Rochlin’s theorem on signatures of spin 4-manifolds via algebraic topology

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Abstract

We give the original proof of Rochlin’s famous theorem on signatures of smooth spin 4-manifolds, which uses techniques from algebraic topology. We have attempted to include enough background and details to make this proof understandable to a geometrically minded topologist. We also include a fairly complete discussion of spin structures on manifolds.

1 Introduction

Let $M^4$ be a closed oriented smooth 4-manifold. All manifolds in this note (including $M^4$) are assumed to be connected.

Intersection form. The cup product map

$$H^2(M^4; \mathbb{Z}) \times H^2(M^4; \mathbb{Z}) \rightarrow H^4(M^4; \mathbb{Z}) \cong \mathbb{Z}$$

descends to an integral bilinear form $\omega(\cdot, \cdot)$ on $V := H^2(M^4; \mathbb{Z})/\text{torsion}$ called the intersection form. Poincaré duality implies that $\omega(\cdot, \cdot)$ is a unimodular integral form, that is, it induces an isomorphism between $V$ and its dual $V^* = \text{Hom}(V, \mathbb{Z})$. It plays a fundamental role in the topology of 4-manifolds. For example, building on work of Whitehead [30], Milnor [17] proved that if $M^4$ is simply-connected, then its homotopy type is determined by $H^2(M^4; \mathbb{Z})$ together with $\omega$.

Signature of forms. One of the basic invariants of $\omega$ is its signature, which is defined as follows. Let $\omega_Q(\cdot, \cdot)$ be the induced form on $V \otimes \mathbb{Q}$. We can diagonalize $\omega_Q$, i.e. choose coordinates on $V \otimes \mathbb{Q}$ such that with respect to these coordinates, we have $\omega_Q(\tilde{v}, \tilde{w}) = \tilde{v}^t M \tilde{w}$ for a diagonal matrix $M$. Since $\omega$ is unimodular, all the diagonal entries of $M$ are nonzero. The signature of $\omega$ is then $r - s$, where $r$ is the number of positive diagonal entries of $M$ and $s$ is the number of negative entries. Neither $r$ nor $s$ depend on the choice of diagonalization.

Signature of 4-manifolds. Define $\sigma(M^4)$ to be the signature of $\omega$. As the following example shows, $\sigma(M^4)$ can achieve arbitrary values.
Example. For \( r, s \geq 0 \), let \( M^4 \) be the connect sum of \( r \) copies of \( \mathbb{CP}^2 \) and \( s \) copies of \( \overline{\mathbb{CP}}^2 \) (here \( \overline{\mathbb{CP}}^2 \) is \( \mathbb{CP}^2 \) with its orientation reversed). Then \( H^2(M^4; \mathbb{Z}) \cong \mathbb{Z}^{r+s} \) and the intersection form on \( H^2(M^4; \mathbb{Z}) \) is represented by a diagonal matrix with \( r \) entries equal to 1 and \( s \) entries equal to \(-1\). In particular, \( \sigma(M^4) = r - s \).

Spin structures and even forms. However, it is definitely not true that \( \omega(\cdot, \cdot) \) can be an arbitrary form. Let \( M^4 \) be a 4-manifold. Recall that \( M^4 \) is orientable if and only if its first Stiefel–Whitney class \( w_1 \in H^1(M^4; \mathbb{Z}/2) \) vanishes. The 4-manifold \( M^4 \) is spin if it is orientable and \( w_2(M^4) \in H^2(M^4; \mathbb{Z}/2) \) also vanishes. We will say more about this in §3 below. In particular, we will show that if \( M^4 \) is spin and closed, then its intersection form \( \omega(\cdot, \cdot) \) is even, i.e. \( \omega(\tilde{v}, \tilde{v}) \) is an even integer for all \( \tilde{v} \in H^2(M^4; \mathbb{Z})/\text{torsion} \). The converse is almost true; for instance, it is true if \( M^4 \) is simply-connected. It is perhaps a little surprising that unimodular integral bilinear forms can be even. Here is an important example.

Example. The \( E_8 \) form is the bilinear form on \( \mathbb{Z}^8 \) defined via the formula \( \omega(\tilde{v}, \tilde{w}) = \tilde{v}^t M \tilde{w} \) with

\[
M = \begin{pmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}.
\]

An easy calculation shows that \( \det(M) = 1 \), so \( \omega \) is unimodular. It is also not hard to see that it is even. Moreover, \( \omega(\tilde{v}, \tilde{v}) > 0 \) for all \( \tilde{v} \in \mathbb{Z}^8 \), so the signature of \( \omega \) is 8.

Divisibility and Rochlin’s theorem. A classical theorem of van der Blij [28] says that the signature of a unimodular even integral bilinear form is divisible by 8. See [20] for a textbook proof of van der Blij’s theorem. The main result proved in this note is the following theorem of Rochlin [25], which strengthens this divisibility for the signatures of smooth closed 4-manifolds. It implies in particular that no smooth closed simply-connected 4-manifold has \( E_8 \) for its intersection form.

Rochlin’s Theorem. If \( M^4 \) is a smooth spin closed 4-manifold, then

\[
\sigma(M^4) \equiv 0 \pmod{16}.
\]

Remark. The condition that \( M^4 \) is spin cannot be replaced with the condition that the intersection form is even. A counterexample is given by the Enriques surface.
Recall that the Hirzebruch Signature Theorem [21, Theorem 19.4] gives a formula for the signature of a $4k$-dimensional manifold in terms of the Pontryagin classes of the manifold. For a 4-manifold $M^4$, this formula is

$$\sigma(M^4) = \frac{1}{3} p_1(M^4)([M^4]).$$

Rochlin’s theorem is thus equivalent to the following theorem.

