Smith theory and Bredon homology

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Abstract

We discuss theorems of P. Smith and Floyd connecting the cohomology of a simplicial complex equipped with an action of a finite p-group to the cohomology of its fixed points. The proof we give is the original one, but phrased using the modern language of Bredon homology (to which we give a concise introduction).

1 Introduction

Let G be a discrete group. A *simplicial G-complex* is a simplicial complex X equipped with an action of G by simplicial automorphisms. Around 1940, Paul Smith proved a number of striking theorems that relate the cohomology of a simplicial G-complex to the cohomology of the subspace X^G of fixed points. See [Sm] for references. In these notes, we discuss and prove one of Smith's main results.

Basic definitions. To state Smith's results, we need some definitions. Fix a prime p and a simplicial complex X.

- The simplicial complex X is *mod-p acyclic* if
$$\mathrm{H}_k(X;\mathbb{F}_p) = \begin{cases} \mathbb{F}_p & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$
Note that this implies that X is nonempty and connected.
- The simplicial complex X is a *mod-p homology n-sphere* if
$$\mathrm{H}_k(X;\mathbb{F}_p) = \begin{cases} \mathbb{F}_p & \text{if } k = 0 \text{ or } k = n, \\ 0 & \text{otherwise.} \end{cases}$$
Note that we are not requiring that X be a manifold nor that it be n-dimensional.
- If all the mod-p homology groups of X are finite-dimensional and only finitely many of them are nonzero, define
$$\chi_p(X) = \sum_{k=1}^{\infty} (-1)^k \dim \mathrm{H}_k(X;\mathbb{F}_p) \in \mathbb{Z}.$$
We emphasize that this is an integer.

Smith's theorem. The following theorem summarizes some of Smith's results.

Theorem A (Smith)**.** *Let p be a prime, let G be a finite p-group, and let X be a finite-dimensional simplicial G-complex. The following then hold.*

(a) If all the mod-p homology groups of X are finite dimensional, then the same is true for X^G and $\chi_p(X) \equiv \chi_p(X^G)$ modulo p.
(b) If X is mod-p acyclic, then so is X^G. In particular, X^G is nonempty.
(c) If X is a mod-p homology n-sphere, then X^G is either empty or a mod-p homology m-sphere for some $0 \leq m \leq n$.

Remark 1.1. The condition that X is a simplicial complex can be somewhat relaxed at the cost of using Čech cohomology and imposing certain technical conditions on the spaces involved. See [B2] for a discussion. □

Remark 1.2. No theorem like Theorem A holds for groups that are not finite p-groups. For instance, if G is a finite cyclic group whose order is not a power of a prime, then building on work of Conner–Floyd [CF], Kister [K] constructed a simplicial action of G on a triangulation X of Euclidean space such that $X^G = \emptyset$. The ultimate result in this direction is a remarkable theorem of Oliver [O] that characterizes the possible fixed-point sets for actions of *arbitrary* finite groups on finite-dimensional contractible complexes. □

Compactness and Euler characteristic. If p is a prime, G is a finite p-group, and X is a **compact** simplicial G-complex, then there is the following elementary argument for why $\chi_p(X) \equiv \chi_p(X^G)$ modulo p. Subdividing X appropriately, we can assume that X/G is the simplicial complex whose k-simplices are the G-orbits of simplices of X. In particular, X^G is a subcomplex of X. This condition will reoccur many times in this paper; we will say that such a simplicial G-complex is *nicely transitive*.

The proof will be by induction on $|G|$. The base case is where $|G| = p$, so G is a cyclic group of order p. This implies that for all simplices Δ of X, the G-orbit $G \cdot \Delta$ either consists of a single simplex (so $\Delta \in X^G$) or consists of p distinct simplices. Adding all of these up, we see that in fact

$$\chi(X) + (p-1)\chi(X^G) = p\chi(X/G);$$

here we can talk about χ rather than χ_p since everything is compact. Reducing everything modulo p, we see that $\chi(X) \equiv \chi(X^G)$ modulo p, as desired.

Now assume that $|G| > p$ and that the result is true for all smaller groups. Since G is a finite p-group, it is nilpotent. This implies that there is a nontrivial proper normal subgroup G' of G. The group G/G' acts on $X^{G'}$, and applying our inductive hypothesis twice we see that

$$\chi(X) \equiv \chi(X^{G'}) \equiv \chi\left(\left(X^{G'}\right)^{G/G'}\right) = \chi(X^G)$$

modulo p, as desired.

Finite-dimensionality. The condition that X is finite-dimensional is essential for all three parts of Theorem A. Let G be a finite p-group (in fact, the arguments below will not use the fact that G is finite nor that it is a p-group).

