.

A bit of reflection shows that the following claim is equivalent to the desired conclusion:

- w contains a subword of the form $\rho^{k-1}\rho'$, where ρ is a cyclic conjugate of r and ρ' is a nontrivial initial segment of ρ.

We will prove this claim by induction on the length of r. The case where r has length 1 is trivial. We remark that in this case the claim is that w contains r^k or r^{-k} as a subword; it is instructive to meditate on where in the induction we "lose" part of the final r factor in r^k. We can thus assume that r has length at least 2 and that the theorem is true for all shorter relators. We now make several reductions:

- Letting $S' \subset S$ be the set of letters that appear in r and $S'' = S \smallsetminus S'$, we have
$$G = \langle S \mid r \rangle = \langle S' \mid r \rangle * F(S'').$$
The word w thus cannot contain any letters from S''. We can thus restrict our attention to $\langle S' \mid r \rangle$. In other words, we can assume without loss of generality that every element of S appears in r.

- If $|S| = 1$, then r must be a nontrivial power of that generator and thus G must be a finite cyclic group. The theorem is trivial in this case, so we can assume without loss of generality that $|S| > 1$. Since every element of S appears in r, this implies that r is not simply a power of a single generator.

As in the proof of the Freiheitsatz (Theorem 2.1), the proof now divides into two cases.

Case 1. *There exists some $t \in S$ that appears in r such that $\sigma_t(r) = 0$.*

Just as in the proof of the Freiheitsatz (Theorem 2.1), this implies that we can decompose G as an HNN extension $\widehat{G} = G' *_\phi$ of a one-relator group $G' = \langle S' \mid (r')^k \rangle$ such that r' is cyclically reduced and shorter than r'. Recall from the proof of Theorem 2.1 that G' has the following properties.

- For a distinguished element $x \in S \smallsetminus \{t\}$ and some $a \le 0 \le b$, the generating set S' consists of formal symbols
$$\{s_i \mid s \in S \smallsetminus \{t\},\ i \in \mathbb{Z}, \text{ and } a \le i \le b \text{ if } s = x\}.$$
For our proof, we choose the distinguished element x as follows. If t is the letter of w that is omitted in v, then choose x arbitrarily. Otherwise, choose x to be the letter of w that is omitted in v.

- Set
$$A = \{s_i \mid s \in S \smallsetminus \{t\},\ i \in \mathbb{Z}, \text{ and } a \le i \le b-1 \text{ if } s = x\}$$
and
$$B = \{s_i \mid s \in S \smallsetminus \{t\},\ i \in \mathbb{Z}, \text{ and } a+1 \le i \le b \text{ if } s = x\}.$$
Then A and B are bases for free subgroups of G'. The isomorphism used to define the HNN extension $\widehat{G}$ is the isomorphism $\phi\colon F(A) \to F(B)$ that takes $s_i \in A$ to $s_{i+1} \in B$.

- Let t be the stable letter of $\widehat{G}$. Define $\zeta\colon F(S' \cup \{t\}) \to F(S)$ via the formulas $\zeta(t) = t$ and
$$\zeta(s_i) = t^i s t^{-i} \qquad (s_i \in S').$$
Also, define $\iota\colon F(S) \to F(S' \cup \{t\})$ via the formulas $\iota(t) = t$ and
$$\iota(s) = s_0 \qquad (s \in S \smallsetminus \{t\}).$$
Then ζ and ι descend to isomorphisms $\widehat{\eta}\colon \widehat{G} \to G$ and $\theta\colon G \to \widehat{G}$ satisfying $\theta \circ \widehat{\eta} = \text{id}$ and $\widehat{\eta} \circ \theta = \text{id}$. Moreover, $\zeta(r') = r$.

Define $w_0 = \iota(w)$ and $v_0 = \iota(v)$, so $\overline{w}_0 = \overline{v}_0$. The proof of Claim 1 now divides into two subcases.

Subcase 1.1. *The letter of w that is omitted in v is t.*

This implies that t appears in w_0 but not in v_0. The usual normal form for HNN extensions implies that w_0 either contains a subword of the form $tw_0't^{-1}$ such that $\overline{w}_0'$ lies in the subgroup generated by A or a subword of the form $t^{-1}w_0't$ such that $\overline{w}_0'$ lies in the subgroup generated by B, and moreover in both cases the letter $t^{\pm 1}$ does not appear in w_0'. Of course, the word w_0' need not be itself a word in A or B; however, we will want to deal with this simple situation first.

This requires introducing some terminology. For words $u_0, u_1 \in F(S')$, say that u_1 is obtained from u_0 by *shifting subscripts* if u_1 can be obtained from u_0 by performing a sequence of the following two moves:

- replacing a subword $tu_0't^{-1}$ of u_0 such that $u_0' \in F(A)$ by the word u_0'' obtained by replacing each letter s_i in u_0 by s_{i+1}, and
- replacing a subword $t^{-1}u_0't$ of u_0 such that $u_0' \in F(B)$ by the word u_0'' obtained by replacing each letter s_i in u_0 by s_{i-1}.

These two moves are special cases of the relations in our HNN extension, so if u_1 is obtained from u_0 by shifting subscripts, then $\overline{u}_0 = \overline{u}_1$. In fact, something even stronger holds: recalling that $\zeta\colon F(S' \cup \{t\}) \to F(S)$ is the homomorphism defined above, we have $\zeta(u_0) = \zeta(u_1)$.

We now return to the above situation. Let $w_1 \in F(S')$ be the word obtained by shifting subscripts in w_0 as many times as possible (since each such shift deletes a t, this process has to stop). We thus have $\overline{w}_1 = \overline{w}_0 = \overline{v}_0$ and $\zeta(w_0) = \zeta(w_1)$. The proof of Subcase 1.1 now divides into two further sub-subcases.