**Rochlin’s Theorem’.** If $M^4$ is a smooth spin closed 4-manifold, then

$$p_1(M^4)([M^4]) \equiv 0 \pmod{48}.$$

**Comments on proofs.** In this note, we will give a variant on the original proof of Rochlin’s theorem, which uses techniques from algebraic topology. See [6] for a French translation of the original Russian paper [25] together with quite a bit of useful commentary. Our viewpoint is strongly inspired by the proof sketched in Kervaire–Milnor’s paper [10]. There are many other proof of Rochlin’s theorem.

- For proofs that use techniques from 4-manifold topology, see [5, 7, 14]; textbook references for these geometric proofs include [11, Chapter XI] and [26, p. 507].

- A novel proof using techniques from 3-manifold topology can be found in [12, Appendix].

- A proof using the Atiyah–Singer index theorem can be found in [13, Chapter IV.1].

**Later developments.** While Rochlin’s Theorem might appear to be a curiosity, it is actually the root of many important developments.

- Freedman [4] proved that there exist closed simply-connected topological 4-manifolds whose intersection form is any given unimodular integral bilinear form. For instance, there exists a simply-connected topological 4-manifold $M^4$ whose intersection form is the $E_8$ form. By Rochlin’s Theorem this 4-manifold cannot be given a smooth structure.

- It is almost (but not quite) known which unimodular integral bilinear forms can be the intersection forms of a smooth simply-connected 4-manifold. The main result here is a theorem of Donaldson [3] which says that if $\omega$ is the intersection form of a smooth simply-connected 4-manifold and $\omega$ is definite (i.e. $\omega(\vec{v}, \vec{v})$ is always nonpositive or always nonnegative), then with respect to some basis $\omega$ is represented by either the identity matrix or the negative of the identity matrix. It is also known whether or not most indefinite forms are realized; the remaining cases are the subject of the famous 11/8-conjecture.
2 The proof of Rochlin’s Theorem

In this section, we will give the proof of Rochlin’s Theorem. Actually, we will prove the equivalent Rochlin’s Theorem’, which asserts that if $M^4$ is a smooth spin closed 4-manifold, then

$$p_1(M^4)([M^4]) \equiv 0 \pmod{48}.$$  

This proof will depend on two key facts which will be proved in subsequent sections.

The tangent bundle. The first ingredient is a description of the tangent bundle of $M^4$. Recall that a manifold $X$ is said to be parallelizable if its tangent bundle $T_X$ is trivial and is said to be almost parallelizable if $X \setminus \{p\}$ is parallelizable for any $p \in X$. The following proposition will be proved in §4 using obstruction theory.

**Proposition 2.1.** Let $M^4$ be a smooth spin closed 4-manifold. Then $M^4$ is almost parallelizable.

Let $B^4 \subset M^4$ be a submanifold which is diffeomorphic to a closed 4-dimensional ball. The result of collapsing $M^4 \setminus \text{Int}(B^4)$ to a point is homeomorphic to a 4-dimensional sphere $S^4$; let $\beta : M^4 \to S^4$ be the resulting quotient map. We will call $\beta$ a ball-collapse map (of course, it depends on various choices, but none of them are important in what follows). Proposition 2.1 implies that $M^4 \setminus \text{Int}(B^4)$ is parallelizable, so there exists a 4-dimensional oriented real vector bundle $E \to S^4$ such that $T_{M^4} = \beta^*(E)$. The induced map $\beta^* : H^4(S^4; \mathbb{Z}) \to H^4(M^4; \mathbb{Z})$ is an isomorphism, so

$$p_1(M^4)([M^4]) = p_1(E)([S^4]).$$

The rest of the proof will focus on understanding $p_1(E)([S^4])$.

Transition to K-theory. The Pontryagin classes are stable characteristic classes, which in our context implies that $p_1(E) = p_1(E \oplus e^k)$ for all $k \geq 0$, where $e^k$ is the $k$-dimensional trivial bundle $S^4 \times \mathbb{R}^k$. This brings us into the realm of K-theory, whose basic definitions we quickly recall. Let $X$ be a compact connected CW-complex. Two oriented real vector bundles $B_1$ and $B_2$ on $X$ define the same stable oriented real vector bundle if there exists some $k_1, k_2 \geq 0$ such that $B_1 \oplus e^{k_1} \cong B_2 \oplus e^{k_2}$. This defines an equivalence relation on the set of oriented real vector bundles on $X$; if $B$ is such a bundle, then we will write $[B]$ for its equivalence class. The reduced oriented K-theory of $X$, denoted $KO(X)$, is the set of stable oriented real vector bundles.

**Remark.** An alternate description of $KO(X)$ is that it is the set of principal $\text{SL}(\mathbb{R})$-bundles on $X$, where $\text{SL}(\mathbb{R})$ is the union of the increasing sequence

$$\text{SL}_1(\mathbb{R}) \subset \text{SL}_2(\mathbb{R}) \subset \text{SL}_3(\mathbb{R}) \subset \cdots$$

of groups.
The set $\widetilde{KO}(X)$ forms an abelian group under connected sum; the identity element is the equivalence class of the trivial bundle. For all $i \geq 1$, the $i^{th}$ Pontryagin class induces a well-defined set map

$$p_i : \widetilde{KO}(X) \to H^{4i}(X;\mathbb{Z}).$$

The $p_i$ are not necessarily homomorphisms. Instead, for $[B_1], [B_2] \in \widetilde{KO}(X)$ we have