- Let X be the universal cover of a $K(G,1)$. The group G then acts freely on X, so $X^G = \emptyset$, contradicting (b), and $0 = \chi_p(X^G) \neq \chi_p(X) = 1$, contradicting (a).
- Generalizing the usual construction of a $K(G,1)$, more elaborate infinite-dimensional counterexamples to (a) and (b) can be constructed as follows. Let K be *any* simplicial complex. Endow K with the trivial G-action. Form X by first equivariantly attaching 1-cells to K that are permuted freely by G to make it connected, then equivariantly attaching 2-cells to make it simply-connected, etc. The result is a contractible simplicial G-complex X with $X^G = K$.
- With a bit more care, an argument like in the previous bullet point can show that any simplicial complex K can be the fixed-point set of a simplicial action of G on a mod-p homology n-sphere for any n (necessarily infinite-dimensional), contradicting (c).

Remark 1.3. L. Jones [J] proved a beautiful converse to part (b) of Theorem A that says that if K is a finite simplicial complex that is mod-p acyclic and G is a finite p-group, then there exists a finite-dimensional contractible simplicial G-complex X such that $X^G \cong K$. □

Stronger result. In fact, what we will actually prove is the following theorem, which is due to Floyd [F]. See below for how to derive Theorem A from it.

Theorem B (Floyd)**.** *Let G be a finite p-group and let X be a finite-dimensional simplicial G-complex whose mod-p homology groups are all finitely generated. The following then hold.*

(i) For all $n \geq 0$, we have

$$\sum_{k=n}^{\infty} \dim \mathrm{H}_k(X^G; \mathbb{F}_p) \leq \sum_{k=n}^{\infty} \dim \mathrm{H}_k(X; \mathbb{F}_p).$$

In particular, all the mod-p homology groups of X^G are finite-dimensional.

(ii) $\chi_p(X^G) \equiv \chi_p(X)$ modulo p.

Derivation of Smith's theorem. Theorem A can be derived from Theorem B as follows. As in Theorem A, let G be a finite p-group and let X be a finite-dimensional simplicial G-complex.

- Part (a) of Theorem A is contained in the conclusions of Theorem B.
- For part (b) of Theorem A, assume that X is mod-p acyclic. Part (i) of Theorem B then says that
$$\sum_{k=0}^{\infty} \dim \mathrm{H}_k(X^G; \mathbb{F}_p) \leq \sum_{k=0}^{\infty} \dim \mathrm{H}_k(X; \mathbb{F}_p) = 1.$$
This implies that either $X^G = \emptyset$ or that X^G is mod-p acyclic. To rule out $X^G = \emptyset$, we use part (ii) of Theorem B to see that
$$\chi_p(X^G) \equiv \chi_p(X) \equiv 1 \pmod{p}.$$
- For part (c) of Theorem A, assume that X is a mod-p homology n-sphere and that $X^G \neq \emptyset$. We must prove that X^G is a mod-p homology m-sphere for some $0 \leq m \leq n$.

Part (i) of Theorem B says that

$$\sum_{k=0}^{\infty} \dim \mathrm{H}_k(X^G; \mathbb{F}_p) \leq \sum_{k=0}^{\infty} \dim \mathrm{H}_k(X; \mathbb{F}_p) = 2. \tag{1.1}$$

The left hand side of (1.1) is thus either 0 or 1 or 2. The case where it is 0 is precisely the case where $X^G = \emptyset$, which is allowed. If the left-hand side of (1.1) is 1, then necessarily the nonzero homology group is the 0^{th} one, so $\chi_p(X^G) = 1$, which contradicts the conclusion in part (ii) of Theorem B that says that modulo p we have

$$\chi_p(X^G) \equiv \chi_p(X) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Finally, assume that the left hand side of (1.1) is 2, so X^G is a mod-p homology m-sphere for some m. To see that $0 \leq m \leq n$, we apply part (i) of Theorem B again to see that

$$\sum_{k=n+1}^{\infty} \dim \mathrm{H}_k(X^G; \mathbb{F}_p) \leq \sum_{k=n+1}^{\infty} \dim \mathrm{H}_k(X; \mathbb{F}_p) = 0,$$

so $\mathrm{H}_k(X^G; \mathbb{F}_p) = 0$ for all $k \geq n+1$.

Outline. We will begin in §2 by reducing Theorem B to the special case of a cyclic group of order p. In §3, we will introduce Bredon homology groups. In §4, we will discuss some algebraic properties of the group ring of a cyclic group of order p. In §5, we will introduce Smith's special homology groups. Part (i) of Theorem B is then proved in §6 and part (ii) in §7.

