Rochlin's theorem on signatures of spin 4-manifolds via algebraic topology

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Abstract

We give the original proof of Rochlin's famous theorem on signatures of smooth spin 4-manifolds, which uses techniques from algebraic topology. We have attempted to include enough background and details to make this proof understandable to a geometrically minded topologist. We also include a fairly complete discussion of spin structures on manifolds.

1 Introduction

Let M^4 be a closed oriented smooth 4-manifold. All manifolds in this note (including M^4) are assumed to be connected.

Intersection form. The cup product map

$$\mathrm{H}^2(M^4;\mathbb{Z}) \times \mathrm{H}^2(M^4;\mathbb{Z}) \longrightarrow \mathrm{H}^4(M^4;\mathbb{Z}) \cong \mathbb{Z}$$

descends to an integral bilinear form $\omega(\cdot,\cdot)$ on $V := \mathrm{H}^2(M^4;\mathbb{Z})/\text{torsion}$ called the *intersection form*. Poincaré duality implies that $\omega(\cdot,\cdot)$ is a *unimodular* integral form, that is, it induces an isomorphism between V and its dual $V^* = \mathrm{Hom}(V,\mathbb{Z})$. It plays a fundamental role in the topology of 4-manifolds. For example, building on work of Whitehead [30], Milnor [17] proved that if M^4 is simply-connected, then its homotopy type is determined by $\mathrm{H}^2(M^4;\mathbb{Z})$ together with ω.

Signature of forms. One of the basic invariants of ω is its *signature*, which is defined as follows. Let $\omega_{\mathbb{Q}}(\cdot,\cdot)$ be the induced form on $V \otimes \mathbb{Q}$. We can diagonalize $\omega_{\mathbb{Q}}$, i.e. choose coordinates on $V \otimes \mathbb{Q}$ such that with respect to these coordinates, we have $\omega_{\mathbb{Q}}(\vec{v},\vec{w}) = \vec{v}^t M \vec{w}$ for a diagonal matrix M. Since ω is unimodular, all the diagonal entries of M are nonzero. The signature of ω is then $r - s$, where r is the number of positive diagonal entries of M and s is the number of negative entries. Neither r nor s depend on the choice of diagonalization.

Signature of 4-manifolds. Define $\sigma(M^4)$ to be the signature of ω. As the following example shows, $\sigma(M^4)$ can achieve arbitrary values.

Example. For $r, s \geq 0$, let M^4 be the connect sum of r copies of $\mathbb{CP}^2$ and s copies of $\overline{\mathbb{CP}}^2$ (here $\overline{\mathbb{CP}}^2$ is $\mathbb{CP}^2$ with its orientation reversed). Then $\mathrm{H}^2(M^4; \mathbb{Z}) \cong \mathbb{Z}^{r+s}$ and the intersection form on $\mathrm{H}^2(M^4; \mathbb{Z})$ is represented by a diagonal matrix with r entries equal to 1 and s entries equal to -1. In particular, $\sigma(M^4) = r - s$.

Spin structures and even forms. However, it is definitely not true that $\omega(\cdot, \cdot)$ can be an arbitrary form. Let M^4 be a 4-manifold. Recall that M^4 is orientable if and only if its first Stiefel–Whitney class $w_1 \in \mathrm{H}^1(M^4; \mathbb{Z}/2)$ vanishes. The 4-manifold M^4 is *spin* if it is orientable and $w_2(M^4) \in \mathrm{H}^2(M^4; \mathbb{Z}/2)$ also vanishes. We will say more about this in §3 below. In particular, we will show that if M^4 is spin and closed, then its intersection form $\omega(\cdot, \cdot)$ is *even*, i.e. $\omega(\vec{v}, \vec{v})$ is an even integer for all $\vec{v} \in \mathrm{H}^2(M^4; \mathbb{Z})/\text{torsion}$. The converse is almost true; for instance, it is true if M^4 is simply-connected. It is perhaps a little surprising that unimodular integral bilinear forms can be even. Here is an important example.

Example. The E_8 form is the bilinear form on $\mathbb{Z}^8$ defined via the formula $\omega(\vec{v}, \vec{w}) = \vec{v}^t M \vec{w}$ with

$$M = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

An easy calculation shows that $\det(M) = 1$, so ω is unimodular. It is also not hard to see that it is even. Moreover, $\omega(\vec{v}, \vec{v}) > 0$ for all $\vec{v} \in \mathbb{Z}^8$, so the signature of ω is 8.

Divisibility and Rochlin's theorem. A classical theorem of van der Blij [28] says that the signature of a unimodular even integral bilinear form is divisible by 8. See [20] for a textbook proof of van der Blij's theorem. The main result proved in this note is the following theorem of Rochlin [25], which strengthens this divisibility for the signatures of smooth closed 4-manifolds. It implies in particular that no smooth closed simply-connected 4-manifold has E_8 for its intersection form.

Rochlin's Theorem. *If M^4 is a smooth spin closed 4-manifold, then*

$$\sigma(M^4) \equiv 0 \pmod{16}.$$

Remark. The condition that M^4 is spin cannot be replaced with the condition that the intersection form is even. A counterexample is given by the Enriques surface.

Recall that the Hirzebruch Signature Theorem [21, Theorem 19.4] gives a formula for the signature of a $4k$-dimensional manifold in terms of the Pontryagin classes of the manifold. For a 4-manifold M^4, this formula is

$$\sigma(M^4) = \frac{1}{3} p_1(M^4)([M^4]).$$

Rochlin's theorem is thus equivalent to the following theorem.

Rochlin's Theorem′. *If M^4 is a smooth spin closed 4-manifold, then*

$$p_1(M^4)([M^4]) \equiv 0 \pmod{48}.$$

Comments on proofs. In this note, we will give a variant on the original proof of Rochlin's theorem, which uses techniques from algebraic topology. See [6] for a French translation of the original Russian paper [25] together with quite a bit of useful commentary. Our viewpoint is strongly inspired by the proof sketched in Kervaire–Milnor's paper [10]. There are many other proof of Rochlin's theorem.