Sub-Subcase 1.1.1. *The letter t does not appear in w_1.*

This implies that w_1 lies in the subgroup G' of $\widehat{G} = G' *_{\phi}$. Our standing assumption in Subcase 1.1 is that t does not appear in v, so v_0 also lies in G'. Finally, some letter s_i with $i \neq 0$ must appear in w_1; since such a letter cannot appear in v_0, we see in particular that v_0 omits some letter appearing in w_1.

Since G' is a one-relator group whose defining relation is shorter than that of G, we can thus apply our inductive hypothesis to the identity $\overline{w}_1 = \overline{v}_0$ to deduce that w_1 contains a subword of the form $\rho_1^{k-1}\rho_1'$, where ρ_1 is a cyclic conjugate of r' and ρ_1' is a nontrivial initial segment of ρ_1. The word $\zeta(\rho_1)$ is a cyclic conjugate of r and $\zeta(\rho_1')$ is a nontrivial initial segment of $\zeta(\rho_1)$. Moreover, the fact that r is cyclically reduced implies that $\zeta(\rho_1)^{k-1}\zeta(\rho_1')$ is a word of the form that we are looking for (i.e. that no cancellation occurs between the various factors in it).

Unfortunately, it is not necessarily the case that $\zeta(\rho_1)^{k-1}\zeta(\rho_1')$ is a subword of $w = \zeta(w_1)$. The problem is that some $t^{\pm 1}$ that the beginning or end of $\zeta(\rho_1)^{k-1}\zeta(\rho_1')$ might cancel in w. This can be solved as follows. Let ρ be the result of cyclically conjugating $\zeta(\rho_1)$ so as to move all the $t^{\pm 1}$ at the beginning of $\zeta(\rho_1)$ to the end and let ρ' be the result of deleting all $t^{\pm 1}$ from the beginning and end of $\zeta(\rho_1')$. Since $\zeta(\rho_1')$ must contain letters other than $t^{\pm 1}$, the word ρ' is a nontrivial initial segment of ρ. Moreover, a moment of thought shows that $\rho^{k-1}\rho'$ is a subword of w. This is the subword we are looking for; for future use, observe that $t^{\pm 1}$ is not the initial or final letter of $\rho^{k-1}\rho'$. This completes the proof of Sub-Subcase 1.1.1.

Sub-Subcase 1.1.2. *The letter t does appear in w_1.*

The usual normal form for HNN extensions implies that w_1 either contains a subword of the form $tw_1't^{-1}$ such that $\overline{w}_1'$ lies in the subgroup generated by A or a subword of the form

$t^{-1}w_1't$ such that $\overline{w}_1'$ lies in the subgroup generated by B, and moreover in both cases the letter $t^{\pm 1}$ does not appear in w_1'. Both of these cases are similar; for concreteness, we will assume that w_1 contains a subword of the form $tw_1't^{-1}$ such that $\overline{w}_1'$ lies in the subgroup generated by A and such that $t^{\pm 1}$ does not appear in w_1'. Since we cannot shift subscripts in w_1, it must be the case that w_1' is *not* a word in the letters A. We can find a word v_1' in the letters A such that $\overline{w}_1' = \overline{v}_1'$. Both w_1' and v_1' are words in S', so $\overline{w}_1' = \overline{v}_1'$ is an identity in G'. Since w_1' contains a letter that is not in A, we see that $\overline{v}_1'$ omits some letter that appears in $\overline{w}_1'$.

Just as in Sub-Subcase 1.1.1, we can now apply our inductive hypothesis to the identity $\overline{w}_1' = \overline{v}_1'$. Following the proof in that case, we find the desired subword $\rho^{k-1}\rho'$ of w_1, and thus of w. Moreover, just as in that case we can arrange for $t^{\pm 1}$ to not be the initial or final letter of $\rho^{k-1}\rho'$. This completes the proof of Sub-Subcase 1.1.2, and thus of Subcase 1.1.

Subcase 1.2. *The letter of w that is omitted in v is different from t.*

Recall from above that in this case the distinguished letter x that goes into the construction of G' is the letter of w that is omitted from v. The equation $\overline{w} = \overline{v}$ implies that $\sigma_t(w) = \sigma_t(v)$; let this common value be $\alpha \in \mathbb{Z}$. We then have $\overline{w_0 t^{-\alpha}} = \overline{v_0 t^{-\alpha}}$. Since $\sigma_t(vt^{-\alpha}) = 0$ and the letter x does not appear in v, it follows that we can shift subscripts in $v_0 t^{-\alpha}$ repeatedly to obtain a word v_1 in which the letter $t^{\pm 1}$ does not appear. We now apply the argument in Subcase 1.2 to the identity $\overline{w_0 t^{-\alpha}} = \overline{v}_1$. The result is a subword $\rho^{k-1}\rho'$ of $wt^{-\alpha}$ of the desired form. We now use the fact highlighted at the end of both sub-subcases of Subcase 1.2 that $t^{\pm 1}$ does not appear at the beginning or end of $\rho^{k-1}\rho'$ to deduce that in fact $\rho^{k-1}\rho'$ is a subword of w, as desired. This completes the proof of Subcase 1.2, and thus of Case 1.