$$p_i([B_1] + [B_2]) = p_i([B_1]) + p_i([B_2]) + \theta,$$

where $\theta \in H^{4i}(X;\mathbb{Z})$ is a linear combination of elements of the form $\theta_1 \cup \theta_2$ with

$$\theta_1, \theta_2 \in \bigoplus_{j=1}^{i-1} H^j(X;\mathbb{Z}).$$

**Calculating the Pontryagin class.** We now return to the bundle $E \to S^4$ constructed above. Our goal is to prove that

$$p_1(E)([S^4]) \equiv 0 \pmod{48}. \quad (2.1)$$

The Bott Periodicity theorem (see [18]) implies that $\widetilde{KO}(S^4) \cong \mathbb{Z}$. Since $H^i(S^4;\mathbb{Z}) = 0$ for $1 \leq i \leq 3$, the first Pontryagin class actually gives a homomorphism

$$p_1 : \mathbb{Z} \cong \widetilde{KO}(S^4) \to H^4(S^4;\mathbb{Z}) \cong \mathbb{Z}.$$

**Claim.** We have $p_1(n) = \nu \cdot n$ for some $\nu \in 2\mathbb{Z}$.

**Proof of claim.** Consider $[B] \in \widetilde{KO}(S^4)$. It is enough to show that $p_1([B])$ is even. By definition, $p_1([B]) = c_2([B_C])$, where $B_C$ is the complexification of $B$. The mod 2 reduction of $c_2([B_C])$ is $w_2((B_C)_\mathbb{R})$, where $(B_C)_\mathbb{R}$ is the real bundle underlying the complex bundle $B_C$. Since $(B_C)_\mathbb{R} \cong B \oplus B$ and $H^1(S^4;\mathbb{Z}/2) = 0$, we deduce that

$$w_2((B_C)_\mathbb{R}) = w_2(B \oplus B) = w_2(B) + w_1(B) \cup w_1(B) + w_2(B) = 2w_2(B) = 0,$$

which implies that $p_1([B]) = c_2([B_C])$ is even. \qed

**Remark.** In fact, one can show that $\nu = 2$ in the above claim, but we will not need this.

**Endgame via the stable J-homomorphism.** The desired identity (2.1) now follows immediately from the Claim and the following proposition, which will be proved in §5.

**Proposition 2.2.** Let $E \to S^4$ be an oriented real vector bundle such that there exists a compact oriented 4-manifold $M^4$ with $T_{M^4} = \beta^*(E)$, where $\beta : M^4 \to S^4$ is a ball-collapse map. Then the element $[E] \in \widetilde{KO}(S^4) \cong \mathbb{Z}$ is divisible by 24.
This is the deepest part of the proof. The key is the stable $J$-homomorphism. Recall that the Freudenthal suspension theorem says that for all $k \geq 0$, the group $\pi_{n+k}(S^n)$ is independent of $n$ for $n > 0$; the stable value is the $k^{th}$ stable stem and is denoted $\pi^S_k$. Calculating $\pi^S_k$ is very difficult. The “first layer” comes from homomorphisms

$$J_k : \widetilde{KO}(S^{k+1}) \to \pi^S_k$$

which were first defined by Whitehead [29] following work of Hopf [9]. We will discuss the stable $J$-homomorphism more in §5. The two facts about it that go into Proposition 2.2 are as follows.

- Letting $[E] \in \widetilde{KO}(S^4)$ be as in the proposition, we have $[E] \in \ker(J_3)$. This will be almost immediate from the definition.

- The image of $J_3$ is isomorphic to $\mathbb{Z}/24$. This is a deep fact, and in some sense is the heart of the reason that Rochlin’s theorem holds.

**Remark.** In fact, $\pi^S_3 \cong \mathbb{Z}/24$, though this is not necessary for our proof.

### 3 Spin 4-manifolds

Before we prove the propositions stated in §2, we need to spend some time discussing general facts about spin manifolds. A good reference that influenced our exposition is [13, §II.2]. In this section, $X$ is an arbitrary connected CW complex.

**Definition of spin structure.** We begin by giving the proper definition of a spin structure; the definition given in the introduction in terms of Stiefel–Whitney classes will then be a theorem (see Corollary 3.2 below). For some $n \geq 2$, let $E \to X$ be an $n$-dimensional oriented real vector bundle and let $F(E) \to X$ be the frame bundle of $E$. Thus $F(E)$ is a principal $\text{SL}_n(\mathbb{R})$-bundle. The group $\text{SL}_n(\mathbb{R})$ deformation retracts onto its maximal compact subgroup $\text{SO}_n(\mathbb{R})$, so

$$\pi_1(\text{SL}_n(\mathbb{R})) = \pi_1(\text{SO}_n(\mathbb{R})) = \begin{cases} \mathbb{Z} & \text{if } n = 2, \\ \mathbb{Z}/2 & \text{otherwise.} \end{cases}$$

Let $\widetilde{\text{SL}}_n(\mathbb{R})$ be the unique connected 2-fold cover of $\text{SL}_n(\mathbb{R})$. A spin structure on $E$ is a principal $\widetilde{\text{SL}}_n(\mathbb{R})$-bundle $\widetilde{F(E)} \to X$ that fits into a commutative diagram

$$\begin{array}{ccc}
\widetilde{\text{SL}}_n(\mathbb{R}) & \longrightarrow & \text{SL}_n(\mathbb{R}) \\
\downarrow & & \downarrow \\
\widetilde{F(E)} & \longrightarrow & F(E) \\
\downarrow & & \downarrow \\
X & & X
\end{array}$$
Here the map $\tilde{F(E)} \to F(E)$ is a 2-fold covering map that restricts to the 2-fold covering map $\tilde{\text{SL}}_n(\mathbb{R}) \to \text{SL}_n(\mathbb{R})$ on each fiber. We say that $E$ is spin if there exists a spin structure on $E$. Finally, if $X$ is an oriented smooth manifold, then we say that $X$ is spin if its tangent bundle is spin; when we say that $X$ is a spin manifold, we mean that $X$ is oriented, smooth, and spin.

