The representation theory of $\mathrm{SL}_n(\mathbb{Z})$

Andrew Putman

Abstract

We give a fairly complete description of the finite-dimensional characteristic 0 representation theory of $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$, following work of Margulis and Lubotzky.

1 Introduction

Fix a field $\mathbf{k}$ of characteristic 0. The goal of this note is to describe the finite-dimensional representations of $\mathrm{SL}_n(\mathbb{Z})$ over $\mathbf{k}$. Since $\mathrm{SL}_2(\mathbb{Z})$ is very close to a free group, its representation theory is quite wild and we will have little to say about it. For $n \geq 3$, however, there is a beautiful answer whose basic idea goes back to Lubotzky's PhD thesis [9], building on work of Margulis.

Rational representations. There are two natural sources of representations of $\mathrm{SL}_n(\mathbb{Z})$. The first are rational representations of the algebraic group $\mathrm{SL}_n(\mathbf{k})$, which can be restricted to the subgroup $\mathrm{SL}_n(\mathbb{Z}) < \mathrm{SL}_n(\mathbf{k})$. These representations are very well-behaved: they decompose into direct sums of irreducible representations, and at least when $\mathbf{k}$ is algebraically closed these irreducible representations are completely understood (a good starting point for this is [7]). As we will prove later, a rational representation V of $\mathrm{SL}_n(\mathbf{k})$ is irreducible if and only if its restriction to $\mathrm{SL}_n(\mathbb{Z})$ is irreducible. The key point is that $\mathrm{SL}_n(\mathbb{Z})$ is Zariski dense in $\mathrm{SL}_n(\mathbf{k})$. It follows that this picture remains unchanged when the representations are restricted to $\mathrm{SL}_n(\mathbb{Z})$.

Finite groups. The second source of representations of $\mathrm{SL}_n(\mathbb{Z})$ come from finite quotients. If $\pi\colon \mathrm{SL}_n(\mathbb{Z}) \to F$ is a surjective map from $\mathrm{SL}_n(\mathbb{Z})$ to a finite group F and V is a representation of F over $\mathbf{k}$, then V is also a representation of $\mathrm{SL}_n(\mathbb{Z})$ via π. For $n \geq 3$, these finite quotients are well-understood: the congruence subgroup property for $\mathrm{SL}_n(\mathbb{Z})$ (proved independently by Bass–Lazard–Serre [1] and Mennicke [11]) says that for every finite quotient $\pi\colon \mathrm{SL}_n(\mathbb{Z}) \to F$, there is some $\ell \geq 2$ such that π factors through the map $\mathrm{SL}_n(\mathbb{Z}) \longrightarrow \mathrm{SL}_n(\mathbb{Z}/\ell)$ that reduces matrices modulo ℓ. It is thus enough to only consider representations of $\mathrm{SL}_n(\mathbb{Z}/\ell)$ for $\ell \geq 2$. These are also well-understood, at least when ℓ is prime.

Profinite completion The finite quotients of $\mathrm{SL}_n(\mathbb{Z})$ form an inverse system of finite groups, and their inverse limit

$$\widehat{\mathrm{SL}_n(\mathbb{Z})} = \varprojlim_{\substack{K \lhd \mathrm{SL}_n(\mathbb{Z}) \\ [\mathrm{SL}_n(\mathbb{Z}):K]<\infty}} \mathrm{SL}_n(\mathbb{Z})/K$$

is the *profinite completion* of $\mathrm{SL}_n(\mathbb{Z})$. Endowing each $\mathrm{SL}_n(\mathbb{Z})/K$ with the discrete topology and $\widehat{\mathrm{SL}_n(\mathbb{Z})}$ with the resulting inverse limit topology, the group $\widehat{\mathrm{SL}_n(\mathbb{Z})}$ is a compact totally disconnected topological group. It is immediate from the definitions that a finite-dimensional representation $\rho\colon \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{GL}(V)$ factors through a finite group if and only if it factors through a continuous homomorphism $\widehat{\mathrm{SL}_n(\mathbb{Z})} \to \mathrm{GL}(V)$, where $\mathrm{GL}(V)$ is endowed with the discrete topology.

Remark 1.1. Consider some $\ell \geq 2$. Letting

$$\ell = p_1^{k_1} \cdots p_m^{k_m},$$

be its prime factorization, the Chinese remainder theorem can be used to prove that

$$\mathrm{SL}_n(\mathbb{Z}/\ell) \cong \mathrm{SL}_n(\mathbb{Z}/p_1^{k_1}) \times \cdots \times \mathrm{SL}_n(\mathbb{Z}/p_m^{k_m}).$$

Using this, the congruence subgroup property implies that

$$\widehat{\mathrm{SL}_n(\mathbb{Z})} \cong \prod_p \varprojlim_k \mathrm{SL}_n(\mathbb{Z}/p^k) \cong \prod_p \mathrm{SL}_n(\mathbb{Z}_p). \qquad \square$$

Combining the two families The main theorem we will discuss says informally that all representations of $\mathrm{SL}_n(\mathbb{Z})$ are built from combinations of the above basic families of representations. The natural inclusions

$$\mathrm{SL}_n(\mathbb{Z}) \hookrightarrow \mathrm{SL}_n(\mathbf{k}) \quad \text{and} \quad \mathrm{SL}_n(\mathbb{Z}) \hookrightarrow \widehat{\mathrm{SL}_n(\mathbb{Z})}$$

combine to give an inclusion

$$\mathrm{SL}_n(\mathbb{Z}) \hookrightarrow \mathrm{SL}_n(\mathbf{k}) \times \widehat{\mathrm{SL}_n(\mathbb{Z})}.$$

Define a *rational representation* of $\mathrm{SL}_n(\mathbf{k}) \times \widehat{\mathrm{SL}_n(\mathbb{Z})}$ to be a finite-dimensional $\mathbf{k}$-vector space V equipped with a homomorphism $\rho\colon \mathrm{SL}_n(\mathbf{k}) \times \widehat{\mathrm{SL}_n(\mathbb{Z})} \to \mathrm{GL}(V)$ such that $\rho|_{\mathrm{SL}_n(\mathbf{k})}$ is rational and $\rho|_{\widehat{\mathrm{SL}_n(\mathbb{Z})}}$ is continuous. We then have the following theorem, which was essentially proved by Lubotzky [9].