Case 2. *We have $\sigma_t(r) \neq 0$ for all $t \in S$ that appear in r.*

Choose distinct $x, t \in S$ and set $\alpha = \sigma_t(r)$ and $\beta = \sigma_x(r)$, so $\alpha, \beta \neq 0$. As in the proof of Case 2 of the Freiheitsatz (Theorem 2.1), set $S_1 = S$ and define a homomorphism $\psi\colon F(S) \to F(S_1)$ via the formula

$$\psi(s) = \begin{cases} t^{\beta} & \text{if } s = t, \\ xt^{-\alpha} & \text{if } s = x, \\ s & \text{otherwise} \end{cases} \qquad (s \in S).$$

Also, set $r_1 = \psi(r)$ and $G_1 = \langle S_1 \mid r_1 \rangle$. We thus have $\sigma_t(r) = 0$, and by an argument similar to that in the proof of Case 2 of the Freiheitsatz (Theorem 2.1) we can use our inductive hypothesis to show that the theorem holds for G_1. This immediately implies that the theorem holds for G as well, completing the proof of Case 2 and thus also completing the proof of Theorem 5.1. □

6 Lyndon's Identity Theorem

We now turn to a theorem of Cohen and Lyndon [CohL] that identifies the "relations between relations" in a one-relator group. Fix a set S, some nontrivial $r \in F(S)$ that is not a proper power, and some $k \geq 1$. The set of relations in the one-relator group $\langle S \mid r^k \rangle$ is the normal closure $\langle\!\langle r^k \rangle\!\rangle$ of r^k in the free group $F(S)$. We wish to identify a basis for the free group $\langle\!\langle r^k \rangle\!\rangle$. The most naive thing to hope for is that there is a set $\Lambda \subset F(S)$ such that $\{xr^kx^{-1} \mid x \in \Lambda\}$ is a basis for $\langle\!\langle r^k \rangle\!\rangle$. If such a Λ existed, then it would have two properties.

- For $x \in \Lambda$, we have $xr^\ell \notin \Lambda$ for all nonzero $\ell \in \mathbb{Z}$. Indeed, $xr^kx^{-1} = (xr^\ell)r^k(xr^\ell)^{-1}$.
- For distinct $x, y \in \Lambda$, we have $xy^{-1} \notin \langle\langle r^k \rangle\rangle$. Indeed, if $xy^{-1} \in \langle\langle r^k \rangle\rangle$, then the purported basis elements xr^kx^{-1} and yr^ky^{-1} would be conjugate.

Cohen and Lyndon proved that such a Λ exists, and moreover it is maximal with respect to the above two properties.

Theorem 6.1 (Cohen–Lyndon, [CohL])**.** *For some $k \geq 1$, let $G = \langle S \mid r^k \rangle$ be a one-relator group such that $r \in F(S)$ is not a proper power. Then there exists a set Λ of $(\langle\langle r^k \rangle\rangle, \langle r \rangle)$-double coset representatives for $F(S)$ such that $\{xr^kx^{-1} \mid x \in \Lambda\}$ is a basis for $\langle\langle r^k \rangle\rangle$.*

Remark. If $k = 1$, then $\langle r \rangle \subset \langle\langle r^k \rangle\rangle$ and thus the set Λ given by Theorem 6.1 consists of a single element of $F(S)$ representing each element of G.

Before we prove this, we point out one important consequence. The *relation module* of a group G with a fixed presentation $G = \langle S \mid R \rangle$ is the abelian group $\frac{\langle\langle R \rangle\rangle}{[\langle\langle R \rangle\rangle, \langle\langle R \rangle\rangle]}$. The action of $F(S)$ on $\langle\langle R \rangle\rangle$ by conjugation descends to an action of G on the relation module, making it into a module over the group ring $\mathbb{Z}[G]$. The following theorem of Lyndon [L] identifies the relation module of a one-relator group. It will play an important role in our proof that the presentation 2-complex of a torsion-free one-relator group is aspherical.

Theorem 6.2 (Identity Theorem)**.** *For some $k \geq 1$, let $G = \langle S \mid r^k \rangle$ be a one-relator group such that $r \in F(S)$ is not a proper power. Then the relation module of G is isomorphic as a $\mathbb{Z}[G]$-module to the coset representation $\mathbb{Z}[G/\langle r \rangle]$. In particular, if $k = 1$ then the relation module is isomorphic to $\mathbb{Z}[G]$.*

Proof. Immediate from Theorem 6.1. □

We now prove Theorem 6.1. Our proof is not the original one; it is inspired by a later proof of Karrass–Solitar [KSo].

Proof of Theorem 6.1. We will prove this by induction on the length of r. The base case is already nontrivial, so we separate it out as a claim.

Claim. *The theorem holds when r has length 1.*

Proof of claim. Replacing r by its inverse if necessary, we can assume that $r \in S$. Define $S' = S \setminus \{r\}$, so we have a decomposition $F(S) = F(S') * \langle r \rangle$. The Bass–Serre tree T associated to this decomposition of $F(S)$ has two kinds of vertices:

- those with stabilizer a conjugate of $F(S')$, and
- those with stabilizer a conjugate of $\langle r \rangle$.

The vertices of the quotient of T by the subgroup $\langle\langle r^k \rangle\rangle$ are thus of two kinds:

- the images of the first kind of vertices, which are in bijection with the $(\langle\langle r^k \rangle\rangle, F(S'))$-double cosets for $F(S)$, and
- the image of the second kind of vertices, which are in bijection with the $(\langle\langle r^k \rangle\rangle, \langle\langle r \rangle\rangle)$-double cosets for $F(S)$.