Remark. The unique 2-fold cover of $\text{SO}_n(\mathbb{R})$ is called the spin group, whence the name spin for the above phenomena.

The vanishing of $w_2$. Recalling that connected 2-fold coverings of connected CW complexes $Y$ are classified by nontrivial elements of $H^1(Y; \mathbb{Z}/2)$, we see that the data of a spin structure is equivalent to the data of an element of $H^1(F(E); \mathbb{Z}/2)$ that restricts to the unique nonzero element $\theta$ of $H^1(\text{SL}_n(\mathbb{R}); \mathbb{Z}/2) = \mathbb{Z}/2$ on each fiber. The bottom left hand corner of the Leray–Serre spectral sequence of the fiber bundle $\text{SL}_n(\mathbb{R}) \to F(E) \to X$ degenerates into an exact sequence of the form

$$0 \to H^1(X; \mathbb{Z}/2) \to H^1(F(E); \mathbb{Z}/2) \to H^1(\text{SL}_n(\mathbb{R}); \mathbb{Z}/2) \xrightarrow{w} H^2(X; \mathbb{Z}/2). \quad (3.1)$$

If you have not seen this piece of algebra before, see [15, Example 1.A]. It follows that $E$ is spin if and only if $w(\theta) = 0$.

Lemma 3.1. With the above notation, we have $w(\theta) = w_2(E)$.

Proof. Define $w'_2(E) = w(\theta)$. To prove that $w'_2(E) = w_2(E)$, we will prove that $w'_2(E)$ is a characteristic class satisfying the axioms characterizing the second Stiefel–Whitney class proved in [21]. This requires three things. In the first two items below, $X$ is an arbitrary CW-complex.

- Let $f : X' \to X$ be a map of CW complexes, let $E \to X$ be an $n$-dimensional oriented real vector bundle, and let $E' \to X'$ be the pullback of $E$. Then we must prove that $w'_2(E') = f^*(w'_2(E))$. This follows immediately from the commutative diagram

$$\begin{array}{ccc}
0 & \to & H^1(X; \mathbb{Z}/2) \\
\downarrow f^* & & \downarrow \quad f^* \\
0 & \to & H^1(F(E); \mathbb{Z}/2) \\
\downarrow & & \downarrow \\
0 & \to & H^1(\text{SL}_n(\mathbb{R}); \mathbb{Z}/2) \\
\downarrow & & \downarrow \\
0 & \to & H^2(X; \mathbb{Z}/2)
\end{array}$$

given by the naturality of the Leray–Serre spectral sequence.

- Let $E \to X$ be an $n$-dimensional oriented real vector bundle and let $m \geq 0$. Then we must prove that $w_2(E \oplus \mathbb{R}^m) = w_2(E)$. The standard upper left hand corner inclusion $\text{SL}_n(\mathbb{R}) \hookrightarrow \text{SL}_{n+m}(\mathbb{R})$ induces an isomorphism on $H^1$ with $\mathbb{Z}/2$-coefficients. The desired result now follows as before from the commutative
We must prove that there exists some $E \rightarrow X$ over some base $X$ such that $w_2(E) \neq 0$. Equivalently, we must prove that there exists some 2-dimensional oriented real vector bundle which is not spin. Let $E \rightarrow S^2$ be the 2-dimensional real vector bundle with Euler number 1. The associated oriented frame bundle $F(E) \rightarrow S^2$ is then fiberwise homotopy equivalent to the Hopf fibration $S^3 \rightarrow S^2$. In particular, we have $H^1(F(E); \mathbb{Z}/2) = H^1(S^3; \mathbb{Z}/2) = 0$, and thus there does not exist a spin structure.

**Corollary 3.2.** For some $n \geq 2$, let $E \rightarrow X$ be an $n$-dimensional oriented real vector bundle. Then $E$ is spin if and only if $w_2(E) = 0$. In particular, if $X$ is an oriented smooth manifold, then $X$ is spin if and only if $w_2(X) = 0$.

**Remark.** It follows from (3.1) that any two spin structures on $E \rightarrow X$ (represented as elements of $H^1(F(E); \mathbb{Z}/2)$) differ by an element of $H^1(X; \mathbb{Z}/2)$. Thus if a spin structure exists, then there is a simply transitive action of $H^1(X; \mathbb{Z}/2)$ on the set of spin structures.

**Even intersection forms.** Our next goal is to show that if $M^4$ is a spin 4-manifold, then its intersection form is even. We first need the following lemma.

**Lemma 3.3.** Let $M^4$ be a smooth 4-manifold. Then every element of $H^2(M^4; \mathbb{Z})$ is Poincaré dual to an embedded oriented surface in $M^4$.