Theorem A (Extending representations)**.** *Let $\mathbf{k}$ be a field of characteristic 0 and let $n \geq 3$. Let V be a finite-dimensional vector space over $\mathbf{k}$ and let $\rho\colon \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{GL}(V)$ be a representation. Then ρ can be uniquely extended to a rational representation $\rho\colon \mathrm{SL}_n(\mathbf{k}) \times \widehat{\mathrm{SL}_n(\mathbb{Z})} \to \mathrm{GL}(V)$. Moreover, if $W \subset V$ is a subspace that is an $\mathrm{SL}_n(\mathbb{Z})$-subrepresentation of V, then W is also an $\mathrm{SL}_n(\mathbf{k}) \times \widehat{\mathrm{SL}_n(\mathbb{Z})}$-subrepresentation.*

Remark 1.2. Theorem A implies that $\mathrm{SL}_n \times \widehat{\mathrm{SL}_n(\mathbb{Z})}$ is the *proalgebraic completion* of $\mathrm{SL}_n(\mathbb{Z})$; see, e.g., [2]. $\square$

Consequences. Theorem A implies that the finite-dimensional representations of $\mathrm{SL}_n(\mathbb{Z})$ over $\mathbf{k}$ are in bijection with the rational representations of $\mathrm{SL}_n(\mathbf{k}) \times \widehat{\mathrm{SL}_n(\mathbb{Z})}$. Moreover, this bijection preserves subrepresentations, and thus restricts to a bijection between the irreducible representations of these groups. Finally, the fact that this bijection preserves subrepresentations implies that it takes direct sums of representations to direct sums of representations.

Now, every rational representation of $\mathrm{SL}_n(\mathbf{k}) \times \widehat{\mathrm{SL}_n(\mathbb{Z})}$ decomposes as a direct sum of irreducible subrepresentations; indeed, such a representation $\rho\colon \mathrm{SL}_n(\mathbf{k}) \times \widehat{\mathrm{SL}_n(\mathbb{Z})} \to \mathrm{GL}(V)$ factors through $\mathrm{SL}_n(\mathbf{k}) \times F$ for some finite group F, making V into a rational representation of the algebraic group $\mathrm{SL}_n(\mathbf{k}) \times F$. It is standard that such representations decompose into direct sums of irreducibles. Combining this with Theorem A, we deduce the following.

Corollary B (Semisimplicity)**.** *Let $\mathbf{k}$ be a field of characteristic 0 and let $n \geq 3$. Then every finite-dimensional representation of $\mathrm{SL}_n(\mathbb{Z})$ over $\mathbf{k}$ decomposes as a direct sum of irreducible representations.*

Irreducible representations. If $\mathbf{k}$ is algebraically closed, then the irreducible rational representations of $\mathrm{SL}_n(\mathbf{k}) \times \widehat{\mathrm{SL}_n(\mathbb{Z})}$ have a simple description. Recall that if G and H are finite groups and $\mathbf{k}$ is an algebraically closed field of characteristic 0, then the irreducible representations of $G \times H$ over $\mathbf{k}$ are precisely the representations of the form $V \otimes W$, where V is an irreducible representation of G and W is an irreducible representation of H. The following theorem says that the same thing holds for $\mathrm{SL}_n(\mathbf{k}) \times \widehat{\mathrm{SL}_n(\mathbb{Z})}$.

Theorem C (Irreducible representations)**.** *Let $\mathbf{k}$ be an algebraically closed field of characteristic 0 and let $n \geq 2$. Then the irreducible rational representations of $\mathrm{SL}_n(\mathbf{k}) \times \widehat{\mathrm{SL}_n(\mathbb{Z})}$ are precisely those of the form $V \otimes W$, where V and W are as follows:*

- *V is an irreducible finite-dimensional rational representation of $\mathrm{SL}_n(\mathbf{k})$.*
- *W is an irreducible finite-dimensional continuous representation of $\widehat{\mathrm{SL}_n(\mathbb{Z})}$ over $\mathbf{k}$.*

Theorem A implies for $n \geq 3$ that the restrictions of these to $\mathrm{SL}_n(\mathbb{Z})$ are precisely the irreducible finite-dimensional representations of $\mathrm{SL}_n(\mathbb{Z})$ over $\mathbf{k}$.

Super-rigidity. The main tool for proving Theorem A is super-rigidity for $\mathrm{SL}_n(\mathbb{Z})$, one version of which is as follows. The use of this theorem is the key place where the assumption that $n \geq 3$ is used.

Theorem D (Super-rigidity)**.** *Let $\mathbf{k}$ be a field of characteristic 0 and let $n \geq 3$. Let V be a finite-dimensional $\mathbf{k}$-vector space and $\rho\colon \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{GL}(V)$ be a representation. Then there exists a rational representation $f\colon \mathrm{SL}_n(\mathbf{k}) \to \mathrm{GL}(V)$ of the algebraic group $\mathrm{SL}_n(\mathbf{k})$ and a finite-index subgroup K of $\mathrm{SL}_n(\mathbb{Z})$ such that $f|_K = \rho|_K$.*

The history of Theorem D is a little complicated. For $\mathbf{k} = \mathbb{Q}$, it was originally proved by Bass–Milnor–Serre [3] using the congruence subgroup property. This proof was later extended to general $\mathbf{k}$ by Ragunathan [14]. Shortly before this, however, Margulis proved his famous super-rigidity theorem for higher-rank lattices [10]. One special case of this result is as follows (see [13, Theorem 16.1.1]):

- For some $n \geq 3$, let Γ be a non-uniform lattice in $\mathrm{SL}_n(\mathbb{R})$ and let $\rho\colon \Gamma \to \mathrm{GL}_m(\mathbb{R})$ be any homomorphism. Then there exists a continuous homomorphism $f\colon \mathrm{SL}_n(\mathbb{R}) \to \mathrm{GL}_m(\mathbb{R})$ and a finite-index subgroup $K < \Gamma$ such that $f|_K = \rho|_K$.