The subgroup $\langle\langle r^k \rangle\rangle$ intersects all conjugates of $F(S')$ trivially and is generated by its intersection with the conjugates of $\langle r \rangle$. Bass–Serre theory thus tells us that there is a set Λ of $(\langle\langle r^k \rangle\rangle, \langle\langle r \rangle\rangle)$-double cosets representatives for $F(S)$ such that

$$\langle\langle r^k \rangle\rangle = \mathop{\ast}_{x \in \Lambda} \langle x r^k x^{-1} \rangle,$$

as desired. □

Now assume that r has length at least 2 and that the theorem is true for all shorter relators. We make several reductions:

- Without loss of generality, we can assume that r is cyclically reduced.
- If $|S| = 1$, then r must be a nontrivial power of that generator and thus G must be a finite cyclic group. The theorem is trivial in this case, so we can assume without loss of generality that $|S| > 1$.

As in the proof of the Freiheitsatz (Theorem 2.1), the proof now divides into two cases, though we modify things slightly since we are not assuming that every element of S appears in r.

Case 1. *There exists some $t \in S$ that appears in r such that $\sigma_t(r) = 0$.*

Just as in the proof of the Freiheitsatz (Theorem 2.1), this implies that we can decompose G as an HNN extension $\widehat{G} = G' \ast_\phi$ of a one-relator group $G' = \langle S' \mid (r')^k \rangle$ such that r' is cyclically reduced and shorter than r'. Recall from the proof of Theorem 2.1 that G' has the following properties.

- For a distinguished element $x \in S \setminus \{t\}$ (which in this proof we can choose arbitrarily) and some $a \leq 0 \leq b$, the generating set S' consists of formal symbols

$$\{s_i \mid s \in S \setminus \{t\},\ i \in \mathbb{Z}, \text{ and } a \leq i \leq b \text{ if } s = x\}.$$

- Set

$$A = \{s_i \mid s \in S \setminus \{t\},\ i \in \mathbb{Z}, \text{ and } a \leq i \leq b-1 \text{ if } s = x\}$$

  and

$$B = \{s_i \mid s \in S \setminus \{t\},\ i \in \mathbb{Z}, \text{ and } a+1 \leq i \leq b \text{ if } s = x\}.$$

  Then A and B are bases for free subgroups of G'. The isomorphism used to define the HNN extension $\widehat{G}$ is the isomorphism $\phi\colon F(A) \to F(B)$ that takes $s_i \in A$ to $s_{i+1} \in B$.

- Let t be the stable letter of $\widehat{G}$. Define $\zeta\colon F(S' \cup \{t\}) \to F(S)$ via the formulas $\zeta(t) = t$ and

$$\zeta(s_i) = t^i s t^{-i} \qquad (s_i \in S').$$

  Then $\zeta(r') = r$, and hence ζ descends to a homomorphism $\widehat{\eta}\colon \widehat{G} \to G$. In fact, $\widehat{\eta}$ is an isomorphism.

Of course, $F(A)$ and $F(B)$ are also subgroups of $F(S')$, so we can also form the HNN extension $\widehat{F} := F(S') \ast_\phi$. The generating set for $F(S')$ is $F(S' \cup \{t\})$ and the homomorphism

ζ respects the relations in $\widehat{F}$, so ζ descends to a homomorphism $\widehat{\zeta}\colon \widehat{F} \to F(S)$ that is easily seen to be an isomorphism. Summing up, we have a commutative diagram

$$\begin{tikzcd} \widehat{F} \arrow[r, "\widehat{\zeta}", "\cong"'] \arrow[d] & F(S) \arrow[d] \\ \widehat{G} \arrow[r, "\widehat{\eta}", "\cong"'] & G. \end{tikzcd}$$

We can thus work entirely in $\widehat{F}$. Writing $\langle\!\langle (r')^k \rangle\!\rangle_{\widehat{F}}$ for the normal closure of $(r')^k$ in $\widehat{F}$, our goal is to find a set Λ of $(\langle\!\langle (r')^k \rangle\!\rangle_{\widehat{F}}, \langle r' \rangle)$-double coset representatives for $\widehat{F}$ such that $\{x(r')^k x^{-1} \mid x \in \Lambda'\}$ is a basis for $\langle\!\langle (r')^k \rangle\!\rangle_{\widehat{F}}$.

To keep our notation straight, we will write $\langle\!\langle (r')^k \rangle\!\rangle_{F(S')}$ for the normal closure of $(r')^k$ in $F(S')$. By induction, we know that there exists a set Λ' of $(\langle\!\langle (r')^k \rangle\!\rangle_{F(S')}, \langle r' \rangle)$-double coset representatives for $F(S')$ such that $\{x(r')^k x^{-1} \mid x \in \Lambda'\}$ is a basis for $\langle\!\langle (r')^k \rangle\!\rangle_{F(S')}$. The vertices of the Bass–Serre tree for the HNN extension $\widehat{F}$ can be identified with the cosets $\widehat{F}/F(S')$. Examining the action of $\langle\!\langle (r')^k \rangle\!\rangle_{\widehat{F}}$ on this Bass–Serre tree, we see that there exists a set $\Omega \subset \widehat{F}$ of $(\langle\!\langle (r')^k \rangle\!\rangle_{\widehat{F}}, F(S'))$-double coset representatives such that

$$\langle\!\langle (r')^k \rangle\!\rangle_{\widehat{F}} = \mathop{\ast}_{x \in \Omega} x \langle\!\langle (r')^k \rangle\!\rangle_{F(S')} x^{-1}. \tag{6.1}$$

This is an ordinary free product with no amalgamation because $\langle\!\langle (r')^k \rangle\!\rangle_{F(S')}$ intersects the subgroup $F(A)$ of $\widehat{F}$ used to construct the HNN extension trivially (a consequence of the fact that $F(A)$ injects into the quotient group $G' = F(S')/\langle\!\langle (r')^k \rangle\!\rangle_{F(S')}$). Combining (6.1) with the defining property of Λ' above, we see that $\{(xy)(r')^k (xy)^{-1} \mid x \in \Omega,\ y \in \Lambda'\}$ is a basis for $\langle\!\langle (r')^k \rangle\!\rangle_{\widehat{F}}$. Setting $\Lambda := \{xy \mid x \in \Omega,\ y \in \Lambda'\}$, it remains to prove the following claim.