- For proofs that use techniques from 4-manifold topology, see [5, 7, 14]; textbook references for these geometric proofs include [11, Chapter XI] and [26, p. 507].
- A novel proof using techniques from 3-manifold topology can be found in [12, Appendix].
- A proof using the Atiyah–Singer index theorem can be found in [13, Chapter IV.1].

Later developments. While Rochlin's Theorem might appear to be a curiosity, it is actually the root of many important developments.

- Freedman [4] proved that there exist closed simply-connected *topological* 4-manifolds whose intersection form is any given unimodular integral bilinear form. For instance, there exists a simply-connected topological 4-manifold M^4 whose intersection form is the E_8 form. By Rochlin's Theorem this 4-manifold cannot be given a smooth structure.
- It is almost (but not quite) known which unimodular integral bilinear forms can be the intersection forms of a smooth simply-connected 4-manifold. The main result here is a theorem of Donaldson [3] which says that if ω is the intersection form of a smooth simply-connected 4-manifold and ω is *definite* (i.e. $\omega(\vec{v}, \vec{v})$ is always nonpositive or always nonnegative), then with respect to some basis ω is represented by either the identity matrix or the negative of the identity matrix. It is also known whether or not most indefinite forms are realized; the remaining cases are the subject of the famous 11/8-conjecture.

2 The proof of Rochlin's Theorem

In this section, we will give the proof of Rochlin's Theorem. Actually, we will prove the equivalent Rochlin's Theorem′, which asserts that if M^4 is a smooth spin closed 4-manifold, then

$$p_1(M^4)([M^4]) \equiv 0 \pmod{48}.$$

This proof will depend on two key facts which will be proved in subsequent sections.

The tangent bundle. The first ingredient is a description of the tangent bundle of M^4. Recall that a manifold X is said to be *parallelizable* if its tangent bundle T_X is trivial and is said to be *almost parallelizable* if $X \setminus \{p\}$ is parallelizable for any $p \in X$. The following proposition will be proved in §4 using obstruction theory.

Proposition 2.1. *Let M^4 be a smooth spin closed 4-manifold. Then M^4 is almost parallelizable.*

Let $B^4 \subset M^4$ be a submanifold which is diffeomorphic to a closed 4-dimensional ball. The result of collapsing $M^4 \setminus \mathrm{Int}(B^4)$ to a point is homeomorphic to a 4-dimensional sphere S^4; let $\beta : M^4 \to S^4$ be the resulting quotient map. We will call β a *ball-collapse map* (of course, it depends on various choices, but none of them are important in what follows). Proposition 2.1 implies that $M^4 \setminus \mathrm{Int}(B^4)$ is parallelizable, so there exists a 4-dimensional oriented real vector bundle $E \to S^4$ such that $T_{M^4} = \beta^*(E)$. The induced map $\beta^* : \mathrm{H}^4(S^4;\mathbb{Z}) \to \mathrm{H}^4(M^4;\mathbb{Z})$ is an isomorphism, so

$$p_1(M^4)([M^4]) = p_1(E)([S^4]).$$

The rest of the proof will focus on understanding $p_1(E)([S^4])$.

Transition to K-theory. The Pontryagin classes are *stable* characteristic classes, which in our context implies that $p_1(E) = p_1(E \oplus \epsilon^k)$ for all $k \geq 0$, where ϵ^k is the k-dimensional trivial bundle $S^4 \times \mathbb{R}^k$. This brings us into the realm of K-theory, whose basic definitions we quickly recall. Let X be a compact connected CW-complex. Two oriented real vector bundles B_1 and B_2 on X define the same *stable oriented real vector bundle* if there exists some $k_1, k_2 \geq 0$ such that $B_1 \oplus \epsilon^{k_1} \cong B_2 \oplus \epsilon^{k_2}$. This defines an equivalence relation on the set of oriented real vector bundles on X; if B is such a bundle, then we will write $[B]$ for its equivalence class. The *reduced oriented K-theory* of X, denoted $\widetilde{KO}(X)$, is the set of stable oriented real vector bundles.

Remark. An alternate description of $\widetilde{KO}(X)$ is that it is the set of principal $\mathrm{SL}(\mathbb{R})$-bundles on X, where $\mathrm{SL}(\mathbb{R})$ is the union of the increasing sequence

$$\mathrm{SL}_1(\mathbb{R}) \subset \mathrm{SL}_2(\mathbb{R}) \subset \mathrm{SL}_3(\mathbb{R}) \subset \cdots$$

of groups.

The set $\widetilde{KO}(X)$ forms an abelian group under connected sum; the identity element is the equivalence class of the trivial bundle. For all $i \geq 1$, the i^{th} Pontryagin class induces a well-defined set map

$$p_i : \widetilde{KO}(X) \to \mathrm{H}^{4i}(X;\mathbb{Z}).$$

The p_i are not necessarily homomorphisms. Instead, for $[B_1], [B_2] \in \widetilde{KO}(X)$ we have

$$p_i([B_1] + [B_2]) = p_i([B_1]) + p_i([B_2]) + \theta,$$

where $\theta \in \mathrm{H}^{4i}(X;\mathbb{Z})$ is a linear combination of elements of the form $\theta_1 \cup \theta_2$ with

$$\theta_1, \theta_2 \in \bigoplus_{j=1}^{i-1} \mathrm{H}^j(X;\mathbb{Z}).$$

Calculating the Pontryagin class. We now return to the bundle $E \to S^4$ constructed above. Our goal is to prove that

$$p_1(E)([S^4]) \equiv 0 \pmod{48}. \tag{2.1}$$

The Bott Periodicity theorem (see [18]) implies that $\widetilde{KO}(S^4) \cong \mathbb{Z}$. Since $\mathrm{H}^i(S^4;\mathbb{Z}) = 0$ for $1 \leq i \leq 3$, the first Pontryagin class actually gives a homomorphism

$$p_1 : \mathbb{Z} \cong \widetilde{KO}(S^4) \to \mathrm{H}^4(S^4;\mathbb{Z}) \cong \mathbb{Z}.$$