On the one hand, this is much more general than Theorem D since it applies to *all* non-uniform lattices, not just $\mathrm{SL}_n(\mathbb{Z})$. On the other hand, it appears to just work for $\mathbf{k} = \mathbb{R}$ and to only give a continuous extension of ρ. However, it is not hard to derive the general case of Theorem D from the above statement. We will not prove Theorem D, but in Appendix A we will show how to derive it from the above version of Margulis's result.

Remark 1.3. In [17, Theorem 6], Steinberg sketches a remarkably elementary proof of Theorem D. Like Bass–Milnor–Serre and Ragunathan's proofs, it uses the congruence subgroup property, but otherwise just relies on elementary facts about linear algebra and algebraic groups. In fact, he states it in a somewhat different way, but he proves a stronger result that is essentially equivalent to Theorem A. □

Outline. We begin in §2 by proving a density result that will be needed for the uniqueness part of Theorem A. Next, in §3 we will prove Theorem A. In §4 we will prove Theorem C, and then finally in our Appendix §A we will show how to derive Theorem D from Margulis's work.

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2 Density

This section is devoted to proving the following lemma, which will play an important role in the proof of Theorem A.

Theorem 2.1 (Density)**.** *Let $\mathbf{k}$ be a field of characteristic 0 and let $n \geq 2$. Endow $\mathrm{SL}_n(\mathbf{k})$ with the Zariski topology and $\widehat{\mathrm{SL}_n(\mathbb{Z})}$ with its usual profinite topology. Then $\mathrm{SL}_n(\mathbb{Z})$ is dense in $\mathrm{SL}_n(\mathbf{k}) \times \widehat{\mathrm{SL}_n(\mathbb{Z})}$.*

For the proof of Theorem 2.1, we will need the following.

Lemma 2.2. *Let $\mathbf{k}$ be a field of characteristic 0 and let $n \geq 2$. Let K be a finite-index subgroup of $\mathrm{SL}_n(\mathbb{Z})$. Then K is Zariski dense in $\mathrm{SL}_n(\mathbf{k})$.*

Proof. Let $G < \mathrm{SL}_n(\mathbf{k})$ be the Zariski closure of K. We must prove that $G = \mathrm{SL}_n(\mathbf{k})$. For distinct $1 \leq i, j \leq n$, let $R_{ij}(\mathbf{k})$ be the root subgroup of $\mathrm{SL}_n(\mathbf{k})$ consisting of matrices obtained from the identity by inserting a single nonzero entry at position (i, j). We thus have $R_{ij}(\mathbf{k}) \cong \mathbf{k}$ as additive groups. Similarly, define the root subgroup $R_{ij}(\mathbb{Z}) \cong \mathbb{Z}$ of $\mathrm{SL}_n(\mathbb{Z})$. The intersection $R_{ij}(\mathbf{k}) \cap K$ is a finite-index subgroup of $R_{ij}(\mathbb{Z})$, and in particular is infinite. Since a one-variable polynomial can only have finitely many zeros, the Zariski closure of $R_{ij}(\mathbf{k}) \cap K$ must be $R_{ij}(\mathbf{k})$. We deduce that $R_{ij}(\mathbf{k}) \subset G$ for all distinct $1 \leq i, j \leq n$. Since $\mathrm{SL}_n(\mathbf{k})$ is generated by these root subgroups, we conclude that $G = \mathrm{SL}_n(\mathbf{k})$, as desired. $\square$

Proof of Theorem 2.1. Let $G < \mathrm{SL}_n(\mathbf{k}) \times \widehat{\mathrm{SL}_n(\mathbb{Z})}$ be the closure of $\mathrm{SL}_n(\mathbb{Z})$. We must prove that $G = \mathrm{SL}_n(\mathbf{k}) \times \widehat{\mathrm{SL}_n(\mathbb{Z})}$. The key to the proof is the following claim.

Claim. *Let F be a finite quotient of $\mathrm{SL}_n(\mathbb{Z})$ and let*

$$\pi\colon\ \mathrm{SL}_n(\mathbf{k}) \times \widehat{\mathrm{SL}_n(\mathbb{Z})} \to \mathrm{SL}_n(\mathbf{k}) \times F$$

be the projection. Then $\pi(G) = \mathrm{SL}_n(\mathbf{k}) \times F$.

Proof of claim. Let $K \lhd \mathrm{SL}_n(\mathbb{Z})$ be the kernel of the projection to F, so $\pi(K) = K \times 1$. Lemma 2.2 implies that K is Zariski dense in $\mathrm{SL}_n(\mathbf{k})$, so we conclude that

$$\mathrm{SL}_n(\mathbf{k}) \times 1 = \overline{K \times 1} \subset \pi(G). \tag{2.1}$$

For all $f \in F$, we can find some element of $\mathrm{SL}_n(\mathbb{Z})$ projecting to f, which implies that there exists some $f' \in \mathrm{SL}_n(\mathbf{k})$ such that $(f', f) \in \pi(G)$. Since $(f', 1) \in \pi(G)$, we deduce that $(1, f) \in \pi(G)$, so

$$1 \times F \subset \pi(G). \tag{2.2}$$

Inclusions 2.1 and 2.2 imply the claim. $\square$

We now turn to proving that $G = \mathrm{SL}_n(\mathbf{k}) \times \widehat{\mathrm{SL}_n(\mathbb{Z})}$. Let

$$\mathcal{K} = \{K \mid K \lhd \mathrm{SL}_n(\mathbb{Z}),\ [\mathrm{SL}_n(\mathbb{Z}) : K] < \infty\}.$$