Acknowledgments. I want to thank Tyler Lawson for explaining the relationship of Smith's special homology groups to Bredon homology in [La].

2 An initial reduction

Let us first recall what we must prove. Let p be a prime, let G be a finite p-group, and let X be a finite-dimensional simplicial G-complex whose mod-p homology groups are all finitely generated. Subdividing X appropriately, we can assume that X is nicely transitive (see the introduction for the definition). For all $n \geq 0$, we must prove that

$$\sum_{k=n}^{\infty} \dim \mathrm{H}_k(X^G; \mathbb{F}_p) \leq \sum_{k=n}^{\infty} \dim \mathrm{H}_k(X; \mathbb{F}_p).$$

We must also prove that $\chi_p(X^G) \equiv \chi_p(X)$ modulo p.

The first reduction is that it is enough to prove this for $|G| = p$. Indeed, assume that we have done this. We will prove the general case by induction on the order of G, the base case being the case where $|G| = p$. A general p-group is nilpotent. If $|G| > p$, there thus exists a nontrivial proper normal subgroup G' of G. The group G/G' acts on $X^{G'}$ and

$$\left(X^{G'}\right)^{G/G'} = X^G.$$

We can thus apply our inductive hypothesis twice, first to the action of G' on X and then to the action of G/G' on $X^{G'}$. The desired conclusions follow.

It thus remains to prove the theorem in the case where $|G| = p$, i.e. the case where G is a cyclic group of order p.

3 Bredon homology

Our main tool for proving Theorem B will be the Smith special homology groups. Their original definition was a little ad-hoc, but it has since been subsumed by the theory of Bredon homology. In this section, we will define Bredon homology. The original reference is [B1].

3.1 Homology with respect to coefficient systems

Let Y be a simplicial complex. In this section, we define a very general notion of coefficient systems on Y with respect to which we can define simplicial homology.

Coefficient systems. Let $\mathcal{P}(Y)$ be the poset of simplices of Y. Regard $\mathcal{P}(Y)$ as a category whose objects are the simplices of Y and where there is a unique morphism $\Delta' \to \Delta$ whenever Δ' is a face of Δ. A *coefficient system* on Y is a contravariant functor $\mathcal{M}$ from $\mathcal{P}(Y)$ to the category of abelian groups. Unwinding this, $\mathcal{M}$ consists of the data of an abelian group $\mathcal{M}(\Delta)$ for all simplices Δ of Y together with maps $\mathcal{M}(\Delta) \to \mathcal{M}(\Delta')$ whenever Δ' is a face of Δ. We will call these maps the *boundary maps*. They must satisfy the obvious compatibility conditions.

Example 3.1. Let R be a commutative ring. Define $\underline{R}$ to be the *constant coefficient system* that assigns R to every simplex of Y and whose boundary maps are all the identity map. □

Example 3.2. Let R be a commutative ring. If Z is a subcomplex of Y, then define $\underline{R}|_Z$ to be the coefficient system that assigns R to every simplex of Y that lies in Z and 0 to every simplex of Y that does not lie in Z. The nonzero boundary maps are all the identity map. □

Homology. Let $\mathcal{M}$ be a coefficient system on Y. The *homology of Y with respect to* $\mathcal{M}$, denoted $\mathrm{H}_\bullet(Y;\mathcal{M})$, is the homology of the chain complex $(C_\bullet(Y;\mathcal{M}), \partial)$ defined as follows. Fix a total ordering on the vertices of Y; this will allow us to consistently make various sign choices, but will not affect $\mathrm{H}_\bullet(Y;\mathcal{M})$. For $k \geq 0$, let $\mathcal{P}_k(Y)$ be the collection of all k-simplices of Y. For $k \geq 0$, define

$$C_k(Y;\mathcal{M}) = \bigoplus_{\Delta \in \mathcal{P}_k(Y)} \mathcal{M}(\Delta).$$

Next, define the boundary map $\partial\colon C_k(Y;\mathcal{M}) \to C_{k-1}(Y;\mathcal{M})$ to be

$$\partial = \partial_0 - \partial_1 + \partial_2 - \cdots \pm \partial_k,$$

where ∂_i is defined as follows. Consider some $\Delta \in \mathcal{P}_k(Y)$. Order the vertices of Δ using the total ordering on the vertices of Y, and let $\Delta_i \in \mathcal{P}_{k-1}(Y)$ be the face of Δ obtained by deleting the i^{th} vertex of Δ. The map ∂_i then takes the factor $\mathcal{M}(\Delta)$ of $C_k(Y;\mathcal{M})$ to the factor $\mathcal{M}(\Delta_i)$ of $C_{k-1}(Y;\mathcal{M})$ via the boundary map

$$\mathcal{M}(\Delta) \to \mathcal{M}(\Delta_i)$$

defined by $\mathcal{M}$.