Claim. *We have $p_1(n) = \nu \cdot n$ for some $\nu \in 2\mathbb{Z}$.*

Proof of claim. Consider $[B] \in \widetilde{KO}(S^4)$. It is enough to show that $p_1([B])$ is even. By definition, $p_1([B]) = c_2([B_{\mathbb{C}}])$, where $B_{\mathbb{C}}$ is the complexification of B. The mod 2 reduction of $c_2([B_{\mathbb{C}}])$ is $w_2([(B_{\mathbb{C}})_{\mathbb{R}}])$, where $(B_{\mathbb{C}})_{\mathbb{R}}$ is the real bundle underlying the complex bundle $B_{\mathbb{C}}$. Since $(B_{\mathbb{C}})_{\mathbb{R}} \cong B \oplus B$ and $\mathrm{H}^1(S^4;\mathbb{Z}/2) = 0$, we deduce that

$$w_2([(B_{\mathbb{C}})_{\mathbb{R}}]) = w_2(B \oplus B) = w_2(B) + w_1(B) \cup w_1(B) + w_2(B) = 2w_2(B) = 0,$$

which implies that $p_1([B]) = c_2([B_{\mathbb{C}}])$ is even. □

Remark. In fact, one can show that $\nu = 2$ in the above claim, but we will not need this.

Endgame via the stable J-homomorphism. The desired identity (2.1) now follows immediately from the Claim and the following proposition, which will be proved in §5.

Proposition 2.2. *Let $E \to S^4$ be an oriented real vector bundle such that there exists a compact oriented 4-manifold M^4 with $T_{M^4} = \beta^*(E)$, where $\beta : M^4 \to S^4$ is a ball-collapse map. Then the element $[E] \in \widetilde{KO}(S^4) \cong \mathbb{Z}$ is divisible by 24.*

This is the deepest part of the proof. The key is the *stable J-homomorphism*. Recall that the Freudenthal suspension theorem says that for all $k \geq 0$, the group $\pi_{n+k}(S^n)$ is independent of n for $n \gg 0$; the stable value is the k^{th} *stable stem* and is denoted π_k^S. Calculating π_k^S is very difficult. The "first layer" comes from homomorphisms

$$J_k : \widetilde{KO}(S^{k+1}) \to \pi_k^S$$

which were first defined by Whitehead [29] following work of Hopf [9]. We will discuss the stable J-homomorphism more in §5. The two facts about it that go into Proposition 2.2 are as follows.

- Letting $[E] \in \widetilde{KO}(S^4)$ be as in the proposition, we have $[E] \in \ker(J_3)$. This will be almost immediate from the definition.
- The image of J_3 is isomorphic to $\mathbb{Z}/24$. This is a deep fact, and in some sense is the heart of the reason that Rochlin's theorem holds.

Remark. In fact, $\pi_3^S \cong \mathbb{Z}/24$, though this is not necessary for our proof.

3 Spin 4-manifolds

Before we prove the propositions stated in §2, we need to spend some time discussing general facts about spin manifolds. A good reference that influenced our exposition is [13, §II.2]. In this section, X is an arbitrary connected CW complex.

Definition of spin structure. We begin by giving the proper definition of a spin structure; the definition given in the introduction in terms of Stiefel–Whitney classes will then be a theorem (see Corollary 3.2 below). For some $n \geq 2$, let $E \to X$ be an n-dimensional oriented real vector bundle and let $F(E) \to X$ be the frame bundle of E. Thus $F(E)$ is a principal $\mathrm{SL}_n(\mathbb{R})$-bundle. The group $\mathrm{SL}_n(\mathbb{R})$ deformation retracts onto its maximal compact subgroup $\mathrm{SO}_n(\mathbb{R})$, so

$$\pi_1(\mathrm{SL}_n(\mathbb{R})) = \pi_1(\mathrm{SO}_n(\mathbb{R})) = \begin{cases} \mathbb{Z} & \text{if } n = 2, \\ \mathbb{Z}/2 & \text{otherwise.} \end{cases}$$

Let $\widetilde{\mathrm{SL}}_n(\mathbb{R})$ be the unique connected 2-fold cover of $\mathrm{SL}_n(\mathbb{R})$. A *spin structure* on E is a principal $\widetilde{\mathrm{SL}}_n(\mathbb{R})$-bundle $\widetilde{F(E)} \to X$ that fits into a commutative diagram

$$\begin{tikzcd}
\widetilde{\mathrm{SL}}_n(\mathbb{R}) \arrow[r] \arrow[d] & \mathrm{SL}_n(\mathbb{R}) \arrow[d] \\
\widetilde{F(E)} \arrow[r] \arrow[dr] & F(E) \arrow[d] \\
& X
\end{tikzcd}$$

Here the map $\widetilde{F(E)} \to F(E)$ is a 2-fold covering map that restricts to the 2-fold covering map $\widetilde{\mathrm{SL}}_n(\mathbb{R}) \to \mathrm{SL}_n(\mathbb{R})$ on each fiber. We say that E is *spin* if there exists a spin structure on E. Finally, if X is an oriented smooth manifold, then we say that X is spin if its tangent bundle is spin; when we say that X is a spin manifold, we mean that X is oriented, smooth, and spin.

Remark. The unique 2-fold cover of $\mathrm{SO}_n(\mathbb{R})$ is called the *spin group*, whence the name spin for the above phenomena.