The group $\widehat{\mathrm{SL}_n(\mathbb{Z})}$ is thus the closed subset of

$$\prod_{K \in \mathcal{K}} \mathrm{SL}_n(\mathbb{Z})/K$$

consisting of all $(a_K)_{K \in \mathcal{K}}$ such that for all $K_1, K_2 \in \mathcal{K}$ with $K_2 < K_1$, the element $a_{K_2} \in \mathrm{SL}_n(\mathbb{Z})/K_2$ projects to $a_{K_1} \in \mathrm{SL}_n(\mathbb{Z})/K_1$ under the projection $\mathrm{SL}_n(\mathbb{Z})/K_2 \to \mathrm{SL}_n(\mathbb{Z})/K_1$. Consider some

$$\lambda = (b, (a_K)_{K \in \mathcal{K}}) \in \mathrm{SL}_n(\mathbf{k}) \times \widehat{\mathrm{SL}_n(\mathbb{Z})}.$$

Our goal is to show that $\lambda \in G$. Enumerate $\mathcal{K}$ as

$$\mathcal{K} = \{K_1, K_2, \ldots\}.$$

For all $r \geq 1$, let $L = K_1 \cap \cdots \cap K_r \in \mathcal{K}$. By the above claim, we can find some $\lambda_r = (b, (a'_K)_{K \in \mathcal{K}}) \in G$ such that $a'_L = a_L$. Since $L < K_i$ for all $1 \leq i \leq r$, it follows that $a_{K'_i} = a_{K_i}$ for all $1 \leq i \leq r$. The sequence $\{\lambda_i\}_{i=1}^{\infty}$ is therefore a sequence of elements of G that converges to λ. Since G is closed, we conclude that $\lambda \in G$, as desired. $\square$

3 Extending representations

We now prove Theorem A.

Proof of Theorem A. We start by recalling what we have to prove. Let $\mathbf{k}$ be a field of characteristic 0 and let $n \geq 3$. Let V be a finite-dimensional vector space over $\mathbf{k}$ and let $\rho\colon \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{GL}(V)$ be a representation. Our goal is to prove that ρ can be uniquely extended to a rational representation $\rho\colon \mathrm{SL}_n(\mathbf{k}) \times \widehat{\mathrm{SL}_n(\mathbb{Z})} \to \mathrm{GL}(V)$. Moreover, this extension should be such that if $W \subset V$ is a subspace that is an $\mathrm{SL}_n(\mathbb{Z})$-subrepresentation of V, then W is also an $\mathrm{SL}_n(\mathbf{k}) \times \widehat{\mathrm{SL}_n(\mathbb{Z})}$-subrepresentation.

The uniqueness of such an extension (if it exists) is an immediate consequence of Theorem 2.1 (Density), which also implies that such an extension must preserve all subrepresentations. All we must do, therefore, is construct our extension. As was pointed out by Lubotzky in [9, p. 680], the argument for this was originally given by Serre in the special case of SL_2 (see [15, p. 502]), and the general case is exactly the same.

Applying Theorem D (Super-rigidity), we get a rational representation $f\colon \mathrm{SL}_n(\mathbf{k}) \to \mathrm{GL}(V)$ and a finite-index subgroup $K < \mathrm{SL}_n(\mathbb{Z})$ such that $f|_K = \rho|_K$. Replacing K by a deeper finite-index subgroup if necessary, we can assume that K is a normal subgroup of $\mathrm{SL}_n(\mathbb{Z})$. To extend ρ to a rational representation of $\mathrm{SL}_n(\mathbf{k}) \times \widehat{\mathrm{SL}_n(\mathbb{Z})}$, it is enough to construct a homomorphism $g\colon \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{GL}(V)$ with the following three properties:

(a) For all $x \in K$, we have $g(x) = 1$, and thus g factors through the finite group $\mathrm{SL}_n(\mathbb{Z})/K$ and induces a continuous representation $\overline{g}\colon \widehat{\mathrm{SL}_n(\mathbb{Z})} \to \mathrm{GL}(V)$.

(b) For all $x \in \mathrm{SL}_n(\mathbf{k})$ and $y \in \mathrm{SL}_n(\mathbb{Z})$, the elements $f(x)$ and $g(y)$ of $\mathrm{GL}(V)$ commute, so $f \times \overline{g}$ is a rational representation of $\mathrm{SL}_n(\mathbf{k}) \times \widehat{\mathrm{SL}_n(\mathbb{Z})}$.

(c) For all $x \in \mathrm{SL}_n(\mathbb{Z})$, we have $\rho(x) = f(x)g(x)$.

For (c) to hold, we must define

$$g(y) = f(y)^{-1}\rho(y) \in \mathrm{GL}(V) \qquad (y \in \mathrm{SL}_n(\mathbb{Z})).$$

The resulting map $g\colon \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{GL}(V)$ is a priori only a set map, but we will prove that it is a homomorphism satisfying (a) and (b) above. This will be a 3 step process.

Claim 1. *For all $y \in \mathrm{SL}_n(\mathbb{Z})$, the image $g(y)$ only depends on the image of y in $\mathrm{SL}_n(\mathbb{Z})/K$.*

Since $\rho|_K = f|_K$, we have $g(x) = 1$ for all $x \in K$. However, since we do not yet know that g is a homomorphism, this is not enough. So consider some $y \in \mathrm{SL}_n(\mathbb{Z})$ and $x \in K$. We must prove that $g(xy) = g(y)$. To do this, we calculate $\rho(xy)$ in two ways:

$$\rho(xy) = f(xy)g(xy) = f(x)f(y)g(xy)$$

and

$$\rho(xy) = \rho(x)\rho(y) = f(x)g(x)f(y)g(y) = f(x)f(y)g(y).$$

Comparing these, we see that $g(xy) = g(y)$, as desired.