Claim. *The set Λ is a set of $(\langle\!\langle (r')^k \rangle\!\rangle_{\widehat{F}}, \langle r' \rangle)$-double coset representatives for $\widehat{F}$.*

Proof of claim. We first prove that Λ contains a representative of every $(\langle\!\langle (r')^k \rangle\!\rangle_{\widehat{F}}, \langle r' \rangle)$-double coset. Consider $z \in \widehat{F}$. Since Ω is a set of $(\langle\!\langle (r')^k \rangle\!\rangle_{\widehat{F}}, F(S'))$-double cosets for $\widehat{F}$, we can find $x \in \Omega$ and $a \in \langle\!\langle (r')^k \rangle\!\rangle_{\widehat{F}}$ and $b \in F(S')$ such that $z = axb$. Next, since Λ' is a set of $(\langle\!\langle (r')^k \rangle\!\rangle_{F(S')}, \langle r' \rangle)$-double cosets for $F(S')$, we can find $y \in \Lambda'$ and $c \in \langle\!\langle (r')^k \rangle\!\rangle_{F(S')}$ and $d \in \langle r' \rangle$ such that $b = cyd$. Combining these facts, we have

$$z = axcyd = (axcx^{-1})xy(d).$$

Since $axcx^{-1} \in \langle\!\langle (r')^k \rangle\!\rangle_{\widehat{F}}$, we conclude that z is in the same $(\langle\!\langle (r')^k \rangle\!\rangle_{\widehat{F}}, \langle r' \rangle)$-double coset as xy, as desired.

We now prove that no two distinct elements of Λ are in the same $(\langle\!\langle (r')^k \rangle\!\rangle_{\widehat{F}}, \langle r' \rangle)$-double coset. Consider $x_1, x_2 \in \Omega$ and $y_1, y_2 \in \Lambda'$ such that x_1y_1 and x_2y_2 are in the same $(\langle\!\langle (r')^k \rangle\!\rangle_{\widehat{F}}, \langle r' \rangle)$-double coset. There thus exists $e \in \langle\!\langle (r')^k \rangle\!\rangle_{\widehat{F}}$ and $f \in \langle r' \rangle$ such that $x_1y_1 = ex_2y_2f$. Rearranging this, we see that

$$x_2^{-1}e^{-1}x_1 = y_2fy_1^{-1} \in F(S').$$

It follows that x_1 and x_2 are in the same $(\langle\!\langle (r')^k \rangle\!\rangle_{\widehat{F}}, F(S'))$-double coset, so $x_1 = x_2$. This implies that $x_2^{-1}x^{-1}x_1 \in \langle\!\langle (r')^k \rangle\!\rangle_{\widehat{F}}$, so by the previous equation we see that $y_2fy_1^{-1} \in F(S') \cap \langle\!\langle (r')^k \rangle\!\rangle_{\widehat{F}} = \langle\!\langle (r')^k \rangle\!\rangle_{F(S')}$. Thus y_1 and y_2 are in the same $(\langle\!\langle (r')^k \rangle\!\rangle_{F(S')}, \langle r' \rangle)$-double coset, so $y_1 = y_2$, as desired. $\square$

This completes the proof of Case 1.

Case 2. *We have $\sigma_t(r) \neq 0$ for all $t \in S$ that appear in r.*

As in the proof of Case 2 of the Freiheitsatz (Theorem 2.1), we can embed $F(S)$ into a free group $F(S_1)$ such that the theorem is true for the one-relator group $G_1 = \langle S_1 \mid r_1 \rangle$. In fact, examining the proof of Lemma 2.2 we see that $F(S_1)$ is the free product of $F(S)$ and $\mathbb{Z}$ amalgamated along a common subgroup, and similarly for G_1. We will need one consequence of this free product with amalgamation description of $F(S_1)$ and G_1:

- For $x \in F(S_1)$, the group $xF(S)x^{-1}$ intersects $F(S)$ nontrivially if and only if $x \in F(S)$.

We now proceed with the proof. As notation, we will write $\langle\langle r^k \rangle\rangle_{F(S)}$ for the normal closure of r^k in $F(S)$ and $\langle\langle r^k \rangle\rangle_{F(S_1)}$ for the normal closure of r^k in $F(S_1)$. Since the theorem is true for G_1, there exists a set Λ_1 of $(\langle\langle r^k \rangle\rangle_{F(S_1)}, \langle r \rangle)$-double cosets for $F(S_1)$ such that

$$\langle\langle r^k \rangle\rangle_{F(S_1)} = \mathop{\ast}_{x \in \Lambda_1} \langle x r^k x^{-1} \rangle. \tag{6.2}$$

Let T be the Bass–Serre tree for $\langle\langle r^k \rangle\rangle_{F(S_1)}$ associated to the free product decomposition 6.2. The vertices of T are in bijection with the set

$$\bigsqcup_{x \in \Lambda_1} \langle\langle r^k \rangle\rangle_{F(S_1)} / \langle x r^k x^{-1} \rangle.$$

The orbits of this under the action of the subgroup $\langle\langle r^k \rangle\rangle_{F(S)}$ are in in bijection with the set

$$\bigsqcup_{x \in \Lambda_1} \langle\langle r^k \rangle\rangle_{F(S)} \backslash \langle\langle r^k \rangle\rangle_{F(S_1)} / \langle x r^k x^{-1} \rangle.$$