The vanishing of $\mathbf{w_2}$. Recalling that connected 2-fold coverings of connected CW complexes Y are classified by nontrivial elements of $\mathrm{H}^1(Y;\mathbb{Z}/2)$, we see that the data of a spin structure is equivalent to the data of an element of $\mathrm{H}^1(F(E);\mathbb{Z}/2)$ that restricts to the unique nonzero element θ of $\mathrm{H}^1(\mathrm{SL}_n(\mathbb{R});\mathbb{Z}/2) = \mathbb{Z}/2$ on each fiber. The bottom left hand corner of the Leray–Serre spectral sequence of the fiber bundle $\mathrm{SL}_n(\mathbb{R}) \to F(E) \to X$ degenerates into an exact sequence of the form

$$0 \longrightarrow \mathrm{H}^1(X;\mathbb{Z}/2) \longrightarrow \mathrm{H}^1(F(E);\mathbb{Z}/2) \longrightarrow \mathrm{H}^1(\mathrm{SL}_n(\mathbb{R});\mathbb{Z}/2) \xrightarrow{w} \mathrm{H}^2(X;\mathbb{Z}/2). \tag{3.1}$$

If you have not seen this piece of algebra before, see [15, Example 1.A]. It follows that E is spin if and only if $w(\theta) = 0$.

Lemma 3.1. *With the above notation, we have $w(\theta) = w_2(E)$.*

Proof. Define $w_2'(E) = w(\theta)$. To prove that $w_2'(E) = w_2(E)$, we will prove that $w_2'(E)$ is a characteristic class satisfying the axioms characterizing the second Stiefel–Whitney class proved in [21]. This requires three things. In the first two items below, X is an arbitrary CW-complex.

- Let $f : X' \to X$ be a map of CW complexes, let $E \to X$ be an n-dimensional oriented real vector bundle, and let $E' \to X'$ be the pullback of E. Then we must prove that $w_2'(E') = f^*(w_2'(E))$. This follows immediately from the commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{H}^1(X;\mathbb{Z}/2) & \longrightarrow & \mathrm{H}^1(F(E);\mathbb{Z}/2) & \longrightarrow & \mathrm{H}^1(\mathrm{SL}_n(\mathbb{R});\mathbb{Z}/2) & \longrightarrow & \mathrm{H}^2(X;\mathbb{Z}/2) \\ & \downarrow f^* & & \downarrow & & \downarrow = & & \downarrow f^* \\ 0 \longrightarrow & \mathrm{H}^1(X';\mathbb{Z}/2) & \longrightarrow & \mathrm{H}^1(F(E');\mathbb{Z}/2) & \longrightarrow & \mathrm{H}^1(\mathrm{SL}_n(\mathbb{R});\mathbb{Z}/2) & \longrightarrow & \mathrm{H}^2(X';\mathbb{Z}/2) \end{array}$$

  given by the naturality of the Leray–Serre spectral sequence.

- Let $E \to X$ be an n-dimensional oriented real vector bundle and let $m \geq 0$. Then we must prove that $w_2'(E \oplus \mathbb{R}^m) = w_2'(E)$. The standard upper left hand corner inclusion $\mathrm{SL}_n(\mathbb{R}) \hookrightarrow \mathrm{SL}_{n+m}(\mathbb{R})$ induces an isomorphism on H^1 with $\mathbb{Z}/2$-coefficients. The desired result now follows as before from the commutative diagram

$$\begin{array}{ccccccccc}
0 \to \mathrm{H}^1(X;\mathbb{Z}/2) & \to & \mathrm{H}^1(F(E\oplus\mathbb{R}^m);\mathbb{Z}/2) & \to & \mathrm{H}^1(\mathrm{SL}_{n+m}(\mathbb{R});\mathbb{Z}/2) & \to & \mathrm{H}^2(X;\mathbb{Z}/2) \\
\downarrow^{=} & & \downarrow & & \downarrow^{\cong} & & \downarrow^{=} \\
0 \to \mathrm{H}^1(X;\mathbb{Z}/2) & \longrightarrow & \mathrm{H}^1(F(E);\mathbb{Z}/2) & \longrightarrow & \mathrm{H}^1(\mathrm{SL}_n(\mathbb{R});\mathbb{Z}/2) & \longrightarrow & \mathrm{H}^2(X;\mathbb{Z}/2)
\end{array}$$

given by the naturality of the Leray–Serre spectral sequence. Here the second vertical map comes from the map $F(E) \to F(E\oplus\mathbb{R}^m)$ induced by the map $E \to E\oplus\mathbb{R}^m$ which for $x \in X$ takes the fiber E_x to $E_x \oplus \mathbb{R}^m$ using the obvious inclusion.

- We must prove that there exists some 2-dimensional oriented real vector bundle $E \to X$ over some base X such that $w_2'(E) \neq 0$. Equivalently, we must prove that there exists some 2-dimensional oriented real vector bundle which is not spin. Let $E \to S^2$ be the 2-dimensional real vector bundle with Euler number 1. The associated oriented frame bundle $F(E) \to S^2$ is then fiberwise homotopy equivalent to the Hopf fibration $S^3 \to S^2$. In particular, we have $\mathrm{H}^1(F(E);\mathbb{Z}/2) = \mathrm{H}^1(S^3;\mathbb{Z}/2) = 0$, and thus there does not exist a spin structure. □

Corollary 3.2. *For some $n \geq 2$, let $E \to X$ be an n-dimensional oriented real vector bundle. Then E is spin if and only if $w_2(E) = 0$. In particular, if X is an oriented smooth manifold, then X is spin if and only if $w_2(X) = 0$.*

Remark. It follows from (3.1) that any two spin structures on $E \to X$ (represented as elements of $\mathrm{H}^1(F(E);\mathbb{Z}/2)$) differ by an element of $\mathrm{H}^1(X;\mathbb{Z}/2)$. Thus if a spin structure exists, then there is a simply transitive action of $\mathrm{H}^1(X;\mathbb{Z}/2)$ on the set of spin structures.

Even intersection forms. Our next goal is to show that if M^4 is a spin 4-manifold, then its intersection form is even. We first need the following lemma.