Example 3.3. Let R be a commutative ring. Then $(C_\bullet(Y;\underline{R}),\partial)$ is the usual simplicial chain complex for Y and $\mathrm{H}_\bullet(Y;\underline{R}) = \mathrm{H}_\bullet(Y;R)$. □

Example 3.4. Let R be a commutative ring and let Z be a subcomplex of Y. Then $\mathrm{H}_\bullet(Y;\underline{R}|_Z) \cong \mathrm{H}_\bullet(Z;R)$. □

Long exact sequences. The collection of all coefficient systems on Y forms an abelian category whose morphisms are natural transformations. Given a morphism of coefficient systems $\mathcal{M}_1 \to \mathcal{M}_2$, there is an induced map $\mathrm{H}_\bullet(Y;\mathcal{M}_1) \to \mathrm{H}_\bullet(Y;\mathcal{M}_2)$ of homology groups induced by the evident map of chain complexes. We then have the following lemma.

Lemma 3.5. *Let Y be a simplicial complex and let*

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_3 \longrightarrow 0$$

be a short exact sequence of coefficient systems on Y. We then have a long exact sequence

$$\cdots \to \mathrm{H}_k(Y;\mathcal{M}_1) \to \mathrm{H}_k(Y;\mathcal{M}_2) \to \mathrm{H}_k(Y;\mathcal{M}_3) \to \mathrm{H}_{k-1}(Y;\mathcal{M}_1) \to \cdots$$

in homology.

Proof. This is the long exact sequence in homology induced by the evident short exact sequence

$$0 \longrightarrow C_\bullet(Y;\mathcal{M}_1) \longrightarrow C_\bullet(Y;\mathcal{M}_2) \longrightarrow C_\bullet(Y;\mathcal{M}_3) \longrightarrow 0$$

of chain complexes. □

3.2 The orbit category and Bredon homology

Let G be a discrete group and let X be a nicely transitive simplicial G-complex. Recall that this means that X/G is the simplicial complex whose simplices are the G-orbits of simplices in X. In this section, we introduce the Bredon homology of X, which is a sort of homology theory that takes into account the G-orbit structure of X.

The orbit category. The *orbit category* of G, denoted $\mathcal{O}_G$, is the category whose objects are transitive G-sets and whose morphisms are G-equivariant maps. Every object of $\mathcal{O}_G$ is isomorphic to G/H for some subgroup H of G, so $\mathcal{O}_G$ is equivalent to the category whose objects are

$$\{G/H \mid H \text{ a subgroup of } G\}$$

and whose morphisms are G-equivariant maps. A *Bredon coefficient system* for G is a covariant functor from $\mathcal{O}_G$ to the category of abelian groups. The collection of all Bredon coefficient systems forms an abelian category whose morphisms are natural transformations.

Example 3.6. Let R be a commutative ring. There is then a Bredon coefficient system $\mathcal{B}_R$ defined by $\mathcal{B}_R(X) = R[X]$ for $X \in \mathcal{O}_G$. Here $R[X]$ is the set of all formal R-linear combinations of finitely many elements of X. $\square$

Example 3.7. Let R be a commutative ring. There is then a Bredon coefficient system $\overline{\mathcal{B}}_R$ defined by $\overline{\mathcal{B}}_R(X) = R$ for all $X \in \mathcal{O}_G$. There is morphism $\mathcal{B}_R \to \overline{\mathcal{B}}_R$ that on $X \in \mathcal{O}_G$ takes

$$\sum_{x \in X} c_x x \in \mathcal{B}_R(X) = R[X]$$

to

$$\sum_{x \in X} c_x \in \overline{\mathcal{B}}_R(X). \qquad \square$$

Example 3.8. Let R be a commutative ring. There is then a Bredon coefficient system $\mathcal{B}_R^G$ defined by

$$\mathcal{B}_R^G(X) = \begin{cases} 0 & \text{if } X \text{ has more than one element,} \\ R & \text{if } X \text{ has one element} \end{cases}$$

for $X \in \mathcal{O}_G$. For $X = G/H$ with H a subgroup of G, this can be written as

$$\mathcal{B}_R^G(G/H) = \begin{cases} 0 & \text{if } H \text{ is a proper subgroup of } G, \\ R & \text{if } H = G. \end{cases}$$

The reason that we denote this with a superscript G will be clear later. For all $X \in \mathcal{O}_G$, we have $\mathcal{B}_R^G(X) \subset \mathcal{B}_R$, and from this we get a morphism $\mathcal{B}_R^G \to \mathcal{B}_R$. $\square$