From the action of this subgroup, we get a free product decomposition of $\langle\langle r^k \rangle\rangle_{F(S)}$. Namely, for all $x \in \Lambda_1$ there exists a set Ω_x of $(\langle\langle r^k \rangle\rangle_{F(S)}, \langle x r^k x^{-1} \rangle)$-double coset representatives for $\langle\langle r^k \rangle\rangle_{F(S_1)}$ such that

$$\langle\langle r^k \rangle\rangle_{F(S)} = \mathop{\ast}_{x \in \Lambda_1, y \in \Omega_x} (\langle\langle r^k \rangle\rangle_{F(S)} \cap \langle (yx) r^k (yx)^{-1} \rangle).$$

By the "one consequence" listed above, the intersection $\langle\langle r^k \rangle\rangle_{F(S)} \cap \langle (yx) r^k (yx)^{-1} \rangle$ is nontrivial if and only if $yx \in \langle\langle r^k \rangle\rangle_{F(S)}$. We conclude that

$$\langle\langle r^k \rangle\rangle_{F(S)} = \mathop{\ast}_{\substack{x \in \Lambda_1, y \in \Omega_x \\ \text{s.t. } yx \in \langle\langle r^k \rangle\rangle_{F(S)}}} \langle (yx) r^k (yx)^{-1} \rangle.$$

It remains to prove the following claim.

Claim. *The set $\Lambda \coloneqq \{yx \mid x \in \Lambda_1,\ y \in \Omega_x, \text{ and } yx \in \langle\langle r^k \rangle\rangle_{F(S)}\}$ is a set of $(\langle\langle r^k \rangle\rangle_{F(S)}, \langle r \rangle)$-double cosets for $F(S)$.*

Proof of claim. Set $\Lambda' = \{yx \mid x \in \Lambda_1,\ y \in \Omega_x\}$. It is enough to prove that Λ' is a set of $(\langle\langle r^k \rangle\rangle_{F(S)}, \langle r \rangle)$-double cosets for $F(S_1)$. We begin by proving that Λ' contains a representative of every double coset. Consider $z \in F(S_1)$. Since Λ_1 is a set of $(\langle\langle r^k \rangle\rangle_{F(S_1)}, \langle r \rangle)$-double cosets for $F(S_1)$, we can find $x \in \Lambda_1$ and $a \in \langle\langle r^k \rangle\rangle_{F(S_1)}$ and $b \in \langle r \rangle$ such that $z = axb$. Next, since Ω_x is a set of $(\langle\!\langle r^k \rangle\!\rangle_{F(S)}, \langle x r^k x^{-1} \rangle)$-double cosets for $\langle\!\langle r^k \rangle\!\rangle_{F(S_1)}$, we can find $y \in \Omega_x$ and $c \in \langle\!\langle r^k \rangle\!\rangle_{F(S)}$ and $d \in \langle r^k \rangle$ such that $a = cy(xdx^{-1})$. Combining our two equalities, we see that

$$z = (cyxdx^{-1})xb = c(yx)(db),$$

which implies that z and $yx \in \Lambda'$ are in the same $(\langle\!\langle r^k \rangle\!\rangle_{F(S)}, \langle r \rangle)$-double coset, as desired.

We now prove that Λ' contains at most one representative of every double coset. Consider $x_1, x_2 \in \Lambda_1$ and $y_1 \in \Omega_{x_1}$ and $y_2 \in \Omega_{x_2}$ such that $y_1 x_1$ and $y_2 x_2$ are in the same $(\langle\!\langle r^k \rangle\!\rangle_{F(S)}, \langle r \rangle)$-double coset. We can thus find $e \in \langle\!\langle r^k \rangle\!\rangle_{F(S)}$ and $f \in \langle r \rangle$ such that

$$y_1 x_1 = e(y_2 x_2) f.$$

Rearranging this, we see that

$$y_2^{-1} e^{-1} y_1 = x_2 f x_1^{-1}.$$

Now, $y_2^{-1} e^{-1} y_1 \in \langle\!\langle r^k \rangle\!\rangle_{F(S_1)}$, and thus $x_2 f x_1^{-1} \in \langle\!\langle r^k \rangle\!\rangle_{F(S_1)}$. This implies that x_1 and x_2 are in the same $(\langle\!\langle r^k \rangle\!\rangle_{F(S_1)}, \langle r \rangle)$-double coset, and thus that $x_1 = x_2$. Set $x = x_1 = x_2$. Our goal now is to prove that $y_1 = y_2$. The first thing to observe is that

$$f = x^{-1}(y_2^{-1} e^{-1} y_1)x \in \langle\!\langle r^k \rangle\!\rangle_{F(S)}.$$

Since we already know that $f \in \langle r \rangle$, we see that we must have $f \in \langle r^k \rangle$. This implies that

$$y_2^{-1} e^{-1} y_1 = x f x^{-1} \in \langle x r^k x^{-1} \rangle.$$

We conclude that y_1 and y_2 are in the same $(\langle\!\langle r^k \rangle\!\rangle_{F(S_1)}, \langle x r^k x^{-1} \rangle)$-double coset, and thus that $y_1 = y_2$, as desired. $\square$

This completes the proof of Case 2 and thus of Theorem 6.1. $\square$

7 Presentation 2-complexes

Let G be a group with a fixed presentation $G = \langle S \mid R \rangle$. The *presentation 2-complex* of G is the CW complex X whose cells are as follows:

- a single 0-cell $*$, and
- a 1-cell e_s for each $s \in S$, and
- a 2-cell f_r for each $r \in R$ attached to the 1-skeleton according to the word r.