**Proof.** Since $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$, there is a natural bijection between $H^2(M^4; \mathbb{Z})$ and the set $[M^4, \mathbb{CP}^\infty]$ of homotopy classes of maps from $M^4$ to $\mathbb{CP}^\infty$ (see [8, Theorem 4.57]). By simplicial approximation, every homotopy class of maps $M^4 \rightarrow \mathbb{CP}^\infty$ has a representative whose image lies in the 4-skeleton of the usual triangulation of $\mathbb{CP}^\infty$, which is $\mathbb{CP}^2$. Consider a map $f : M^4 \rightarrow \mathbb{CP}^2$. Homotoping $f$, we can assume that $f$ is smooth. Then for a regular value $x \in \mathbb{CP}^2$, the preimage $f^{-1}(x)$ is an embedded surface in $M^4$ which is Poincaré dual to the cohomology class represented by $f$. □

We now prove the following key result.
Lemma 3.4. Let $M^4$ be a smooth closed oriented 4-manifold. Consider an element $\lambda \in H^2(M^4; \mathbb{Z}/2)$ which is in the image of the map $H^2(M^4; \mathbb{Z}) \to H^2(M^4; \mathbb{Z}/2)$. Then $\lambda \cup w_2(M^4) = \lambda \cup \lambda$ in $H^4(M^4; \mathbb{Z}/2) \cong \mathbb{Z}/2$.

Proof. By Lemma 3.3, we can find an embedded oriented surface $\Sigma$ in $M^4$ which is Poincaré dual to an element of $H^2(M^4; \mathbb{Z})$ which projects to $\lambda$. To keep our notation straight, we will denote by $[\Sigma] \in H_2(\Sigma; \mathbb{Z}/2)$ the $\mathbb{Z}/2$-fundamental class in $\Sigma$ and by $[\Sigma] \in H_2(M^4; \mathbb{Z}/2)$ the $\mathbb{Z}/2$-fundamental class in $M^4$. Chasing through the definitions, the lemma is equivalent to the assertion that $w_2(M^4)([\Sigma]) \in \mathbb{Z}/2$ equals the algebraic self-intersection number of $\Sigma$ in $M^4$ modulo 2. Let $T_{M^4}$ be the tangent bundle of $M^4$. Also, let $T_{\Sigma}$ and $N_{\Sigma/M}$ be the tangent and normal bundles of $\Sigma$, respectively. We thus have $T_{M^4}|_{\Sigma} = T_{\Sigma} \oplus N_{\Sigma/M}$, so

$$w_2(M^4)([\Sigma]) = w_2(T_{M^4}|_{\Sigma})([\Sigma]) = w_2(T_{\Sigma})([\Sigma]) + w_2(N_{\Sigma/M})([\Sigma]) + (w_1(T_{\Sigma}) \cup w_1(N_{\Sigma/M}))(0).$$

Since $\Sigma$ is orientable, we have $w_1(T_{\Sigma}) = 0$. Also, $w_2(T_{\Sigma})([\Sigma])$ and $w_2(N_{\Sigma/M})([\Sigma])$ are the mod 2 reductions of the Euler numbers of $T_{\Sigma}$ and $N_{\Sigma/M}$, respectively. The Euler characteristic of $\Sigma$ is even, so $w_2(T_{\Sigma})([\Sigma]) = 0$. We conclude that $w_2(M^4)([\Sigma])$ equals the mod 2 reduction of the Euler number of $N_{\Sigma/M}$.

Let $\theta$ be a section of $N_{\Sigma/M}$ with isolated simple zeros. The signed count of these zeros is the Euler number of $N_{\Sigma/M}$. Identifying $N_{\Sigma/M}$ with a tubular neighborhood of $\Sigma$ in $M$, the section $\theta$ becomes a surface $\Sigma'$ that is homotopic to $\Sigma$. The zeros of $\theta$ are in bijection with the intersections of $\Sigma'$ and $\Sigma$, and the signs of those intersections are the same as the signs of the zeros. The Euler number of $N_{\Sigma/M}$ is thus equal to the signed count of the intersections of $\Sigma$ and $\Sigma'$, i.e. the algebraic self-intersection number of $\Sigma$. The lemma follows. \hfill $\Box$

Corollary 3.5. Let $M^4$ be a smooth closed oriented 4-manifold. Let $\omega(\cdot, \cdot)$ be the intersection form on $H^2(M^4; \mathbb{Z})/\text{torsion}$. If $w_2(M^4) = 0$, then $\omega(\cdot, \cdot)$ is even. Conversely, if $\omega(\cdot, \cdot)$ is even and $H_1(M^4; \mathbb{Z})$ has no 2-torsion, then $w_2(M^4) = 0$.

Proof. The first assertion of the corollary follows immediately from Lemma 3.4. For the second assertion, the condition that $H_1(M^4; \mathbb{Z})$ has no 2-torsion implies that the map $H^2(M^4; \mathbb{Z}) \to H^2(M^4; \mathbb{Z}/2)$ is surjective. Combining this with Lemma 3.4 and the fact that $\omega(\cdot, \cdot)$ is even, we deduce that $\lambda \cup w_2(M^4) = 0$ for all $\lambda \in H^2(M^4; \mathbb{Z}/2)$. By Poincaré duality, this implies that $w_2(M^4) = 0$. \hfill $\Box$

Remark. Without the assumption that $H_1(M^4; \mathbb{Z})$ has no 2-torsion, there do exist examples of smooth closed oriented 4-manifolds $M^4$ whose intersection forms are even but where $w_2(M^4) \neq 0$. Indeed, there even exist examples which are smooth complex projective varieties (e.g. the Enriques surface).
4 The tangent bundles of compact spin smooth 4-manifolds

In this section, we prove Proposition 2.1, which asserts that if \( M^4 \) is a smooth spin closed 4-manifold, then \( M^4 \) is almost parallelizable. Letting \( p \in M^4 \) be a point and \( N^4 = M^4 \setminus \{p\} \), this is equivalent to saying that \( N^4 \) is parallelizable.

