Realizing homology classes by simple closed curves

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Abstract

We give an efficient proof that primitive classes in the first homology group of a surface can be realized by simple closed curves.

Let Σ_g be a closed oriented genus g surface. An element $h \in \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ is *primitive* if it cannot be written as $h = nh'$ for some $h' \in \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ and $n \geq 2$. In this note, we give a short proof of the following theorem. For an oriented closed curve γ on Σ_g, write $[\gamma]$ for the associated element of $\mathrm{H}_1(\Sigma_g;\mathbb{Z})$.

Theorem 0.1. *A nonzero element $h \in \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ can be written as $h = [\gamma]$ for a simple closed curve γ if and only if h is primitive.*

The earliest references we are aware of for Theorem 0.1 are 1976 papers of Meyerson [Mey] and Schafer [S]. Our proof is a simplification of one due to Meeks–Patrusky [MeeP].

Proof of Theorem 0.1. First assume that $h = [\gamma]$ for an oriented simple closed curve γ. Since separating curves on Σ_g are nullhomologous and $h \neq 0$, the curve γ must be nonseparating. There thus exists an oriented closed curve δ that intersects γ once; indeed, cutting Σ_g open along γ results in a connected surface with two boundary components, and we can construct δ by connecting these two boundary components by an arc and then gluing them back together. Letting $i(\cdot,\cdot)$ be the algebraic intersection pairing, we thus have $i(h,[\delta]) = i([\gamma],[\delta]) = \pm 1$. If $h = nh'$ for some $n \in \mathbb{Z}$ and $h' \in \mathrm{H}_1(\Sigma_g;\mathbb{Z})$, then

$$\pm 1 = i(h,[\delta]) = i(nh',[\delta]) = ni(h',[\delta]).$$

This forces $n = \pm 1$, so we conclude that h is primitive, as desired.

Assume now that $h \in \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ is a primitive homology class. We will construct an oriented simple closed curve γ such that $h = [\gamma]$ as follows. Observe first that we can find an oriented closed curve γ' (not necessarily simple) such that $h = [\gamma']$. Resolving all the self-intersections of γ' as in Figure 1 produces a collection $\gamma_1, \ldots, \gamma_n$ of disjoint oriented simple closed curves such that

$$h = [\gamma_1] + \cdots + [\gamma_n]. \tag{0.1}$$

Choose a representation as in (0.1) such that n is as small as possible. We will prove that $n = 1$.

Consider a component S of Σ_g cut open along the γ_i. Each boundary component of S is one of the γ_i; orient these boundary components via the orientations on the γ_i. We claim that S must have exactly two boundary components. Indeed, observe the following.

- If S only had one boundary component, then that boundary component would be nullhomologous and we could delete the resulting γ_i from (0.1), contradicting the minimality of n.
- If S had at least three boundary components, then S would lie on the same side (left or right) of two of them. Let γ_i and γ_j be these two boundary components. Since S lies on the same side of γ_i and γ_j, we must have $i \neq j$. As is shown in Figure 2, there exists a simple closed oriented curve δ in S such that $[\delta] = [\gamma_i] + [\gamma_j]$. We can thus replace γ_i and γ_j in (0.1) with δ, again contradicting the minimality of n.

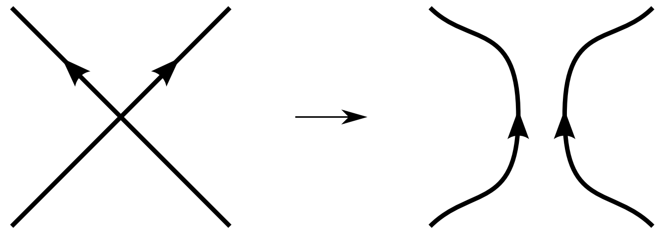

Figure 1: *Resolving the self-intersections of γ' results in a collection of disjoint simple closed curves in the same homology class as γ'.*

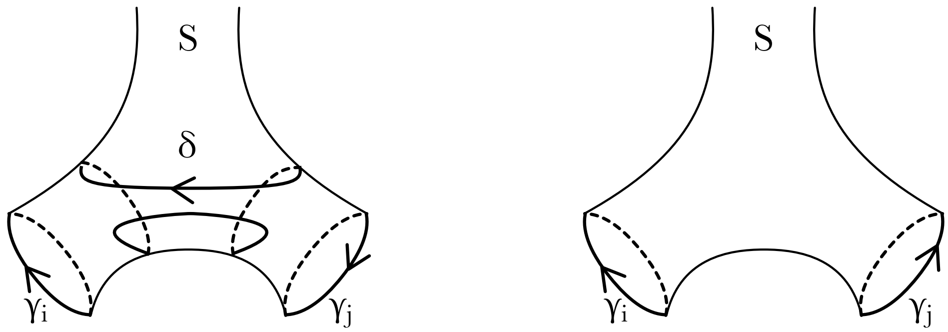

Figure 2: *On the left, S lies to the right of both γ_i and γ_j, and δ is an oriented simple closed curve in S satisfying $[\delta] = [\gamma_i] + [\gamma_j]$. On the right, S lies to the right of γ_i and to the left of γ_j, and $[\gamma_i] = [\gamma_j]$.*

The above argument actually gives more: not only does S have two boundary components, but S lies on different sides of these components. These two boundary components are thus homologous curves (see Figure 2). Since this holds for all components of Σ_g cut open along the γ_i, we deduce that in fact all the γ_i are homologous. Letting h' be their common homology class, we have

$$h = [\gamma_1] + \cdots + [\gamma_n] = nh'.$$

Since h is primitive, we conclude that $n = 1$, as desired. $\square$

References

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