Lemma 3.3. *Let M^4 be a smooth 4-manifold. Then every element of $\mathrm{H}^2(M^4;\mathbb{Z})$ is Poincaré dual to an embedded oriented surface in M^4.*

Proof. Since $\mathbb{CP}^\infty$ is a $K(\mathbb{Z},2)$, there is a natural bijection between $\mathrm{H}^2(M^4;\mathbb{Z})$ and the set $[M^4,\mathbb{CP}^\infty]$ of homotopy classes of maps from M^4 to $\mathbb{CP}^\infty$ (see [8, Theorem 4.57]). By simplicial approximation, every homotopy class of maps $M^4 \to \mathbb{CP}^\infty$ has a representative whose image lies in the 4-skeleton of the usual triangulation of $\mathbb{CP}^\infty$, which is $\mathbb{CP}^2$. Consider a map $f : M^4 \to \mathbb{CP}^2$. Homotoping f, we can assume that f is smooth. Then for a regular value $x \in \mathbb{CP}^2$, the preimage $f^{-1}(x)$ is an embedded surface in M^4 which is Poincaré dual to the cohomology class represented by f. □

We now prove the following key result.

Lemma 3.4. *Let M^4 be a smooth closed oriented 4-manifold. Consider an element $\lambda \in \mathrm{H}^2(M^4;\mathbb{Z}/2)$ which is in the image of the map $\mathrm{H}^2(M^4;\mathbb{Z}) \to \mathrm{H}^2(M^4;\mathbb{Z}/2)$. Then $\lambda \cup w_2(M^4) = \lambda \cup \lambda$ in $\mathrm{H}^4(M^4;\mathbb{Z}/2) \cong \mathbb{Z}/2$.*

Proof. By Lemma 3.3, we can find an embedded oriented surface Σ in M^4 which is Poincaré dual to an element of $\mathrm{H}^2(M^4;\mathbb{Z})$ which projects to λ. To keep our notation straight, we will denote by $[\Sigma] \in \mathrm{H}_2(\Sigma;\mathbb{Z}/2)$ the $\mathbb{Z}/2$-fundamental class in Σ and by $\overline{[\Sigma]} \in \mathrm{H}_2(M^4;\mathbb{Z}/2)$ the $\mathbb{Z}/2$-fundamental class in M^4. Chasing through the definitions, the lemma is equivalent to the assertion that $w_2(M^4)(\overline{[\Sigma]}) \in \mathbb{Z}/2$ equals the algebraic self-intersection number of Σ in M^4 modulo 2. Let T_{M^4} be the tangent bundle of M^4. Also, let T_Σ and $N_{\Sigma/M}$ be the tangent and normal bundles of Σ, respectively. We thus have $T_{M^4}|_\Sigma = T_\Sigma \oplus N_{\Sigma/M}$, so

$$\begin{aligned} w_2(M^4)(\overline{[\Sigma]}) &= w_2(T_{M^4}|_\Sigma)([\Sigma]) \\ &= w_2(T_\Sigma)([\Sigma]) + w_2(N_{\Sigma/M})([\Sigma]) + (w_1(T_\Sigma) \cup w_1(N_{\Sigma/M}))([\Sigma]). \end{aligned}$$

Since Σ is orientable, we have $w_1(T_\Sigma) = 0$. Also, $w_2(T_\Sigma)([\Sigma])$ and $w_2(N_{\Sigma/M})([\Sigma])$ are the mod 2 reductions of the Euler numbers of T_Σ and $N_{\Sigma/M}$, respectively. The Euler characteristic of Σ is even, so $w_2(T_\Sigma)([\Sigma]) = 0$. We conclude that $w_2(M^4)(\overline{[\Sigma]})$ equals the mod 2 reduction of the Euler number of $N_{\Sigma/M}$.

Let θ be a section of $N_{\Sigma/M}$ with isolated simple zeros. The signed count of these zeros is the Euler number of $N_{\Sigma/M}$. Identifying $N_{\Sigma/M}$ with a tubular neighborhood of Σ in M, the section θ becomes a surface Σ' that is homotopic to Σ. The zeros of θ are in bijection with the intersections of Σ' and Σ, and the signs of those intersections are the same as the signs of the zeros. The Euler number of $N_{\Sigma/M}$ is thus equal to the signed count of the intersections of Σ and Σ', i.e. the algebraic self-intersection number of Σ. The lemma follows. □

Corollary 3.5. *Let M^4 be a smooth closed oriented 4-manifold. Let $\omega(\cdot,\cdot)$ be the intersection form on $\mathrm{H}^2(M^4;\mathbb{Z})$/torsion. If $w_2(M^4) = 0$, then $\omega(\cdot,\cdot)$ is even. Conversely, if $\omega(\cdot,\cdot)$ is even and $\mathrm{H}_1(M^4;\mathbb{Z})$ has no 2-torsion, then $w_2(M^4) = 0$.*

Proof. The first assertion of the corollary follows immediately from Lemma 3.4. For the second assertion, the condition that $\mathrm{H}_1(M^4;\mathbb{Z})$ has no 2-torsion implies that the map $\mathrm{H}^2(M^4;\mathbb{Z}) \to \mathrm{H}^2(M^4;\mathbb{Z}/2)$ is surjective. Combining this with Lemma 3.4 and the fact that $\omega(\cdot,\cdot)$ is even, we deduce that $\lambda \cup w_2(M^4) = 0$ for all $\lambda \in \mathrm{H}^2(M^4;\mathbb{Z}/2)$. By Poincaré duality, this implies that $w_2(M^4) = 0$. □

Remark. Without the assumption that $\mathrm{H}_1(M^4;\mathbb{Z})$ has no 2-torsion, there do exist examples of smooth closed oriented 4-manifolds M^4 whose intersection forms are even but where $w_2(M^4) \neq 0$. Indeed, there even exist examples which are smooth complex projective varieties (e.g. the Enriques surface).