Let \( T_N^4 \) be the tangent bundle of \( N^4 \), let \( F(T_N^4) \) be the oriented frame bundle of \( T_N^4 \), and let \( \tilde{F}(T_N^4) \) be the fiberwise 2-fold cover of \( F(T_N^4) \) provided by the spin structure, so the fibers of \( \tilde{F}(T_N^4) \) are \( \tilde{SL}_4(\mathbb{R}) \). To prove that \( T_N^4 \) is a trivial bundle, it is enough to show that \( \tilde{F}(T_N^4) \) is a trivial bundle. We will do this using obstruction theory; see [2, Chapter 7] and [8, p. 415] for two different expositions of this (the point of view of [2, Chapter 7] is more elementary). Fix a triangulation of \( N^4 \). The group \( \tilde{SL}_4(\mathbb{R}) \) is 1-connected by construction. Moreover, \( \pi_2(\tilde{SL}_4(\mathbb{R})) = 0 \); indeed, \( \pi_2(G) = 0 \) for every Lie group \( G \). This follows from [18, Theorem 21.7]; see also [23]. Of course, this can also be proved for \( SL_4(\mathbb{R}) \) and thus for \( \tilde{SL}_4(\mathbb{R}) \) by elementary methods. We deduce that \( \tilde{SL}_4(\mathbb{R}) \) is 2-connected. The first possible obstruction to trivializing \( \tilde{F}(T_N^4) \) thus lies in

\[
H^4(N^4; \pi_3(\tilde{SL}_4(\mathbb{R}))).
\]

Here there might be a nontrivial monodromy action of \( \pi_1(N^4) \) on the \( \pi_3 \) of the fiber \( \tilde{SL}_4(\mathbb{R}) \), so \( \pi_3(\tilde{SL}_4(\mathbb{R})) \) in this cohomology group should be regarded as a local coefficient system. However, since \( N^4 \) is a noncompact 4-manifold, we have \( H^4(N^4; V) = 0 \) for all local coefficient systems \( V \). This follows from the appropriate version of Poincaré-Lefschetz duality for local coefficient systems (here we must use locally finite homology since \( N^4 \) is noncompact; see the remark below for an alternate approach). The above obstruction therefore vanishes and \( \tilde{F}(T_N^4) \) can be trivialized over the entire 4-skeleton of \( N^4 \), i.e. over all of \( N^4 \).

**Remark.** An alternate way of seeing that \( H^4(N^4; V) = 0 \) in the above proof is to show that \( N^4 \) is homotopy equivalent to a 3-dimensional CW complex. This kind of thing holds in great generality: if \( X \) is a smooth noncompact \( n \)-manifold, then \( X \) is homotopy equivalent to an \((n-1)\)-dimensional CW complex (see, e.g., [22, Theorem 2.2], which proves this by constructing a proper Morse function with no local maxima).

5 The stable J-homomorphism

In this section, we prove Proposition 2.2. As we said after the statement of Proposition 2.2, the key will be the stable J-homomorphism \( J_k : \tilde{KO}(S^{k+1}) \to \pi_k^S \). This will require a preliminary discussion of classifying spaces for groups, the K-theory of spheres, and the Pontryagin–Thom construction.
Classifying spaces for groups. Let $G$ be a topological group. A classifying space for $G$ is a topological space $BG$ together with a principal $G$-bundle $EG \to BG$ such that for all CW complexes $X$, there is a bijection between the set $[X, BG]$ of homotopy classes of maps from $X$ to $BG$ and the set of principal $G$-bundles on $X$. Given a map $f : X \to BG$, the associated principal $G$-bundle on $X$ is the pullback $f^*(EG)$. The base $BG$ of a principal $G$-bundle $EG \to BG$ forms a classifying space for $G$ if and only if $EG$ is contractible (see [27, §19]. From this, one can show that if $BG$ is a classifying space for $G$, then $\Omega BG$ is homotopy equivalent to $G$ (see [8, Proposition 4.66]; here $\Omega BG$ denotes the based loop-space of $BG$). In other words, $BG$ is a “de-looping” of $G$. This implies in particular that $BG$ is simply-connected if $G$ is connected. Milnor [16] proved that all topological groups have classifying spaces.

K-theory of spheres. To define $J_k$, we will need to understand $\widetilde{KO}(S^{k+1})$. We will give a somewhat abstract description of the necessary result; see the second remark below for a more hands-on point of view. Recall that $\widetilde{KO}(S^{k+1})$ consists of the set of principal $\text{SL}(\mathbb{R})$-bundles. The classifying space $\text{BSL}(\mathbb{R})$ is the direct limit of the classifying spaces $\text{BSL}_n(\mathbb{R})$. The maps $\text{SL}_n(\mathbb{R}) \times \text{SL}_m(\mathbb{R}) \to \text{SL}_{n+m}(\mathbb{R})$ defined via the formula

$$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

induce maps $\text{BSL}_n(\mathbb{R}) \times \text{BSL}_m(\mathbb{R}) \to \text{BSL}_{n+m}(\mathbb{R})$. Passing to the direct limit, we get a map $\text{BSL}(\mathbb{R}) \times \text{BSL}(\mathbb{R}) \to \text{BSL}(\mathbb{R})$. This turns $\text{BSL}(\mathbb{R})$ into a topological monoid. By the definition of a classifying space, we have

$$\widetilde{KO}(S^{k+1}) \cong [S^{k+1}, \text{BSL}(\mathbb{R})].$$

The abelian group structure on $\widetilde{KO}(S^{k+1})$ is induced by the monoid structure on $\text{BSL}(\mathbb{R})$ (which while not commutative itself is commutative up to homotopy). The key computation now is

$$\widetilde{KO}(S^{k+1}) \cong [S^{k+1}, \text{BSL}(\mathbb{R})] \cong \pi_{k+1}(\text{BSL}(\mathbb{R})) \cong \pi_k(\Omega \text{BSL}(\mathbb{R})) \cong \pi_k(\text{SL}(\mathbb{R})).$$

The second isomorphism here follows from the fact that $\text{BSL}(\mathbb{R})$ is simply-connected, which itself is a consequence of the fact that $\text{SL}(\mathbb{R})$ is connected.