Bredon homology. Let $\mathcal{M}$ be a Bredon coefficient system for G. As in the first paragraph, let X be a nicely transitive simplicial G-complex Each simplex Δ of X/G is thus naturally identified with a transitive G-set (namely, the set of simplices of X which are identified by G to form Δ), so we can evaluate $\mathcal{M}$ on Δ to get an abelian group $\mathcal{M}(\Delta)$. If Δ' is a face of Δ, then a moments thought shows that there is a G-equivariant map from the transitive G-set corresponding to Δ to the transitive G-set corresponding to Δ'. The associated maps $\mathcal{M}(\Delta) \to \mathcal{M}(\Delta')$ form the boundary maps for a coefficient system on X/G that we will denote by $\mathcal{C}(\mathcal{M})$. The *Bredon homology* of X with coefficients in $\mathcal{M}$, denoted $\mathrm{H}_\bullet(X; \mathcal{M})$ is defined to be $\mathrm{H}_\bullet(X/G; \mathcal{C}(\mathcal{M}))$. This is functorial under morphisms of Bredon coefficient systems.

Example 3.9. If R is a commutative ring, then $\mathrm{H}_\bullet(X; \mathcal{B}_R) \cong \mathrm{H}_\bullet(X; R)$; indeed, the chain complex computing $\mathrm{H}_\bullet(X; \mathcal{B}_R) = \mathrm{H}_\bullet(X/G; \mathcal{C}(\mathcal{B}_R))$ is precisely the simplicial chain complex of X with coefficients in R. $\square$

Example 3.10. If R is a commutative ring, then $\mathrm{H}_\bullet(X; \overline{\mathcal{B}}_R) \cong \mathrm{H}_\bullet(X/G; R)$. Under the identifications of this example and Example 3.9, the map $\mathrm{H}_\bullet(X; \mathcal{B}_R) \to \mathrm{H}_\bullet(X; \overline{\mathcal{B}}_R)$ induced by the morphism $\mathcal{B}_R \to \overline{\mathcal{B}}_R$ from Example 3.7 equals the map $\mathrm{H}_\bullet(X; R) \to \mathrm{H}_\bullet(X/G; R)$ induced by the quotient map $X \to X/G$. □

Example 3.11. If R is a commutative ring, then $\mathrm{H}_\bullet(X; \mathcal{B}_R^G) \cong \mathrm{H}_\bullet(X^G; R)$. Here X^G is subcomplex of fixed points for the action of G on X. This is the reason we used the superscript to denote $\mathcal{B}_R^G$. Under the identifications of this example and Example 3.9, the map $\mathrm{H}_\bullet(X; \mathcal{B}_R^G) \to \mathrm{H}_\bullet(X; \mathcal{B}_R)$ induced by the morphism $\mathcal{B}_R^G \to \mathcal{B}_R$ from Example 3.8 equals the map $\mathrm{H}_\bullet(X^G; R) \to \mathrm{H}_\bullet(X; R)$ induced by the inclusion map $X^G \hookrightarrow X$. □

Long exact sequences. The following lemma follows immediately from the definitions and Lemma 3.5.

Lemma 3.12. *Let G be a discrete group and let*

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_3 \longrightarrow 0$$

be a short exact sequence of Bredon coefficient systems for G. Then for any nicely transitive simplicial G-complex, we have a long exact sequence

$$\cdots \to \mathrm{H}_k(X; \mathcal{M}_1) \to \mathrm{H}_k(X; \mathcal{M}_2) \to \mathrm{H}_k(X; \mathcal{M}_3) \to \mathrm{H}_{k-1}(X; \mathcal{M}_1) \to \cdots$$

of Bredon homology groups.

4 The group ring of a cyclic group

Let p be a prime and let G be a cyclic group of order p. In the next section, we will define the Smith special homology groups of a nicely transitive simplicial G-complex X to be certain Bredon homology groups of X. To define the needed Bredon coefficient systems, we first need some preliminary results about the group ring $\mathbb{F}_p[G]$.

Let t be the generator of G, so $\mathbb{F}_p[G] \cong \mathbb{F}_p[t]/(t^p - 1)$. Define

$$\tau = 1 - t \in \mathbb{F}_p[G],$$
$$\sigma = 1 + t + t^2 + \cdots + t^{p-1} \in \mathbb{F}_p[G].$$

The following sequence of results relate τ and σ.