4 The tangent bundles of compact spin smooth 4-manifolds

In this section, we prove Proposition 2.1, which asserts that if M^4 is a smooth spin closed 4-manifold, then M^4 is almost parallelizable. Letting $p \in M^4$ be a point and $N^4 = M^4 \smallsetminus \{p\}$, this is equivalent to saying that N^4 is parallelizable.

Let T_{N^4} be the tangent bundle of N^4, let $F(T_{N^4})$ be the oriented frame bundle of T_{N^4}, and let $\widetilde{F(T_{N^4})}$ be the fiberwise 2-fold cover of $F(T_{N^4})$ provided by the spin structure, so the fibers of $\widetilde{F(T_{N^4})}$ are $\widetilde{\mathrm{SL}}_4(\mathbb{R})$. To prove that T_{N^4} is a trivial bundle, it is enough to show that $\widetilde{F(T_{N^4})}$ is a trivial bundle. We will do this using obstruction theory; see [2, Chapter 7] and [8, p. 415] for two different expositions of this (the point of view of [2, Chapter 7] is more elementary). Fix a triangulation of N^4. The group $\widetilde{\mathrm{SL}}_4(\mathbb{R})$ is 1-connected by construction. Moreover, $\pi_2(\widetilde{\mathrm{SL}}_4(\mathbb{R})) = 0$; indeed, $\pi_2(G) = 0$ for every Lie group G. This follows from [18, Theorem 21.7]; see also [23]. Of course, this can also be proved for $\mathrm{SL}_4(\mathbb{R})$ and thus for $\widetilde{\mathrm{SL}}_4(\mathbb{R})$ by elementary methods. We deduce that $\widetilde{\mathrm{SL}}_4(\mathbb{R})$ is 2-connected. The first possible obstruction to trivializing $\widetilde{F(T_{N^4})}$ thus lies in

$$\mathrm{H}^4(N^4; \pi_3(\widetilde{\mathrm{SL}}_4(\mathbb{R}))).$$

Here there might be a nontrivial monodromy action of $\pi_1(N^4)$ on the π_3 of the fiber $\widetilde{\mathrm{SL}}_4(\mathbb{R})$, so $\pi_3(\widetilde{\mathrm{SL}}_4(\mathbb{R}))$ in this cohomology group should be regarded as a local coefficient system. However, since N^4 is a noncompact 4-manifold, we have $\mathrm{H}^4(N^4; V) = 0$ for all local coefficient systems V. This follows from the appropriate version of Poincaré-Lefschetz duality for local coefficient systems (here we must use locally finite homology since N^4 is noncompact; see the remark below for an alternate approach). The above obstruction therefore vanishes and $\widetilde{F(T_{N^4})}$ can be trivialized over the entire 4-skeleton of N^4, i.e. over all of N^4.

Remark. An alternate way of seeing that $\mathrm{H}^4(N^4; V) = 0$ in the above proof is to show that N^4 is homotopy equivalent to a 3-dimensional CW complex. This kind of thing holds in great generality: if X is a smooth noncompact n-manifold, then X is homotopy equivalent to an $(n-1)$-dimensional CW complex (see, e.g., [22, Theorem 2.2], which proves this by constructing a proper Morse function with no local maxima).

5 The stable J-homomorphism

In this section, we prove Proposition 2.2. As we said after the statement of Proposition 2.2, the key will be the stable J-homomorphism $J_k : \widetilde{KO}(S^{k+1}) \to \pi_k^S$. This will require a preliminary discussion of classifying spaces for groups, the K-theory of spheres, and the Pontryagin–Thom construction.

Classifying spaces for groups. Let G be a topological group. A *classifying space* for G is a topological space BG together with a principal G-bundle $EG \to BG$ such that for all CW complexes X, there is a bijection between the set $[X, BG]$ of homotopy classes of maps from X to BG and the set of principal G-bundles on X. Given a map $f : X \to BG$, the associated principal G-bundle on X is the pullback $f^*(EG)$. The base BG of a principal G-bundle $EG \to BG$ forms a classifying space for G if and only if EG is contractible (see [27, §19]. From this, one can show that if BG is a classifying space for G, then ΩBG is homotopy equivalent to G (see [8, Proposition 4.66]; here ΩBG denotes the based loop-space of BG). In other words, BG is a "de-looping" of G. This implies in particular that BG is simply-connected if G is connected. Milnor [16] proved that all topological groups have classifying spaces.

K-theory of spheres. To define J_k, we will need to understand $\widetilde{KO}(S^{k+1})$. We will give a somewhat abstract description of the necessary result; see the second remark below for a more hands-on point of view. Recall that $\widetilde{KO}(S^{k+1})$ consists of the set of principal $\mathrm{SL}(\mathbb{R})$-bundles. The classifying space $B\,\mathrm{SL}(\mathbb{R})$ is the direct limit of the classifying spaces $B\,\mathrm{SL}_n(\mathbb{R})$. The maps $\mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_m(\mathbb{R}) \to \mathrm{SL}_{n+m}(\mathbb{R})$ defined via the formula

$$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \tag{5.1}$$

induce maps $B\,\mathrm{SL}_n(\mathbb{R}) \times B\,\mathrm{SL}_m(\mathbb{R}) \to B\,\mathrm{SL}_{n+m}(\mathbb{R})$. Passing to the direct limit, we get a map $B\,\mathrm{SL}(\mathbb{R}) \times B\,\mathrm{SL}(\mathbb{R}) \to B\,\mathrm{SL}(\mathbb{R})$. This turns $B\,\mathrm{SL}(\mathbb{R})$ into a topological monoid. By the definition of a classifying space, we have

$$\widetilde{KO}(S^{k+1}) \cong [S^{k+1}, B\,\mathrm{SL}(\mathbb{R})].$$

The abelian group structure on $\widetilde{KO}(S^{k+1})$ is induced by the monoid structure on $B\,\mathrm{SL}(\mathbb{R})$ (which while not commutative itself is commutative up to homotopy). The key computation now is

$$\widetilde{KO}(S^{k+1}) \cong [S^{k+1}, B\,\mathrm{SL}(\mathbb{R})] \cong \pi_{k+1}(B\,\mathrm{SL}(\mathbb{R})) \cong \pi_k(\Omega B\,\mathrm{SL}(\mathbb{R})) \cong \pi_k(\mathrm{SL}(\mathbb{R})).$$

The second isomorphism here follows from the fact that $B\,\mathrm{SL}(\mathbb{R})$ is simply-connected, which itself is a consequence of the fact that $\mathrm{SL}(\mathbb{R})$ is connected.