Remark. One might worry that the isomorphism $\widetilde{KO}(S^{k+1}) \cong \pi_k(\text{SL}(\mathbb{R}))$ is only a bijection of sets but does not respect the additive structure. That it does respect the additive structure can be proved using the standard Eckmann–Hilton argument as follows. Consider $x, y \in \pi_k(\text{SL}(\mathbb{R}))$ which correspond to elements $X, Y \in \widetilde{KO}(S^{k+1})$. There exist $n, m \geq 1$ such that $x \in \pi_k(\text{SL}_n(\mathbb{R}))$ and $y \in \pi_k(\text{SL}_m(\mathbb{R}))$ (here we are abusing notation and regarding $\pi_k(\text{SL}_n(\mathbb{R}))$ and $\pi_k(\text{SL}_m(\mathbb{R}))$ as subgroups of $\pi_k(\text{SL}(\mathbb{R})))$. For $a \in \pi_k(\text{SL}_n(\mathbb{R}))$ and $b \in \pi_k(\text{SL}_m(\mathbb{R}))$, let $a \ast b \in \pi_k(\text{SL}_{n+m}(\mathbb{R}))$ be the loop obtained by applying the map (5.1) pointwise. It is clear from the definitions that $x \ast y$
represents $X + Y \in \text{KO}(S^{k+1})$. Letting $\cdot$ denote the product in $\pi_k(\text{SL}(\mathbb{R}))$, we then have

$$x \cdot y = (x \cdot 1) \cdot (1 \cdot y) = (x \cdot 1) \cdot (1 \cdot y) = x \cdot y,$$

as desired.

Remark. A more pedestrian perspective on the isomorphism $\text{KO}(S^{k+1}) \cong \pi_k(\text{SL}(\mathbb{R}))$ is as follows. Consider a principal $\text{SL}(\mathbb{R})$-bundle $B$ on $S^{k+1}$. Letting $D_+$ and $D_-$ be the upper and lower hemispheres of $S^{k+1}$, the restrictions of $B$ to $D_+$ and $D_-$ are trivial. Since $D_+ \cap D_- = S^k$, the bundle $B$ can thus be described as $(D_+ \times \mathbb{R}^\infty) \cup (D_- \times \mathbb{R}^\infty)/\sim$, where $\sim$ identifies $(x, \tilde{v}) \in \partial D_+ \times \mathbb{R}^\infty$ with $(x, f(x)(\tilde{v})) \in \partial D_- \times \mathbb{R}^\infty$ for some map $f : S^k \to \text{SL}(\mathbb{R})$. The homotopy class of $f$ is then the element of $\pi_k(\text{SL}(\mathbb{R}))$ associated to $B$. It is called the clutching function for $B$.

The Pontryagin–Thom construction. For proofs of the results we discuss in this paragraph, see [19, §7]. Fix $n \geq 1$ and $k \geq 0$. Our goal is to give a “geometric” description of $\pi_{n+k}(S^n)$. This group depends on the choice of a basepoint. We will regard the sphere as the one-point compactification of Euclidean space; the basepoint will be the point at infinity. We thus want to determine

$$\pi_{n+k}(S^n, \infty) = [(S^{n+k}, \infty), (S^n, \infty)].$$

If $X$ is a smooth manifold (possibly with boundary), then a framed submanifold of $X$ consists of the following data.

- A smooth compact properly embedded submanifold $M$ of $X$. Contrary to our assumptions elsewhere in this note, we do not assume that $M$ is connected; in fact, we allow $M$ to be empty.
- A framing of the normal bundle $N_{X/M}$ of $M$ in $X$, that is, a bundle isomorphism

$$f : M \times \mathbb{R}^p \longrightarrow N_{X/M},$$

where $p$ is the codimension of $M$ in $X$.

Define $\Omega_k^{\text{frame}}(S^{n+k}, \infty)$ to be the set of $k$-dimensional framed submanifolds $M^k$ of $S^{n+k}$ such that $\infty \notin M^k$ (including the empty manifold) modulo the equivalence relation of cobordism, which is defined as follows.

- If $M_0$ and $M_1$ are $k$-dimensional framed submanifolds of $S^{n+k} \setminus \{\infty\}$, then $M_0$ and $M_1$ are cobordant if there exists a framed $(k+1)$-dimensional submanifold $C$ of $(S^{n+k} \setminus \{\infty\}) \times [0,1]$ such that for $i = 0, 1$, we have $C \cap (S^{n+k} \times i) = M_i \times i$ and the framing of $C$ on $C \cap (S^{n+k} \times i)$ agrees with the framing on $M_i$.

The key fact is the following theorem of Pontryagin.
Theorem 5.1 (Pontryagin). For $n \geq 1$ and $k \geq 0$, we have

$$\pi_{n+k}(S^n, \infty) = \Omega^\text{frame}_k(S^{n+k}, \infty).$$

This is an isomorphism of groups, where the group operation on $\Omega^\text{frame}_k(S^{n+k}, \infty)$ is disjoint union.