Lemma 4.1. *The kernel of the map $\mathbb{F}_p[G] \to \tau \cdot \mathbb{F}_p[G]$ that multiplies elements by τ is 1-dimensional and spanned by σ. In particular, $\sigma \cdot \mathbb{F}_p[G]$ is 1-dimensional.*

Proof. A general element x of $\mathbb{F}_p[G]$ is of the form

$$x = a_0 + a_1 t + a_2 t^2 + \cdots + a_{p-1} t^{p-1}$$

for some $a_i \in \mathbb{F}_p$. We then have

$$\tau x = (a_0 - a_{p-1}) + (a_1 - a_0)t + (a_2 - a_1)t^2 + \cdots + (a_{p-1} - a_{p-2})t^{p-1}.$$

This is zero if and only if all the a_i are equal, i.e. if and only if x is a multiple of σ. □

Lemma 4.2. *$\tau^{p-1} = \sigma$.*

Proof. The binomial theorem says that

$$\tau^{p-1} = (1-t)^{p-1} = \sum_{i=0}^{p-1} \binom{p-1}{i} (-1)^i t^i.$$

We must therefore prove that

$$\binom{p-1}{i} (-1)^i \equiv 1 \pmod{p}$$

for $1 \leq i \leq p-1$. For this, we calculate:

$$\binom{p-1}{i} (-1)^i = \frac{(p-1)(p-2)\cdots(p-i)}{(i)(i-1)\cdots(1)} (-1)^i \equiv (-1)^i \frac{(1)(2)\cdots(i)}{(i)(i-1)\cdots(1)} (-1)^i = 1. \qquad \square$$

Lemma 4.3. *For all $0 \leq i \leq p-1$, we have $\sigma \in \tau^i \cdot \mathbb{F}_p[G]$.*

Proof. Lemma 4.2 says that $\sigma = \tau^{p-1} = \tau^i \cdot \tau^{p-1-i}$. $\square$

Lemma 4.4. *For all $0 \leq i \leq p-1$, we have a short exact sequence*

$$0 \longrightarrow \sigma \cdot \mathbb{F}_p[G] \longrightarrow \tau^i \cdot \mathbb{F}_p[G] \longrightarrow \tau^{i+1} \cdot \mathbb{F}_p[G] \longrightarrow 0.$$

Proof. Immediate from Lemmas 4.1 and 4.3. $\square$

Lemma 4.5. *For all $0 \leq i \leq p$, the subspace $\tau^i \cdot \mathbb{F}_p[t]$ of $\mathbb{F}_p[t]$ is $(p-i)$-dimensional.*

Proof. The space $\tau^0 \cdot \mathbb{F}_p[t] = \mathbb{F}_p[t]$ is p-dimensional, and Lemma 4.4 implies that for $0 \leq i \leq p-1$ we have

$$\dim\left(\tau^i \cdot \mathbb{F}_p[G]\right) = \dim\left(\sigma \cdot \mathbb{F}_p[G]\right) + \dim\left(\tau^{i+1} \cdot \mathbb{F}_p[G]\right) = 1 + \dim\left(\tau^{i+1} \cdot \mathbb{F}_p[G]\right).$$

The lemma follows. $\square$

Lemma 4.6. *Let $\rho = \tau^i$ with $1 \leq i \leq p-1$. Set $\overline{\rho} = \tau^{p-i}$. We then have a short exact sequence*

$$0 \longrightarrow \overline{\rho} \cdot \mathbb{F}_p[G] \longrightarrow \mathbb{F}_p[G] \longrightarrow \rho \cdot \mathbb{F}_p[G] \longrightarrow 0.$$

Proof. The evident left hand map is injective and the evident right hand map is surjective. Moreover, using Lemma 4.5 we have that

$$\dim\left(\overline{\rho} \cdot \mathbb{F}_p[G]\right) + \dim\left(\rho \cdot \mathbb{F}_p[G]\right) = (p-i) + i = p = \dim\left(\mathbb{F}_p[G]\right).$$

Exactness follows. $\square$

5 The Smith special homology groups

Let p be a prime and let G be a cyclic group of order p. All homology groups in this section have $\mathbb{F}_p$-coefficients.

Bredon coefficient systems. Since there are only two subgroups of G (namely, the trivial subgroup 0 and G itself), the orbit category of G is equivalent to the two-object category $\{G, 0\}$ whose morphisms consist of:

- $\mathrm{Hom}(G, G) = G$, and
- $\mathrm{Hom}(0, 0) = \{\mathrm{id}\}$, and
- A single morphism $G \to 0$.

To give a Bredon coefficient system for G, therefore, it is enough to give the following data:

- A G-module V, a trivial G-module W, and a G-equivariant map $V \to W$.

Let us specialize Examples 3.6–3.11 to G and discuss them in this language. Let X be a nicely transitive simplicial G-complex.