We thus have $\pi_1(X, *) \cong G$. If G is a surface group $G = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$, then X is the usual CW-complex structure on a genus g surface. A surface of positive genus is aspherical, and is thus an Eilenberg-MacLane space for a surface group. The following theorem generalizes this to other one-relator groups.

Theorem 7.1. *Let $G = \langle S \mid r \rangle$ be a one-relator group such that $r \in F(S)$ is not a proper power. Then the presentation 2-complex for G is aspherical, and thus is an Eilenberg-MacLane space for G.*

The condition that r is not a proper power is necessary; indeed, Theorem 4.1 says that if r can be expressed as a proper power, then G has nontrivial torsion and thus cannot have a finite-dimensional Eilenberg-MacLane space. The following is an immediate corollary of Theorem 7.1.

Corollary 7.2. *Torsion-free one-relator groups have cohomological dimension at most* 2.

As we will soon see, Theorem 7.1 is a consequence of the Identity Theorem (Theorem 6.2), and in fact is implicit in Lyndon's original paper [L]. However, it was first made explicit by Cockcroft [Coc]. For a more topological proof than the one we give, see [DV] (which actually derives the Identity Theorem from Theorem 7.1). Also, for a discussion of what other kinds of groups have aspherical presentation 2-complexes, see [ChColH].

Proof of Theorem 7.1. Let X be the presentation 2-complex for G and let $\widetilde{X}$ be its universal cover. Our goal is to show that $\widetilde{X}$ is contractible, i.e. that $\pi_k(\widetilde{X}) = 0$ for all $k \geq 1$. Since $\widetilde{X}$ is a simply-connected 2-dimensional CW complex, Hurewicz's theorem implies that it is enough to prove that $\mathrm{H}_2(\widetilde{X};\mathbb{Z}) = 0$. Let $C_*(\widetilde{X};\mathbb{Z})$ be the cellular chain complex of $\widetilde{X}$, which takes the form

$$0 \longrightarrow C_2(\widetilde{X};\mathbb{Z}) \xrightarrow{\partial_2} C_1(\widetilde{X};\mathbb{Z}) \xrightarrow{\partial_1} C_0(\widetilde{X};\mathbb{Z}) \longrightarrow 0.$$

We must show that ∂_2 is injective.

The action of G on $\widetilde{X}$ makes each $C_k(\widetilde{X};\mathbb{Z})$ into a $\mathbb{Z}[G]$-module. Fix a basepoint $\widetilde{*} \in \widetilde{X}^{(0)}$. Since there is exactly one G-orbit of 0-cells of $\widetilde{X}$, there is an isomorphism $\mathbb{Z}[G] \cong C_0(\widetilde{X};\mathbb{Z})$ that takes $\nu \in \mathbb{Z}[G]$ to $\nu \cdot \widetilde{*}$. For each $s \in S$, let $e_s \in X^{(1)}$ be the associated oriented loop in X and let $\widetilde{e}_s \in \widetilde{X}^{(1)}$ be the lift of e_s that starts at $\widetilde{*}$. Let $(\mathbb{Z}[G])^S$ be the set of tuples of $\mathbb{Z}[G]$ indexed by elements of S such that only finitely many entries are nonzero. Just like for $C_0(\widetilde{X};\mathbb{Z})$, there is an isomorphism $(\mathbb{Z}[G])^S \cong C_1(\widetilde{X};\mathbb{Z})$ that takes $(\nu_s)_{s\in S} \in (\mathbb{Z}[G])^S$ to $\sum_{s\in S} \nu_s \cdot \widetilde{e}_s$.

Let f be the 2-cell of X. Just like for the 0-cells and the 1-cells, there is an isomorphism $\mathbb{Z}[G] \cong C_2(\widetilde{X};\mathbb{Z})$ which depends on a choice lift of f. To pin down the lift we want to choose, we must make a quick digression into the Fox free differential calculus (see [F] for more details). Recall that for each $s \in S$, there is a uniquely defined function $\frac{\partial}{\partial s}\colon \mathbb{Z}[F(S)] \to \mathbb{Z}[F(S)]$ that satisfies the following conditions.

- The function $\frac{\partial}{\partial s}$ is $\mathbb{Z}$-linear.
- For all $w_1, \ldots, w_k \in F(S)$, we have
$$\frac{\partial}{\partial s}(w_1 \cdots w_k) = \frac{\partial}{\partial s}(w_1) + w_1 \frac{\partial}{\partial s}(w_2) + w_1 w_2 \frac{\partial}{\partial s}(w_3) + \cdots + w_1 w_2 \cdots w_{k-1} \frac{\partial}{\partial s}(w_k).$$
- For all $t \in S$, we have
$$\frac{\partial}{\partial s}(t) = \begin{cases} 1 & \text{if } t = s, \\ 0 & \text{if } t \neq s. \end{cases}$$

These conditions imply that
$$\frac{\partial}{\partial s}(s^{-1}) = -s^{-1}.$$

Meditating on these rules, we see that the "obvious" lift $\widetilde{f}$ of f to $\widetilde{X}$ satisfies the following condition (see Figure 1):

- Under our identification of $C_1(\widetilde{X};\mathbb{Z})$ with $(\mathbb{Z}[G])^S$, the boundary $\partial_2(\widetilde{f})$ of $\widetilde{f}$ has s-coordinate $\overline{\frac{\partial}{\partial s}(r)} \in \mathbb{Z}[G]$ for all $s \in S$, where the overline denotes the image of this element of $\mathbb{Z}[F(S)]$ in $\mathbb{Z}[G]$.