Claim 2. *Condition (b) holds: for all $x \in \mathrm{SL}_n(\mathbf{k})$ and $y \in \mathrm{SL}_n(\mathbb{Z})$, the elements $f(x)$ and $g(y)$ of $\mathrm{GL}(V)$ commute.*

Fixing some $y \in \mathrm{SL}_n(\mathbb{Z})$, the set Λ of elements of $\mathrm{GL}(V)$ that commute with $g(y)$ is Zariski closed. We will prove that $f(\mathrm{SL}_n(\mathbf{k})) \subset \Lambda$. Lemma 2.2 implies that the finite-index subgroup $K < \mathrm{SL}_n(\mathbb{Z})$ is Zariski dense in $\mathrm{SL}_n(\mathbf{k})$, so it is enough to prove that for all $x \in K$ we have $f(x) \in \Lambda$, i.e. that $f(x)$ and $g(y)$ commute. To do this, we calculate $\rho(yx)$ in two ways:

$$\rho(yx) = f(yx)g(yx) = f(y)f(x)g(y),$$

where the second equality uses previously proven fact that $g(yx) = g(y)$, and

$$\rho(yx) = \rho(y)\rho(x) = f(y)g(y)f(x)g(x) = f(y)g(y)f(x),$$

where the third equality uses the fact that $g(x) = 1$ since $x \in K$. Comparing these, we see that $f(x)g(y) = g(y)f(x)$, as desired.

Claim 3. *The map g is a homomorphism, and thus by the first claim (a) holds.*

For $y, y' \in \mathrm{SL}_n(\mathbb{Z})$, we can apply the previous claim to see that

$$\rho(yy') = \rho(y)\rho(y') = f(y)g(y)f(y')g(y') = f(y)f(y')g(y)g(y').$$

Since

$$\rho(yy') = f(yy')g(yy') = f(y)f(y')g(yy'),$$

we conclude that $g(yy') = g(y)g(y')$, as desired. $\square$

4 Classifying the irreducible representations

We now prove Theorem C.

Proof of Theorem C. We start by recalling what we must prove. Let $\mathbf{k}$ be an algebraically closed field of characteristic 0 and let $n \geq 2$. Consider a rational representation U of $\mathrm{SL}_n(\mathbf{k}) \times \widehat{\mathrm{SL}_n(\mathbb{Z})}$. Since U is a continuous representation of the profinite completion $\widehat{\mathrm{SL}_n(\mathbb{Z})}$, the action of $\widehat{\mathrm{SL}_n(\mathbb{Z})}$ on U factors through the action of a finite group F. We must prove that U is irreducible if and only if $U \cong V \otimes W$, where V and W are as follows:

- V is an irreducible finite-dimensional rational representation of $\mathrm{SL}_n(\mathbf{k})$.
- W is an irreducible finite-dimensional representation of F.

We prove the two directions of this result separately.

Claim. *Assume that $U \cong V \otimes W$, where V and W are as above. Then U is irreducible.*

In fact, for this claim it is not important that V is a rational representation of $\mathrm{SL}_n(\mathbf{k})$, only that it is irreducible. By assumption, V and W are simple modules over the group rings $\mathbf{k}[\mathrm{SL}_n(\mathbf{k})]$ and $\mathbf{k}[F]$, respectively. We remark that by $\mathbf{k}[\mathrm{SL}_n(\mathbf{k})]$ we mean simply the ordinary group ring, not the ring of regular functions on the affine variety $\mathrm{SL}_n(\mathbf{k})$. Since $\mathbf{k}$ is algebraically closed, we can apply the Jacobson Density Theorem (see [5, Theorem 3.2.2] or [12]) and see that the resulting ring maps $\phi\colon \mathbf{k}[\mathrm{SL}_n(\mathbf{k})] \to \mathrm{End}(V)$ and $\psi\colon \mathbf{k}[F] \to \mathrm{End}(W)$ are surjections. It follows that the ring map

$$\mathbf{k}[\mathrm{SL}_n(\mathbf{k}) \times F] \cong \mathbf{k}[\mathrm{SL}_n(\mathbf{k})] \otimes \mathbf{k}[F] \xrightarrow{\phi\otimes\psi} \mathrm{End}(V) \otimes \mathrm{End}(W) \cong \mathrm{End}(V \otimes W)$$

is a surjection and thus that $V \otimes W$ is a simple $\mathbf{k}[\mathrm{SL}_n(\mathbf{k}) \times F]$-module, as desired.

Claim. *Assume that U is irreducible. Then $U \cong V \otimes W$, where V and W are as above.*

First regard U as a representation of F. Since F is finite, U decomposes as a direct sum of isotypic components. Since the action of $\mathrm{SL}_n(\mathbf{k})$ on U commutes with the action of F, it must preserve these isotypic components. Since U was assumed to be irreducible, it follows that U must have a single F-isotypic component, i.e. that $U \cong W^{\oplus m}$ for some irreducible F-representation W and some $m \geq 0$. Consider the map

$$\Psi\colon\ \mathrm{Hom}_F(W, U) \otimes W \to U$$

defined via the formula $\Psi(\rho \otimes \vec{w}) = \rho(\vec{w})$. Since $U \cong W^{\oplus m}$, this map is surjective. Also, since $\mathbf{k}$ is algebraically closed we can apply Schur's Lemma to see that

$$\mathrm{Hom}_F(W, U) = \mathrm{Hom}_F(W, W^{\oplus m}) \cong \mathbf{k}^m.$$

We deduce that Ψ is a surjective map between vector spaces of the same dimension, so Ψ is an isomorphism.