Remark. One might worry that the isomorphism $\widetilde{KO}(S^{k+1}) \cong \pi_k(\mathrm{SL}(\mathbb{R}))$ is only a bijection of sets but does not respect the additive structure. That it does respect the additive structure can be proved using the standard Eckmann–Hilton argument as follows. Consider $x, y \in \pi_k(\mathrm{SL}(\mathbb{R}))$ which correspond to elements $X, Y \in \widetilde{KO}(S^{k+1})$. There exist $n, m \geq 1$ such that $x \in \pi_k(\mathrm{SL}_n(\mathbb{R}))$ and $y \in \pi_k(\mathrm{SL}_m(\mathbb{R}))$ (here we are abusing notation and regarding $\pi_k(\mathrm{SL}_n(\mathbb{R}))$ and $\pi_k(\mathrm{SL}_m(\mathbb{R}))$ as subgroups of $\pi_k(\mathrm{SL}(\mathbb{R}))$). For $a \in \pi_k(\mathrm{SL}_n(\mathbb{R}))$ and $b \in \pi_k(\mathrm{SL}_m(\mathbb{R}))$, let $a * b \in \pi_k(\mathrm{SL}_{n+m}(\mathbb{R}))$ be the loop obtained by applying the map (5.1) pointwise. It is clear from the definitions that $x * y$

represents $X + Y \in \widetilde{KO}(S^{k+1})$. Letting $\cdot$ denote the product in $\pi_k(\mathrm{SL}(\mathbb{R}))$, we then have

$$x * y = (x \cdot 1) * (1 \cdot y) = (x * 1) \cdot (1 * y) = x \cdot y,$$

as desired.

Remark. A more pedestrian perspective on the isomorphism $\widetilde{KO}(S^{k+1}) \cong \pi_k(\mathrm{SL}(\mathbb{R}))$ is as follows. Consider a principal $\mathrm{SL}(\mathbb{R})$-bundle B on S^{k+1}. Letting D_+ and D_- be the upper and lower hemispheres of S^{k+1}, the restrictions of B to D_+ and D_- are trivial. Since $D_+ \cap D_- = S^k$, the bundle B can thus be described as $(D_+ \times \mathbb{R}^\infty) \sqcup (D_- \times \mathbb{R}^\infty)/\sim$, where $\sim$ identifies $(x, \vec{v}) \in \partial D_+ \times \mathbb{R}^\infty$ with $(x, f(x)(\vec{v})) \in \partial D_- \times \mathbb{R}^\infty$ for some map $f : S^k \to \mathrm{SL}(\mathbb{R})$. The homotopy class of f is then the element of $\pi_k(\mathrm{SL}(\mathbb{R}))$ associated to B. It is called the *clutching function* for B.

The Pontryagin–Thom construction. For proofs of the results we discuss in this paragraph, see [19, §7]. Fix $n \geq 1$ and $k \geq 0$. Our goal is to give a "geometric" description of $\pi_{n+k}(S^n)$. This group depends on the choice of a basepoint. We will regard the sphere as the one-point compactification of Euclidean space; the basepoint will be the point at infinity. We thus want to determine

$$\pi_{n+k}(S^n, \infty) = [(S^{n+k}, \infty), (S^n, \infty)].$$

If X is a smooth manifold (possibly with boundary), then a *framed submanifold* of X consists of the following data.

- A smooth compact properly embedded submanifold M of X. Contrary to our assumptions elsewhere in this note, we do not assume that M is connected; in fact, we allow M to be empty.
- A *framing* of the normal bundle $N_{X/M}$ of M in X, that is, a bundle isomorphism

$$\mathfrak{f} : M \times \mathbb{R}^p \longrightarrow N_{X/M},$$

  where p is the codimension of M in X.

Define $\Omega_k^{\mathrm{frame}}(S^{n+k}, \infty)$ to be the set of k-dimensional framed submanifolds M^k of S^{n+k} such that $\infty \notin M^k$ (including the empty manifold) modulo the equivalence relation of *cobordism*, which is defined as follows.

- If M_0 and M_1 are k-dimensional framed submanifolds of $S^{n+k} \setminus \{\infty\}$, then M_0 and M_1 are cobordant if there exists a framed $(k+1)$-dimensional submanifold C of $(S^{n+k} \setminus \{\infty\}) \times [0,1]$ such that for $i = 0, 1$, we have $C \cap (S^{n+k} \times i) = M_i \times i$ and the framing of C on $C \cap (S^{n+k} \times i)$ agrees with the framing on M_i.

The key fact is the following theorem of Pontryagin.

Theorem 5.1 (Pontryagin). *For $n \geq 1$ and $k \geq 0$, we have*

$$\pi_{n+k}(S^n, \infty) = \Omega_k^{frame}(S^{n+k}, \infty).$$

This is an isomorphism of groups, where the group operation on $\Omega_k^{frame}(S^{n+k}, \infty)$ is disjoint union.