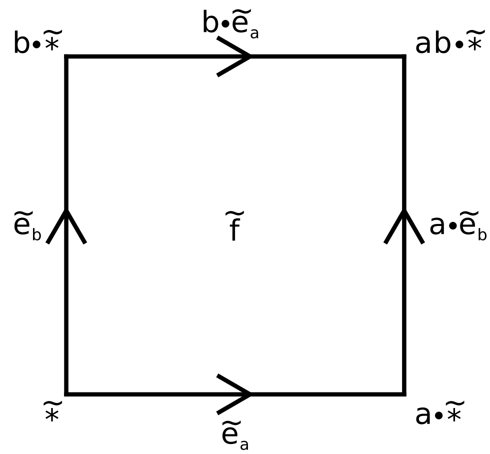

Figure 1: *For the group $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1}\rangle$, this is the lift $\widetilde{f}$ of the 2-cell of the presentation 2-complex. Its boundary is $\widetilde{e}_a + a\widetilde{e}_b - b\widetilde{e}_a - \widetilde{e}_b = (1-b)\widetilde{e}_a + (a-1)\widetilde{e}_b = \frac{\partial}{\partial a}(aba^{-1}b^{-1})\widetilde{e}_a + \frac{\partial}{\partial b}(aba^{-1}b^{-1})\widetilde{e}_b$.*

We now make the connection to the Identity Theorem (Theorem 6.2). Consider $\nu \in \ker(\partial_2)$. Write $\nu = \sum_{i=1}^{n} \epsilon_i \overline{x}_i$, where $\epsilon_i = \pm 1$ and $x_i \in F(S)$. Set

$$w = \prod_{i=1}^{n} x_i r^{\epsilon_i} x_i^{-1} \in F(S).$$

Using the fact that $\overline{x_i r^{\epsilon_i} x_i^{-1}} = 1$ for all i, we then have for all $s \in S$ that

$$\begin{aligned}
\overline{\frac{\partial}{\partial s}(w)} &= \sum_{i=1}^{n} \overline{\frac{\partial}{\partial s}(x_i r^{\epsilon_i} x_i^{-1})} \\
&= \sum_{i=1}^{n} \left(\overline{\frac{\partial}{\partial s}(x_i)} + \overline{x}_i \overline{\frac{\partial}{\partial s}(r^{\epsilon_i})} - \overline{x_i r^{\epsilon_i} x_i^{-1}}\,\overline{\frac{\partial}{\partial s}(x_i)} \right) \\
&= \left(\sum_{i=1}^{n} \epsilon_i \overline{x}_i \right) \frac{\partial}{\partial s}(r) \\
&= \nu \frac{\partial}{\partial s}(r).
\end{aligned}$$

This is the s-coordinate of $\partial_2(\nu)$, and thus vanishes. Since $\overline{\frac{\partial}{\partial s}(w)} = 0$ for all $s \in S$, we can apply a theorem of Schumann [Schum] and Blanchfield [B] (see [F, Theorem 4.9] for a modern proof) to deduce that $w \in [\langle\!\langle r \rangle\!\rangle, \langle\!\langle r \rangle\!\rangle]$. The Identity Theorem (Theorem 6.2) thus implies that the terms of ν must cancel in pairs, i.e. that $\nu = 0$, as desired. □

References

[B] R. C. Blanchfield, Applications of free differential calculus to the theory of groups, Senior Thesis, Princeton University, 1949.

[ChColH] I. M. Chiswell, D. J. Collins and J. Huebschmann, Aspherical group presentations, Math. Z. 178 (1981), no. 1, 1–36.

[Coc] W. H. Cockcroft, On two-dimensional aspherical complexes, Proc. London Math. Soc. (3) 4 (1954), 375–384.

[CohL] D. E. Cohen and R. C. Lyndon, Free bases for normal subgroups of free groups, Trans. Amer. Math. Soc. 108 (1963), 526–537.

[DV] E. Dyer and A. T. Vasquez, Some small aspherical spaces, J. Austral. Math. Soc. 16 (1973), 332–352.

[F] R. H. Fox, Free differential calculus. I. Derivation in the free group ring, Ann. of Math. (2) 57 (1953), 547–560.

[KMaSo] A. Karrass, W. Magnus and D. Solitar, Elements of finite order in groups with a single defining relation, Comm. Pure Appl. Math. 13 (1960), 57–66.

[KSo] A. Karrass and D. Solitar, On a theorem of Cohen and Lyndon about free bases for normal subgroups, Canad. J. Math. 24 (1972), 1086–1091.

[L] R. C. Lyndon, Cohomology theory of groups with a single defining relation, Ann. of Math. (2) 52 (1950), 650–665.

[Ma1] W. Magnus, Über diskontinuierliche Gruppen mit einer definierenden Relation. (Der Freiheitssatz), J. Reine Angew. Math. 163 (1930), 141–165.

[Ma2] W. Magnus, Das Identitätsproblem für Gruppen mit einer definierenden Relation, Math. Ann. 106 (1932), no. 1, 295–307.

[Ma3] W. Magnus, Max Dehn, Math. Intelligencer 1 (1978/79), no. 3, 132–143.

[McSchup] J. McCool and P. E. Schupp, On one relator groups and HNN extensions, J. Austral. Math. Soc. 16 (1973), 249–256.

[N] B. B. Newman, Some results on one-relator groups, Bull. Amer. Math. Soc. 74 (1968), 568–571

[P] A. N. Platonov. An isoperimetric function of the Baumslag–Gersten group. Vestnik Moskov. Univ. Ser. I Mat. Mekh., 3:12-17, 70, 2004. Translation in Moscow Univ. Math. Bull. 59 (2004).

[Schum] H.-G. Schumann, Über Moduln und Gruppenbilder, Math. Ann. 114 (1937), no. 1, 385–413.

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