The commuting actions of $\mathrm{SL}_n(\mathbf{k})$ and F on U thus can be transported via Ψ to give commuting actions of $\mathrm{SL}_n(\mathbf{k})$ and F on $\mathrm{Hom}_F(W, U) \otimes W$. These actions are easily understood:

- The group F acts trivially on $\mathrm{Hom}_F(W, U)$ and acts on W as the given representation. Indeed, for $f \in F$ and $\rho \in \mathrm{Hom}_F(W, U)$ and $\vec{w} \in W$ we have

$$f \cdot \Psi(\rho \otimes \vec{w}) = f \cdot \rho(\vec{w}) = \rho(f \cdot \vec{w}) = \Psi(\rho \otimes f \cdot \vec{w}).$$

- The group $\mathrm{SL}_n(\mathbf{k})$ acts on $\mathrm{Hom}_F(W, U)$ by postcomposition (via its action on U) and acts trivially on W. Indeed, for $x \in \mathrm{SL}_n(\mathbf{k})$ and $\rho \in \mathrm{Hom}_F(W, U)$ and $\vec{w} \in W$ we have

$$x \cdot \Psi(\rho \otimes \vec{w}) = x \cdot \rho(\vec{w}) = (x \cdot \rho)(\vec{w}) = \Psi(x \cdot \rho \otimes \vec{w}).$$

Since U was assumed to be irreducible, it follows that $V := \mathrm{Hom}_F(W, U)$ must be an irreducible $\mathrm{SL}_n(\mathbf{k})$-module. What is more, since U was assumed to be a rational $\mathrm{SL}_n(\mathbf{k})$-representation and V can be realized as an $\mathrm{SL}_n(\mathbf{k})$-subrepresentation of U (namely, for any nonzero $\vec{w} \in W$ as the subspace $\Psi(U \times \vec{w})$), it follows that V is a rational representation of $\mathrm{SL}_n(\mathbf{k})$. The decomposition $U \cong V \otimes W$ is precisely the one we claimed must exist. $\square$

A Appendix: On super-rigidity

In this section, we will show how to derive Theorem D (Super-rigidity) from a version of Margulis's super-rigidity theorem. The special case of Margulis's theorem we will start with will be as follows:

Theorem A.1 ([13, Theorem 16.1.1])**.** *For some $n \geq 3$, let $\rho\colon \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{GL}_m(\mathbb{R})$ be any homomorphism. Then there exists a continuous homomorphism $f\colon \mathrm{SL}_n(\mathbb{R}) \to \mathrm{GL}_m(\mathbb{R})$ and a finite-index subgroup $K < \mathrm{SL}_n(\mathbb{Z})$ such that $f|_K = \rho|_K$.*

Remark A.2. In the above reference, one can in fact replace $\mathrm{SL}_n(\mathbb{Z})$ with any non-uniform lattice in $\mathrm{SL}_n(\mathbb{R})$. There are more general versions of Margulis's theorem for any irreducible lattice in a higher rank Lie group, but they are far more complicated to state. □

There are two differences between Theorem A.1 and Theorem D:

- Theorem D concerns arbitrary fields $\mathbf{k}$ of characteristic 0, not just $\mathbb{R}$.
- In Theorem D, the extended homomorphism $f\colon \mathrm{SL}_n(\mathbf{k}) \to \mathrm{GL}_m(\mathbf{k})$ is a rational representation of the algebraic group $\mathrm{SL}_n(\mathbf{k})$, while in Theorem A.1 the homomorphism $f\colon \mathrm{SL}_n(\mathbb{R}) \to \mathrm{GL}_m(\mathbb{R})$ is just continuous.

We will deal with the second issue first. The first step is to upgrade f to a Lie group homomorphism, i.e. a homomorphism that is everywhere smooth:

Theorem A.3. *Let $f\colon G \to H$ be a continuous homomorphism between real Lie groups. Then f is smooth.*

Proof. This is a nontrivial but standard fact about Lie groups, so we will omit the proof. See [18, Proposition 2.4.6] for an accessible account of it. □

We next upgrade f to a rational representation. Before we explain how to do this, we review some cautionary examples.

Example A.4. Here are some examples of nonalgebraic Lie group homomorphisms between $\mathbb{R}$-algebraic groups:

- The homomorphism $f\colon \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^\times$ defined via the formula $f(x) = \det(x)^{\sqrt{2}}$. This kind of phenomenon indicates that it is important to restrict one's self to perfect groups.
- Issues can still occur for perfect (or even semisimple) groups. For example, consider the adjoint representation of $\mathrm{SL}_3(\mathbb{R})$ obtained via the derivative of the conjugation action:

  $$\mathrm{ad}\colon \mathrm{SL}_3(\mathbb{R}) \longrightarrow \mathrm{GL}(\mathfrak{sl}_3(\mathbb{R})) \cong \mathrm{GL}_8(\mathbb{R}).$$

  The kernel of ad is the center of $\mathrm{SL}_3(\mathbb{R})$, which is trivial. It follows that ad is an isomorphism onto its image, which is a closed subgroup G of $\mathrm{GL}_8(\mathbb{R})$. The inverse

  $$\mathrm{ad}^{-1}\colon G \longrightarrow \mathrm{SL}_3(\mathbb{R})$$

is certainly smooth; however, it is not algebraic since if it was, then we could extend scalars to $\mathbb{C}$ and deduce that the complexified adjoint representation

$$\mathrm{ad}_{\mathbb{C}}\colon \mathrm{SL}_3(\mathbb{C}) \longrightarrow \mathrm{GL}(\mathfrak{sl}_3(\mathbb{C})) \cong \mathrm{GL}_8(\mathbb{C})$$

is an isomorphism, which is false since $\mathrm{SL}_3(\mathbb{C})$ has a nontrivial center (of order 3). See [8] for a discussion of this kind of phenomenon. □

Despite these examples, the following result still holds.

