# Algebraicity of matrix entries of representations

Andrew Putman

**Abstract**

We prove that with respect to an appropriate basis, the matrices associated to complex representations of finite groups have entries lying in algebraic number rings.

Fix a finite group $G$. All representations of $G$ discussed in this note are finite dimensional. Given an $n$-dimensional $\mathbb{C}$-representation $V$ of $G$, we will say that $V$ is defined over a subring $R$ of $\mathbb{C}$ if there is an $R[G]$-module $V'$ with the following two properties:

- $V'$ is a rank-$n$ free $R$-module.
- There is an isomorphism $V' \otimes_R \mathbb{C} \cong V$ of $\mathbb{C}[G]$-modules.

A more pedestrian way of saying this is that with respect an appropriate basis for $V$, the matrices representing the action of $G$ on $V$ have entries in $\mathrm{GL}_n(R) \subset \mathrm{GL}_n(\mathbb{C})$.

Recall that a number ring $\mathcal{O}_{\mathbf{k}}$ is the ring of integers in an algebraic number field $\mathbf{k}$, i.e. a finite extension of $\mathbb{Q}$. The goal of this note is to prove the following standard theorem.

**Theorem A.** *For all $\mathbb{C}$-representations $V$ of $G$, there exists a number ring $\mathcal{O}_{\mathbf{k}}$ such that $V$ is defined over $\mathcal{O}_{\mathbf{k}}$.*

*Proof.* The proof will have two steps.

**Step 1.** *There exists an algebraic number field $\mathbf{k}$ such that $V$ is defined over $\mathbf{k}$.*

We remark that Brauer proved that $V$ can be defined over a cyclotomic extension of $\mathbb{Q}$ containing the $m^{\text{th}}$ roots of unity for $m$ the least common multiple of the orders of the elements of $G$; see [1, Theorem 24]. We will give a much softer argument that does not give an explicit $\mathbf{k}$.

Since $G$ is finite, it is enough to show that $V$ is defined over $\overline{\mathbb{Q}}$. Without loss of generality, $V$ is irreducible. This implies that $V$ is isomorphic to a subrepresentation of the regular representation $\mathbb{C}[G]$. The regular representation $\overline{\mathbb{Q}}[G]$ over $\overline{\mathbb{Q}}$ can be decomposed as a direct sum of irreducible representations, and the only thing that might go wrong is that this decomposition might not be fine enough, i.e. there is an irreducible subrepresentation $W$ of $\overline{\mathbb{Q}}[G]$ such that $W \otimes \mathbb{C}$ is reducible and $V$ is isomorphic to a proper subrepresentation of $W \otimes \mathbb{C}$.

Assume that this happens. Let $n = \dim(V)$, and define

$$X = \{U \in \mathrm{Gr}_n(W) \mid g(U) = U \text{ for all } g \in G\}.$$

Thus $X$ is a closed subvariety of the Grassmannian $\mathrm{Gr}_n(W)$, which is a projective variety defined over $\overline{\mathbb{Q}}$. Our assumption that $W$ is irreducible implies that $X(\overline{\mathbb{Q}}) = \emptyset$, while our assumption that $W \otimes \mathbb{C}$ is reducible and $V$ is isomorphic to a proper subrepresentation of $W \otimes \mathbb{C}$ implies that $X(\mathbb{C}) \neq \emptyset$.

This is impossible. Indeed, if $Y \subset \mathbb{A}^m_{\overline{\mathbb{Q}}}$ is an open affine subset of $X$ defined by a radical ideal $I \subset \overline{\mathbb{Q}}[x_0, \ldots, x_m]$, then since $Y(\overline{\mathbb{Q}}) = \emptyset$ the Nullstellensatz says that $I = \overline{\mathbb{Q}}[x_0, \ldots, x_m]$. This is preserved when we extend scalars to $\mathbb{C}$, i.e. we have

$$Y(\mathbb{C}) = V(I \otimes \mathbb{C}) = V(\mathbb{C}[x_0, \ldots, x_m]) = \emptyset.$$

The desired result follows.

**Step 2.** *There exists a finite extension* $\mathbf{k}'$ *of* $\mathbf{k}$ *such that* $V$ *is defined over* $\mathcal{O}_{\mathbf{k}'}$.

By the previous step, there exists an action of $G$ on $\mathbf{k}^n$ such that $V \cong \mathbf{k}^n \otimes \mathbb{C}$ as $\mathbb{C}[G]$-modules. Let $L \subset \mathbf{k}^n$ be $\mathcal{O}_{\mathbf{k}}$-submodule spanned by the $G$-orbit of the standard lattice $\mathcal{O}_{\mathbf{k}}^n \subset \mathbf{k}^n$, so $L$ is a finitely generated $\mathcal{O}_{\mathbf{k}}$-submodule of $\mathbf{k}^n$ that is preserved by $G$ such that $L \otimes \mathbf{k} = \mathbf{k}^n$. If we had an $\mathcal{O}_{\mathbf{k}}$-module isomorphism $L \cong \mathcal{O}_{\mathbf{k}}^n$, then we would be done. Unfortunately, this need not be true; however, the classification of finitely generated modules over a Dedekind domain shows that

$$L \cong I_1 \oplus \cdots \oplus I_n$$

for some nonzero ideals $I_1, \ldots, I_n \subset \mathcal{O}_{\mathbf{k}}$.

The problem is that the $I_i$ might not be principal ideals, i.e. they might define nonzero elements $[I_i]$ of the class group $\mathrm{cl}(\mathcal{O}_{\mathbf{k}})$. The class group is a finite abelian group, so we can find $k_1, \ldots, k_n \geq 1$ such that $k_i[I_i] = 0$. Pick the $k_i$ to be the minimal such integers. By definition, the ideal $I_i^{k_i}$ is principal, i.e. there exists some $a_i \in \mathcal{O}_{\mathbf{k}}$ such that

$$I_i^{k_i} = (a_i).$$

Define

$$\mathbf{k}' = \mathbf{k}[a_1^{1/k_1}, \ldots, a_n^{1/k_n}],$$

so

$$I_i \otimes_{\mathcal{O}_{\mathbf{k}}} \mathcal{O}_{\mathbf{k}'} = (a_i^{1/k_i}) \subset \mathcal{O}_{\mathbf{k}'}.$$

Setting

$$L' = L \otimes_{\mathcal{O}_{\mathbf{k}}} \mathcal{O}_{\mathbf{k}'} \subset (\mathbf{k}')^n,$$

we then have

$$L' \cong (I_1 \otimes \mathcal{O}_{\mathbf{k}'}) \oplus \cdots \oplus (I_n \otimes \mathcal{O}_{\mathbf{k}'}) \cong \mathcal{O}_{\mathbf{k}'}^n.$$

We conclude that our original representation is defined over $\mathcal{O}_{\mathbf{k}'}$, as desired. □

## References


[1] J.-P. Serre, *Linear representations of finite groups*, translated from the second French edition by Leonard L. Scott, Springer-Verlag, New York, 1977.



Andrew Putman
Department of Mathematics
University of Notre Dame
255 Hurley Hall
Notre Dame, IN 46556
andyp@nd.edu