We refer to [19, §7] for the proof, but to clarify what is going on we indicate the construction of a map $f : (S^{n+k}, \infty) \to (S^n, \infty)$ from a $k$-dimensional framed submanifold $M^k$ of $S^{n+k}$ such that $\infty \notin M^k$ (this construction is known as the Pontryagin–Thom construction). Let $U \subset S^{n+k}$ be a tubular neighborhood of $M^k$ such that $\infty \notin U$. The framing on the normal bundle of $M^k$ then induces a homeomorphism

$$\theta : U \to M^k \times \mathbb{R}^n.$$

Let $\nu : U \to \mathbb{R}^n$ be the composition of $\theta$ with the projection $M^k \times \mathbb{R}^n \to \mathbb{R}^n$. Then $f$ is the map defined via the formula

$$f(x) = \begin{cases} 
\nu(x) & \text{if } x \in U, \\
\infty & \text{otherwise}.
\end{cases}$$

Observe that $f(\infty) = \infty$. The construction of $f$ depends on various choices, but varying these choices results in homotopic $f$.

The stable J-homomorphism. We finally come to the construction of the stable J-homomorphism

$$J_k : \widetilde{KO}(S^{k+1}) \to \pi^S_k.$$

Consider $[B] \in \widetilde{KO}(S^{k+1})$. As discussed above, $[B]$ corresponds to an element of $\pi_k(\text{SL}(\mathbb{R}))$. Represent this element via a map $\psi : S^k \to \text{SL}_n(\mathbb{R})$ for some $n \gg 0$. The image $J_k([B]) \in \pi^S_k$ will be the image in $\pi^S_k$ of the element of $\pi_{n+k}(S^n)$ represented by the following framed submanifold of $S^{n+k} \setminus \{\infty\} \cong \mathbb{R}^{n+k}$.

- The manifold will be image in $\mathbb{R}^{n+k}$ of the unit $k$-sphere in $\mathbb{R}^{k+1}$. Denote this by $S$.

- For the framing, let $\{\vec{e}_1, \ldots, \vec{e}_{n+k}\}$ be the standard basis for $\mathbb{R}^{n+k}$ and let $n : S \to \mathbb{R}^{k+1} \subset \mathbb{R}^{n+k}$ be the outward facing unit normal vector. We thus get a framing $f_0 : S \times \mathbb{R}^n \to N_{\mathbb{R}^{n+k}/S}$ defined via the formula

$$f_0(p, c_1, \ldots, c_n) = (p, c_1(p) + \sum_{i=2}^n c_i \vec{e}_{k+i}).$$

This is not the framing we are looking for; indeed, $S$ with this framing represents the trivial element of $\Omega^\text{frame}_k(S^{n+k}, \infty)$ (easy check!). Instead, the framing we
want is the result of “twisting” this trivial framing via \(\psi : S^k \to \text{SL}_n(\mathbb{R})\). More precisely, the framing we want is the framing \(f : S \times \mathbb{R}^n \to N_{\mathbb{R}^n+k} \) defined via the formula
\[
f(p, \bar{v}) = f_0(p, \psi(p)(\bar{v})).
\]

It is an easy exercise to see that this is a well-defined group homomorphism. The stable J-homomorphism was first introduced by Whitehead [29] following work of Hopf [9]. Determining its image is quite nontrivial. In complete generality, this was accomplished by Adams [1] assuming the truth of the Adams conjecture, which was later proved by Quillen [24]. We will not need the general statement, but only the following special case which was proved by Rochlin [25].

**Theorem 5.2 (Rochlin).** The image of \(J_3 : \overline{KO}(S^4) \to \pi_3^S\) is isomorphic to \(\mathbb{Z}/24\).

**Endgame.** All the pieces are now in place for the proof of Proposition 2.2. Let us first recall its statement. Let \(E \to S^4\) be an oriented real vector bundle such that there exists a compact oriented 4-manifold \(M^4\) with \(T_{M^4} = \beta^*(E)\), where \(\beta : M^4 \to S^4\) is a ball-collapse map. We must prove that the element \([E] \in \overline{KO}(S^4) \cong \mathbb{Z}\) is divisible by 24. By Theorem 5.2, this is equivalent to proving that \([E] \in \ker(J_3)\). As we will see, this is almost formal.

Let \(B \subset M^4\) be the 3-dimensional ball used to construct the ball-collapse map \(\beta\) and let \(\overline{M} = M^4 \setminus \text{Int}(B)\). For some large \(n \gg 0\), let \(S \subset \mathbb{R}^n\) be the 3-sphere used to construct \(J_3\). By choosing \(n\) large enough, we can ensure that the following hold.

- There is a proper embedding \(i : \overline{M} \to \mathbb{R}^n \times [0, 1]\) such that \(i(\partial \overline{M}) = S \times 0 \subset \mathbb{R}^n \times 0\). This follows from Whitney’s embedding theorem.

- Let \(N\) be the normal bundle of \(\overline{M}\) in \(\mathbb{R}^n \times [0, 1]\). Then \(N\) is a trivial bundle. Indeed, we have \(T_{\overline{M}} \oplus N \cong \overline{M} \times \mathbb{R}^{n+1}\), so \([T_{\overline{M}}] + [N] = 0\) in \(\overline{KO}(\overline{M})\). But we already know that \(T_{\overline{M}}\) is a trivial bundle, so \([N] = 0\). Increasing \(n\) more if necessarily, we can ensure that \(N\) is actually a trivial bundle.

Choose a framing of \(N\). The restriction of this framing to \(\partial \overline{M} = S\) can be obtained by twisting the trivial framing of \(S\) (as in the construction of the J-homomorphism) via an element of \(\pi_3(\text{SL}_{n-3}(\mathbb{R}))\) which represents \([-E] \in \overline{KO}(S^4)\) (the negative sign is here since we have switched from the tangent bundle to the normal bundle). We have exhibited an explicit cobordism from our framing of \(S\) to the empty manifold, so we conclude that \(J_3([-E]) = 0\) and hence \(J_3([E]) = 0\).

**References**


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