We refer to [19, §7] for the proof, but to clarify what is going on we indicate the construction of a map $f : (S^{n+k}, \infty) \to (S^n, \infty)$ from a k-dimensional framed submanifold M^k of S^{n+k} such that $\infty \notin M^k$ (this construction is known as the *Pontryagin–Thom construction*). Let $U \subset S^{n+k}$ be a tubular neighborhood of M^k such that $\infty \notin U$. The framing on the normal bundle of M^k then induces a homeomorphism

$$\theta : U \xrightarrow{\theta} M^k \times \mathbb{R}^n.$$

Let $\nu : U \to \mathbb{R}^n$ be the composition of θ with the projection $M^k \times \mathbb{R}^n \to \mathbb{R}^n$. Then f is the map defined via the formula

$$f(x) = \begin{cases} \nu(x) & \text{if } x \in U, \\ \infty & \text{otherwise.} \end{cases}$$

Observe that $f(\infty) = \infty$. The construction of f depends on various choices, but varying these choices results in homotopic f.

The stable J-homomorphism. We finally come to the construction of the stable J-homomorphism

$$J_k : \widetilde{KO}(S^{k+1}) \longrightarrow \pi_k^S.$$

Consider $[B] \in \widetilde{KO}(S^{k+1})$. As discussed above, $[B]$ corresponds to an element of $\pi_k(\mathrm{SL}(\mathbb{R}))$. Represent this element via a map $\psi : S^k \to \mathrm{SL}_n(\mathbb{R})$ for some $n \gg 0$. The image $J_k([B]) \in \pi_k^S$ will be the image in π_k^S of the element of $\pi_{n+k}(S^n)$ represented by the following framed submanifold of $S^{n+k} \setminus \{\infty\} \cong \mathbb{R}^{n+k}$.

- The manifold will be image in $\mathbb{R}^{n+k}$ of the unit k-sphere in $\mathbb{R}^{k+1}$. Denote this by S.
- For the framing, let $\{\vec{e}_1, \ldots, \vec{e}_{n+k}\}$ be the standard basis for $\mathbb{R}^{n+k}$ and let $\mathfrak{n} : S \to \mathbb{R}^{k+1} \subset \mathbb{R}^{n+k}$ be the outward facing unit normal vector. We thus get a framing $\mathfrak{f}_0 : S \times \mathbb{R}^n \to N_{\mathbb{R}^{n+k}/S}$ defined via the formula

  $$\mathfrak{f}_0(p, c_1, \ldots, c_n) = (p, c_1(p) + \sum_{i=2}^{n} c_i \vec{e}_{k+i}).$$

  This is not the framing we are looking for; indeed, S with this framing represents the trivial element of $\Omega_k^{\text{frame}}(S^{n+k}, \infty)$ (easy check!). Instead, the framing we want is the result of "twisting" this trivial framing via $\psi : S^k \to \mathrm{SL}_n(\mathbb{R})$. More precisely, the framing we want is the framing $\mathfrak{f} : S \times \mathbb{R}^n \to N_{\mathbb{R}^{n+k}/S}$ defined via the formula

$$\mathfrak{f}(p, \vec{v}) = \mathfrak{f}_0(p, \psi(p)(\vec{v})).$$

It is an easy exercise to see that this is a well-defined group homomorphism.

The stable J-homomorphism was first introduced by Whitehead [29] following work of Hopf [9]. Determining its image is quite nontrivial. In complete generality, this was accomplished by Adams [1] assuming the truth of the Adams conjecture, which was later proved by Quillen [24]. We will not need the general statement, but only the following special case which was proved by Rochlin [25].

Theorem 5.2 (Rochlin)**.** *The image of $J_3 : \widetilde{KO}(S^4) \to \pi_3^S$ is isomorphic to $\mathbb{Z}/24$.*

Endgame. All the pieces are now in place for the proof of Proposition 2.2. Let us first recall its statement. Let $E \to S^4$ be an oriented real vector bundle such that there exists a compact oriented 4-manifold M^4 with $T_{M^4} = \beta^*(E)$, where $\beta : M^4 \to S^4$ is a ball-collapse map. We must prove that the element $[E] \in \widetilde{KO}(S^4) \cong \mathbb{Z}$ is divisible by 24. By Theorem 5.2, this is equivalent to proving that $[E] \in \ker(J_3)$. As we will see, this is almost formal.

Let $B \subset M^4$ be the 3-dimensional ball used to construct the ball-collapse map β and let $\widehat{M}^4 = M^4 \setminus \mathrm{Int}(B)$. For some large $n \gg 0$, let $S \subset \mathbb{R}^n$ be the 3-sphere used to construct J_3. By choosing n large enough, we can ensure that the following hold.

- There is a proper embedding $i : \widehat{M}^4 \to \mathbb{R}^n \times [0,1]$ such that $i(\partial \widehat{M}^4) = S \times 0 \subset \mathbb{R}^n \times 0$. This follows from Whitney's embedding theorem.
- Let N be the normal bundle of $\widehat{M}^4$ in $\mathbb{R}^n \times [0,1]$. Then N is a trivial bundle. Indeed, we have $T_{\widehat{M}^4} \oplus N \cong \widehat{M}^4 \times \mathbb{R}^{n+1}$, so $[T_{\widehat{M}^4}] + [N] = 0$ in $\widetilde{KO}(\widehat{M}^4)$. But we already know that $T_{\widehat{M}^4}$ is a trivial bundle, so $[N] = 0$. Increasing n more if necessarily, we can ensure that N is actually a trivial bundle.

Choose a framing of N. The restriction of this framing to $\partial \widehat{M}^4 = S$ can be obtained by twisting the trivial framing of S (as in the construction of the J-homomorphism) via an element of $\pi_3(\mathrm{SL}_{n-3}(\mathbb{R}))$ which represents $-[E] \in \widetilde{KO}(S^4)$ (the negative sign is here since have switched from the tangent bundle to the normal bundle). We have exhibited an explicit cobordism from our framing of S to the empty manifold, so we conclude that $J_3(-[E]) = 0$ and hence $J_3([E]) = 0$.

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