Lemma A.5. *Every Lie group homomorphism $f\colon \mathrm{SL}_n(\mathbb{R}) \to \mathrm{GL}_m(\mathbb{R})$ is $\mathbb{R}$-algebraic. Similarly, every complex Lie group homomorphism $F\colon \mathrm{SL}_n(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ is $\mathbb{C}$-algebraic.*

Proof. We start by reducing to the complex case. The derivative of a real Lie group homomorphism $f\colon \mathrm{SL}_n(\mathbb{R}) \to \mathrm{GL}_m(\mathbb{R})$ at the identity is a map $\mathfrak{sl}_n(\mathbb{R}) \to \mathfrak{gl}_m(\mathbb{R})$ of real Lie algebras. Since $\mathfrak{sl}_n(\mathbb{R}) \otimes \mathbb{C} \cong \mathfrak{sl}_n(\mathbb{C})$ and $\mathfrak{gl}_m(\mathbb{R}) \otimes \mathbb{C} = \mathfrak{gl}_m(\mathbb{C})$, we can tensor this map with $\mathbb{C}$ and get a map $\mathfrak{sl}_n(\mathbb{C}) \to \mathfrak{gl}_m(\mathbb{C})$ of complex Lie algebras. This is the derivative at the identify of a homomorphism $F\colon \mathrm{SL}_n(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ of complex Lie groups that fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{SL}_n(\mathbb{R}) \arrow[r, "f"] \arrow[d, hook] & \mathrm{GL}_m(\mathbb{R}) \arrow[d, hook] \\
\mathrm{SL}_n(\mathbb{C}) \arrow[r, "F"] & \mathrm{GL}_m(\mathbb{C}).
\end{tikzcd}$$

To prove that f is a $\mathbb{R}$-algebraic homomorphism, it is enough to prove that F is a $\mathbb{C}$-algebraic homomorphism.

It remains to prove that every complex Lie group homomorphism $F\colon \mathrm{SL}_n(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ is $\mathbb{C}$-algebraic. Let

$$\Lambda = \{(x, F(x)) \mid x \in \mathrm{SL}_n(\mathbb{C})\} \subset \mathrm{SL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C})$$

be the graph of F. There are thus two projections

$$\pi_1\colon \Lambda \xrightarrow{\cong} \mathrm{SL}_n(\mathbb{C}) \quad \text{and} \quad \pi_2\colon \Lambda \longrightarrow \mathrm{GL}_m(\mathbb{C}),$$

and F factors as

$$\mathrm{SL}_n(\mathbb{C}) \xrightarrow{\pi_1^{-1}} \Lambda \xrightarrow{\pi_2} \mathrm{GL}_m(\mathbb{C}). \tag{A.1}$$

The Lie algebra of the subgroup Λ of the $\mathbb{C}$-algebraic group $\mathrm{SL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C})$ is isomorphic to $\mathfrak{sl}_n(\mathbb{C})$. Not all connected Lie subgroups of $\mathbb{C}$-algebraic groups are algebraic subgroups, and the Lie algebras of the ones that are are called *algebraic Lie algebras*. A basic result about algebraic groups is that over an algebraically closed field of characteristic 0 like $\mathbb{C}$, all perfect Lie subalgebras are algebraic (see [4, Corollary 7.9]). Since $\mathfrak{sl}_n(\mathbb{C})$ is perfect, we conclude that Λ is a $\mathbb{C}$-algebraic subgroup of $\mathrm{SL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C})$.

The projection $\pi_1\colon \Lambda \to \mathrm{SL}_n(\mathbb{C})$ is a bijective algebraic map between algebraic groups. Bijective maps between algebraic varieties need not be isomorphisms; however, they are if the target is smooth (or even normal) and the varieties are defined over an algebraically closed field of characteristic 0 (see [16]; the key tool here is Zariski's Main Theorem). Since algebraic groups are automatically smooth, we conclude that π_1 is an isomorphism of algebraic varieties. Its inverse π_1^{-1} is thus algebraic. Since the projection π_2 is also algebraic, we conclude from (A.1) that F is an algebraic map. $\square$

We now prove the complex case of Theorem D:

Theorem A.6. *For some $n \geq 3$, let $\rho\colon \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{GL}_m(\mathbb{C})$ be any homomorphism. Then there exists a rational representation $f\colon \mathrm{SL}_n(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ and a finite-index subgroup $K < \mathrm{SL}_n(\mathbb{Z})$ such that $f|_K = \rho|_K$.*

Proof of Theorem A.6. Forgetting its complex structure, the group $\mathrm{GL}_m(\mathbb{C})$ is a Zariski closed subgroup of the $\mathbb{R}$-algebraic group $\mathrm{GL}_{2m}(\mathbb{R})$. Applying Theorem A.1 to the composition of $\rho\colon \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{GL}_m(\mathbb{C})$ with the inclusion $\mathrm{GL}_m(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2m}(\mathbb{R})$, we get a continuous homomorphism $f\colon \mathrm{SL}_n(\mathbb{R}) \to \mathrm{GL}_{2m}(\mathbb{R})$ and a finite-index subgroup $K < \mathrm{SL}_n(\mathbb{Z})$ such that $f|_K = \rho|_K$. Combining Theorem A.3 and Lemma A.5, we see that f is a rational representation of $\mathrm{SL}_n(\mathbb{R})$. Lemma 2.2 says that K is Zariski dense in $\mathrm{SL}_n(\mathbb{R})$, so since $\rho(K)$ is contained in the $\mathbb{R}$-algebraic subgroup $\mathrm{GL}_m(\mathbb{C})$ of $\mathrm{GL}_{2m}(\mathbb{R})$, we see that the image of f lies in $\mathrm{GL}_m(\mathbb{C})$, so we can regard f as a $\mathbb{R}$-algebraic homomorphism $f\colon \mathrm{SL}_n(\mathbb{R}) \to \mathrm{GL}_m(\mathbb{C})$. The derivative of f at the identity is an $\mathbb{R}$-linear Lie algebra homomorphism $\mathfrak{sl}_n(\mathbb{R}) \to \mathfrak{gl}_m(\mathbb{C})$. We can factor this Lie algebra homomorphism through a $\mathbb{C}$-linear Lie algebra homomorphism

$$\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{sl}_n(\mathbb{R}) \otimes \mathbb{C} \to \mathfrak{gl}_m(\mathbb{C}),$$

which is the derivative at the identity of a map $F\colon \mathrm{SL}_n(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ of complex Lie algebras that fits into a commutative diagram

$$\begin{tikzcd} \mathrm{SL}_n(\mathbb{R}) \arrow[r, "f"] \arrow[d, hook] & \mathrm{GL}_m(\mathbb{C}). \\ \mathrm{SL}_n(\mathbb{C}) \arrow[ur, "F"] & \end{tikzcd}$$

Another application of Lemma A.5 shows that $F\colon \mathrm{SL}_n(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ is a rational representation, and $F|_K = f|_K = \rho|_K$, as desired. $\square$

Before we prove our main result, we need one final lemma showing how to recognize the field of definition of a rational representation.