Example 5.1. The coefficient system $\mathcal{B}_{\mathbb{F}_p}$ corresponds to the augmentation map $\epsilon \colon \mathbb{F}_p[G] \to \mathbb{F}_p$ and satisfies $\mathrm{H}_\bullet(X; \mathcal{B}_{\mathbb{F}_p}) = \mathrm{H}_\bullet(X)$. □

Example 5.2. The coefficient system $\overline{\mathcal{B}}_{\mathbb{F}_p}$ corresponds to identity map $\mathbb{F}_p \to \mathbb{F}_p$ and satisfies $\mathrm{H}_\bullet(X; \overline{\mathcal{B}}_{\mathbb{F}_p}) = \mathrm{H}_\bullet(X/G)$. □

Example 5.3. The coefficient system $\mathcal{B}^G_{\mathbb{F}_p}$ corresponds to $0 \to \mathbb{F}_p$ and satisfies $\mathrm{H}_\bullet(X; \mathcal{B}^G_{\mathbb{F}_p}) = \mathrm{H}_\bullet(X^G)$. □

Morphisms. If for $i = 1, 2$ we have that $\mathcal{M}_i$ is the Bredon coefficient system for G corresponding to $V_i \to W_i$, then a morphism $\mathcal{M}_1 \to \mathcal{M}_2$ of Bredon coefficient systems is precisely a commutative diagram

$$\begin{tikzcd} V_1 \arrow[r] \arrow[d] & V_2 \arrow[d] \\ W_1 \arrow[r] & W_2. \end{tikzcd}$$

Here are some examples:

Example 5.4. The morphism $\mathcal{B}_{\mathbb{F}_p} \to \overline{\mathcal{B}}_{\mathbb{F}_p}$ that induces the map $\mathrm{H}_\bullet(X) \to \mathrm{H}_\bullet(X/G)$ corresponds to the commutative diagram

$$\begin{tikzcd} \mathbb{F}_p[G] \arrow[r, "\epsilon"] \arrow[d, "\epsilon"'] & \mathbb{F}_p \arrow[d, "\mathrm{id}"] \\ \mathbb{F}_p \arrow[r, "\mathrm{id}"] & \mathbb{F}_p. \end{tikzcd}$$

□

Example 5.5. The morphism $\mathcal{B}^G_{\mathbb{F}_p} \to \mathcal{B}_{\mathbb{F}_p}$ that induces the map $\mathrm{H}_\bullet(X^G) \to \mathrm{H}_\bullet(X)$ corresponds to the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{F}_p \arrow[d, "\mathrm{id}"'] \\ \mathbb{F}_p \arrow[r, "\mathrm{id}"] & \mathbb{F}_p. \end{tikzcd}$$
□

Smith special homology groups. Let $t, \tau, \sigma \in \mathbb{F}_p[G]$ be as in §4. Recalling that $\mathcal{B}_{\mathbb{F}_p}$ corresponds to

$$\mathbb{F}_p[G] \to \mathbb{F}_p,$$

for $\rho = \tau^i$ with $1 \leq i \leq p$ there is a submodule $\mathcal{B}^\rho_{\mathbb{F}_p}$ of $\mathcal{B}_{\mathbb{F}_p}$ corresponding to

$$\rho \cdot \mathbb{F}_p[G] \to 0.$$

The 0 appears here since $\rho \cdot \mathbb{F}_p = (1-t)^i \cdot \mathbb{F}_p = 0$, which uses the fact that $\mathbb{F}_p$ is the trivial $\mathbb{F}_p[G]$-module. If X is a nicely transitive simplicial G-complex, then the *Smith special homology groups* of X of index ρ, denoted $\mathrm{H}^\rho_\bullet(X)$, are the Bredon homology groups $\mathrm{H}_\bullet(X; \mathcal{B}^\rho_{\mathbb{F}_p})$.

Long exact sequences. The Smith special homology groups of different indices are related by two long exact sequences. The first is the following. In its statement, recall from Lemma 4.2 that $\sigma = \tau^{p-1}$.

Lemma 5.6. *Let p be a prime, let G be a cyclic group of order p, and let X be a nicely transitive simplicial G-complex. For all $1 \leq i \leq p-1$, we have a long exact sequence*

$$\cdots \to \mathrm{H}^\sigma_k(X) \to \mathrm{H}^{\tau^i}_k(X) \to \mathrm{H}^{\tau^{i+1}}_k(X) \to \mathrm{H}^\sigma_{k-1}(X) \to \cdots$$

Remark 5.7. Since $\tau^p = 0$, we have $\mathrm{H}^{\tau^p}_k(X) = 0$, so for $i = p-1$ this reduces to the trivial fact that $\mathrm{H}^\sigma_k(X) \cong \mathrm{H}^{\tau^{p-1}}_k(X)$. □

