Realizing homology classes by simple closed curves

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Abstract

We give an efficient proof that primitive classes in the first homology group of a surface can be realized by simple closed curves.

Let $\Sigma_g$ be a closed oriented genus $g$ surface. An element $h \in H_1(\Sigma_g; \mathbb{Z})$ is primitive if it cannot be written as $h = nh'$ for some $h' \in H_1(\Sigma_g; \mathbb{Z})$ and $n \geq 2$. In this note, we give a short proof of the following theorem. For an oriented closed curve $\gamma$ on $\Sigma_g$, write $[\gamma]$ for the associated element of $H_1(\Sigma_g; \mathbb{Z})$.

**Theorem 0.1.** A nonzero element $h \in H_1(\Sigma_g; \mathbb{Z})$ can be written as $h = [\gamma]$ for a simple closed curve $\gamma$ if and only if $h$ is primitive.

The earliest references we are aware of for Theorem 0.1 are 1976 papers of Meyerson [Mey] and Schafer [S]. Our proof is a simplification of one due to Meeks–Patrusky [MeeP].

**Proof of Theorem 0.1.** First assume that $h = [\gamma]$ for an oriented simple closed curve $\gamma$. Since separating curves on $\Sigma_g$ are nullhomologous and $h \neq 0$, the curve $\gamma$ must be nonseparating. There thus exists an oriented closed curve $\delta$ that intersects $\gamma$ once; indeed, cutting $\Sigma_g$ open along $\gamma$ results in a connected surface with two boundary components, and we can construct $\delta$ by connecting these two boundary components by an arc and then gluing them back together. Letting $i(\cdot, \cdot)$ be the algebraic intersection pairing, we thus have $i(h, [\delta]) = i([\gamma], [\delta]) = \pm 1$. If $h = nh'$ for some $n \in \mathbb{Z}$ and $h' \in H_1(\Sigma_g; \mathbb{Z})$, then

$$\pm 1 = i(h, [\delta]) = i(nh', [\delta]) = ni(h', [\delta]).$$

This forces $n = \pm 1$, so we conclude that $h$ is primitive, as desired.

Assume now that $h \in H_1(\Sigma_g; \mathbb{Z})$ is a primitive homology class. We will construct an oriented simple closed curve $\gamma$ such that $h = [\gamma]$ as follows. Observe first that we can find an oriented closed curve $\gamma'$ (not necessarily simple) such that $h = [\gamma']$. Resolving all the self-intersections of $\gamma'$ as in Figure 1 produces a collection $\gamma_1, \ldots, \gamma_n$ of disjoint oriented simple closed curves such that

$$h = [\gamma_1] + \cdots + [\gamma_n]. \quad (0.1)$$

Choose a representation as in (0.1) such that $n$ is as small as possible. We will prove that $n = 1$.

Consider a component $S$ of $\Sigma_g$ cut open along the $\gamma_i$. Each boundary component of $S$ is one of the $\gamma_i$; orient these boundary components via the orientations on the $\gamma_i$. We claim that $S$ must have exactly two boundary components. Indeed, observe the following.

- If $S$ only had one boundary component, then that boundary component would be nullhomologous and we could delete the resulting $\gamma_i$ from (0.1), contradicting the minimality of $n$.
- If $S$ had at least three boundary components, then $S$ would lie on the same side (left or right) of two of them. Let $\gamma_i$ and $\gamma_j$ be these two boundary components. Since $S$ lies on the same side of $\gamma_i$ and $\gamma_j$, we must have $i \neq j$. As is shown in Figure 2, there exists a simple closed oriented curve $\delta$ in $S$ such that $[\delta] = [\gamma_i] + [\gamma_j]$. We can thus replace $\gamma_i$ and $\gamma_j$ in (0.1) with $\delta$, again contradicting the minimality of $n$.

1
Figure 1: Resolving the self-intersections of $\gamma'$ results in a collection of disjoint simple closed curves in the same homology class as $\gamma'$.

Figure 2: On the left, $S$ lies to the right of both $\gamma_i$ and $\gamma_j$, and $\delta$ is an oriented simple closed curve in $S$ satisfying $[\delta] = [\gamma_i] + [\gamma_j]$. On the right, $S$ lies to the right of $\gamma_i$ and to the left of $\gamma_j$, and $[\gamma_i] = [\gamma_j]$.

The above argument actually gives more: not only does $S$ have two boundary components, but $S$ lies on different sides of these components. These two boundary components are thus homologous curves (see Figure 2). Since this holds for all components of $\Sigma_g$ cut open along the $\gamma_i$, we deduce that in fact all the $\gamma_i$ are homologous. Letting $h'$ be their common homology class, we have

$$h = [\gamma_1] + \cdots + [\gamma_n] = nh'.$$

Since $h$ is primitive, we conclude that $n = 1$, as desired. \qed

References