Lemma A.7. *Let $f\colon \mathrm{SL}_n(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ be a rational representation, let $K < \mathrm{SL}_n(\mathbb{Z})$ be a finite-index subgroup, and let $\mathbf{k}$ be a subfield of $\mathbb{C}$ such that $f(K) \subset \mathrm{GL}_m(\mathbf{k})$. Then f is defined over $\mathbf{k}$, and thus restricts to a rational representation $\mathrm{SL}_n(\mathbf{k}) \to \mathrm{GL}_m(\mathbf{k})$.*

Proof. Let $\mathbb{C}[\mathrm{SL}_n]$ be the $\mathbb{C}$-algebra of regular functions $\mathrm{SL}_n(\mathbb{C}) \to \mathbb{C}$ and let $\mathbf{k}[\mathrm{SL}_n]$ be the $\mathbf{k}$-algebra of such regular functions that are defined over $\mathbf{k}$, i.e. by polynomials in the entries of $\mathrm{SL}_n(\mathbb{C})$ with coefficients in $\mathbf{k}$. Since SL_n is an algebraic group defined over $\mathbb{Q}$, we have

$$\mathbb{C}[\mathrm{SL}_n] = \mathbf{k}[\mathrm{SL}_n] \otimes_{\mathbf{k}} \mathbb{C}. \tag{A.2}$$

Letting $h \in \mathbb{C}[\mathrm{SL}_n]$ be one of the matrix coefficients of f, our goal is to show that $h \in \mathbf{k}[\mathrm{SL}_n]$.

Since $\mathbb{C}$ is an algebraically closed field of characteristic 0, the field $\mathbf{k}$ is precisely the set of elements of $\mathbb{C}$ that are invariant under all elements of $\mathrm{Aut}(\mathbb{C}/\mathbf{k})$. Combining this with (A.2), we see that $\mathbf{k}[\mathrm{SL}_n]$ is the set of all elements of $\mathbb{C}[\mathrm{SL}_n]$ that are invariant under all elements of $\mathrm{Aut}(\mathbb{C}/\mathbf{k})$. Considering some $\lambda \in \mathrm{Aut}(\mathbb{C}/\mathbf{k})$, we see that our goal is equivalent to showing that $\lambda \circ h = h$.

By assumption, $\lambda \circ h$ and h agree on all elements of K. Lemma 2.2 implies that K is Zariski dense in $\mathrm{SL}_n(\mathbb{C})$, so this implies that in fact $\lambda \circ h = h$, as desired. □

We finally prove Theorem D.

Proof of Theorem D. We start by recalling what we must prove. Let $\mathbf{k}$ be a field of characteristic 0 and let $n \geq 3$. For some m, let $\rho\colon \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{GL}_m(\mathbf{k})$ be a representation. We must prove that there exists a rational representation $f\colon \mathrm{SL}_n(\mathbf{k}) \to \mathrm{GL}_m(\mathbf{k})$ of the algebraic group $\mathrm{SL}_n(\mathbf{k})$ and a finite-index subgroup K of $\mathrm{SL}_n(\mathbb{Z})$ such that $f|_K = \rho|_K$.

Since $\mathrm{SL}_n(\mathbb{Z})$ is a finitely generated group, there exists a subfield $\mathbf{k}'$ of $\mathbf{k}$ with the following two properties:

(a) For all $x \in \mathrm{SL}_n(\mathbb{Z})$, the matrix entries of $\rho(x)$ lie in $\mathbf{k}'$.
(b) The field $\mathbf{k}'$ is a finitely generated $\mathbb{Q}$-algebra.

By (a), there exists a homomorphism $\rho'\colon \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{GL}_m(\mathbf{k}')$ such that ρ factors as

$$\mathrm{SL}_n(\mathbb{Z}) \xrightarrow{\rho'} \mathrm{GL}_m(\mathbf{k}') \hookrightarrow \mathrm{GL}_m(\mathbf{k}).$$

Since $\mathbf{k}'$ is a finitely generated $\mathbb{Q}$-algebra, it is isomorphic to a subfield of $\mathbb{C}$. Identifying $\mathbf{k}'$ with this subfield of $\mathbb{C}$ allows us to identify the image of ρ' with a subfield of $\mathbb{C}$. We can thus apply Theorem A.6 and obtain a rational representation $f'\colon \mathrm{SL}_n(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ and a finite-index subgroup $K < \mathrm{SL}_n(\mathbb{Z})$ such that $f'|_K = \rho'|_K$. Since

$$f'(K) = \rho'(K) \subset \mathrm{GL}_m(\mathbf{k}') \subset \mathrm{GL}_m(\mathbb{C}),$$

we can apply Lemma A.7 to deduce that f' is defined over $\mathbf{k}'$, and thus restricts to a rational representation $f''\colon \mathrm{SL}_n(\mathbf{k}') \to \mathrm{GL}_m(\mathbf{k}')$. Extending scalars from $\mathbf{k}'$ to $\mathbf{k}$, we obtain a rational representation $f\colon \mathrm{SL}_n(\mathbf{k}) \to \mathrm{GL}_m(\mathbf{k})$ such that

$$f|_K = f'|_K = \rho'|_K = \rho|_K,$$

as desired. □

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Andrew Putman
Department of Mathematics
University of Notre Dame
164 Hurley Hall
Notre Dame, IN 46556
`andyp@nd.edu`