Representations of products

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Abstract

We give a character-theory free proof that for finite groups G and H, the irreducible representations of $G \times H$ over an algebraically closed field of characteristic 0 consist of tensor products of irreducible representations of G and H. Because our proof avoids character theory, it generalizes to many other similar situations.

The goal of this note is to prove the following theorem.

Theorem A. *Let G and H be finite groups and let $\mathbf{k}$ be an algebraically closed field of characteristic 0. The irreducible representations of $G \times H$ over $\mathbf{k}$ are precisely those of the form $V \otimes W$, where V and W are irreducible representations of G and H, respectively.*

Here is an example to show that the assumption that $\mathbf{k}$ is algebraically closed is necessary:

Example. Consider the action of $\mathbb{Z}/6$ on $\mathbb{R}^2$ where the generator acts as rotation by $e^{2\pi i/6}$. This is an irreducible representation; however, despite the fact that $\mathbb{Z}/6 \cong \mathbb{Z}/3 \times \mathbb{Z}/2$, it does not decompose into the tensor product of representations of $\mathbb{Z}/3$ and $\mathbb{Z}/2$.

The standard proof of Theorem A (say, from [2]) uses character theory. This has two downsides:

- It is indirect, and gives little insight into why the theorem is true.
- It does not generalize to other situations, for instance to rational representations of products of reductive Lie groups.

We will give a direct proof. The only special fact we will use about finite groups is Maschke's theorem saying that in characteristic 0, their representations decompose uniquely into direct sums of irreducible representations, so this proof applies in many other situations as well.

Proof of Theorem A. We divide the proof into two parts.

Claim 1. *Let V (resp. W) be an irreducible representation of G (resp. H) over $\mathbf{k}$. Then $V \otimes W$ is an irreducible representation of $G \times H$.*

By assumption, V and W are simple modules over the group rings $\mathbf{k}[G]$ and $\mathbf{k}[H]$, respectively. Since $\mathbf{k}$ is algebraically closed, we can apply the Jacobson Density Theorem (see, e.g. [1]) and see that the resulting ring maps $\phi\colon \mathbf{k}[G] \to \mathrm{End}_{\mathbf{k}}(V)$ and $\psi\colon \mathbf{k}[H] \to \mathrm{End}_{\mathbf{k}}(W)$ are surjections. It follows that the ring map

$$\mathbf{k}[G \times H] \cong \mathbf{k}[G] \otimes \mathbf{k}[H] \xrightarrow{\phi \otimes \psi} \mathrm{End}_{\mathbf{k}}(V) \otimes \mathrm{End}_{\mathbf{k}}(W) \cong \mathrm{End}_{\mathbf{k}}(V \otimes W)$$

is a surjection and thus that $V \otimes W$ is a simple $\mathbf{k}[G \times H]$-module, as desired.

Claim 2. *Assume that U is an irreducible representation of $G \times H$ over $\mathbf{k}$. Then $U \cong V \otimes W$, where V (resp. W) is an irreducible representation of G (resp. H) over $\mathbf{k}$.*

First regard U as a representation of H. As such, U decomposes as a direct sum of H-isotypic components. Since the action of G on U commutes with the action of H, it must preserve these isotypic components. Since U was assumed to be irreducible, it follows that U must have a single H-isotypic component, i.e. that $U \cong W^{\oplus m}$ for some irreducible H-representation W and some $m \geq 0$. Consider the $\mathbf{k}$-linear map

$$\Psi\colon \operatorname{Hom}_H(W, U) \otimes W \to U$$

defined via the formula $\Psi(\rho \otimes \vec{w}) = \rho(\vec{w})$. Since $U \cong W^{\oplus m}$, this map is surjective. Also, since $\mathbf{k}$ is algebraically closed we can apply Schur's Lemma to see that

$$\operatorname{Hom}_H(W, U) = \operatorname{Hom}_H(W, W^{\oplus m}) \cong \mathbf{k}^m.$$

We deduce that Ψ is a surjective linear map between vector spaces of the same dimension, so Ψ is an isomorphism.

The commuting actions of G and H on U thus can be transported via Ψ to give commuting actions of G and H on $\operatorname{Hom}_H(W, U) \otimes W$. These actions are easily understood:

- The group H acts trivially on $\operatorname{Hom}_H(W, U)$ and acts on W as the given representation. Indeed, for $h \in H$ and $\rho \in \operatorname{Hom}_H(W, U)$ and $\vec{w} \in W$ we have

$$h \cdot \Psi(\rho \otimes \vec{w}) = h \cdot \rho(\vec{w}) = \rho(h \cdot \vec{w}) = \Psi(\rho \otimes h \cdot \vec{w}).$$

- The group G acts on $\operatorname{Hom}_H(W, U)$ by postcomposition (via its action on U) and acts trivially on W. Indeed, for $g \in G$ and $\rho \in \operatorname{Hom}_H(W, U)$ and $\vec{w} \in W$ we have

$$g \cdot \Psi(\rho \otimes \vec{w}) = g \cdot \rho(\vec{w}) = (g \cdot \rho)(\vec{w}) = \Psi(g \cdot \rho \otimes \vec{w}).$$

Since U was assumed to be irreducible, it follows that $V := \operatorname{Hom}_H(W, U)$ must be an irreducible G-representation. The decomposition $U \cong V \otimes W$ is precisely the one we claimed must exist. □

References

[1] A. Putman, The Jacobson density theorem, informal note.

[2] J.-P. Serre, *Linear representations of finite groups*, translated from the second French edition by Leonard L. Scott, Springer-Verlag, New York, 1977.

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