Proof of Lemma 5.6. We claim that there is a short exact sequence

$$0 \longrightarrow \mathcal{B}^\sigma_{\mathbb{F}_p} \longrightarrow \mathcal{B}^{\tau^i}_{\mathbb{F}_p} \longrightarrow \mathcal{B}^{\tau^{i+1}}_{\mathbb{F}_p} \longrightarrow 0$$

of Bredon coefficient modules for G. Indeed, translated in the above language, this corresponds to the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \sigma \cdot \mathbb{F}_p[G] \arrow[r] \arrow[d] & \tau^i \cdot \mathbb{F}_p[G] \arrow[r] \arrow[d] & \tau^{i+1} \cdot \mathbb{F}_p[G] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

whose two rows are exact, the first by Lemma 4.4 and the second trivially. Now apply Lemma 3.12. □

The second long exact sequence relating the different Smith special homology groups is as follows. The reader should note the appearance of X^G; this exact sequence will be the key to relating the topology of X and X^G.

Lemma 5.8. *Let p be a prime, let G be a cyclic group of order p, and let X be a nicely transitive simplicial G-complex. Fix some $\rho = \tau^i$ with $1 \leq i \leq p-1$, and let $\overline{\rho} = \tau^{p-i}$. We then have a long exact sequence*

$$\cdots \to \mathrm{H}_k^{\overline{\rho}}(X) \oplus \mathrm{H}_k(X^G) \to \mathrm{H}_k(X) \to \mathrm{H}_k^{\rho}(X) \to \mathrm{H}_{k-1}^{\overline{\rho}}(X) \oplus \mathrm{H}_{k-1}(X^G) \to \cdots$$

Proof. We claim that there is a short exact sequence

$$0 \longrightarrow \mathcal{B}_{\mathbb{F}_p}^{\overline{\rho}} \oplus \mathcal{B}_{\mathbb{F}_p}^{G} \longrightarrow \mathcal{B}_{\mathbb{F}_p} \longrightarrow \mathcal{B}_{\mathbb{F}_p}^{\rho} \longrightarrow 0$$

of Bredon coefficient modules for G. Indeed, translated in the above language, this corresponds to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \overline{\rho} \cdot \mathbb{F}_p[G] \arrow[r] \arrow[d, "0"'] & \mathbb{F}_p[G] \arrow[r] \arrow[d, "\epsilon"'] & \rho \cdot \mathbb{F}_p[G] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{F}_p \arrow[r, "\mathrm{id}"] & \mathbb{F}_p \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

whose two rows are exact, the first by Lemma 4.6 and the second trivially. Now apply Lemma 3.12. □

6 The inequality

Let p be a prime. In this section, all homology groups will have $\mathbb{F}_p$ coefficients. Our goal in this section is to use the Smith special homology groups to prove part (i) of Theorem B for the case of cyclic groups of order p, which we observed in §2 implies the general case. In fact, we will prove the following stronger proposition, whose additional conclusions will be important later when we study Euler characteristics. Let $\tau \in \mathbb{F}_p[G]$ be as in §5.

Proposition 6.1. *Let G be a cyclic group of order p and let X be a finite-dimensional nicely transitive simplicial G-complex such that all the mod-p homology groups of X are finite-dimensional. For some $1 \leq i \leq p-1$, set $\rho = \tau^i$. The following then hold.*

- *All the Smith special homology groups $\mathrm{H}_k^{\rho}(X)$ are finite-dimensional.*
- *For all $n \geq 0$, we have*

$$\sum_{k=n}^{\infty} \dim \mathrm{H}_k(X^G) \leq \left(\sum_{k=n}^{\infty} \dim \mathrm{H}_k\left(X\right) \right) - \dim \mathrm{H}_n^{\rho}(X).$$

In particular, all the $\mathrm{H}_k(X^G)$ are finite-dimensional.

Proof. Set $\overline{\rho} = \tau^{p-i}$. For all k, Lemma 5.8 gives a long exact sequence which contains the segment

$$\mathrm{H}_{k+1}^{\rho}(X) \longrightarrow \mathrm{H}_k^{\overline{\rho}}(X) \oplus \mathrm{H}_k(X^G) \longrightarrow \mathrm{H}_k(X). \tag{6.1}$$

Setting

$$a_i = \dim\left(\mathrm{H}_i^{\rho}\left(X\right)\right) \quad \text{and} \quad \overline{a}_i = \dim\left(\mathrm{H}_i^{\overline{\rho}}\left(X\right)\right),$$

we deduce from (6.1) that

$$\overline{a}_k + \dim\left(\mathrm{H}_k\left(X^G\right)\right) \leq a_{k+1} + \dim\left(\mathrm{H}_k\left(X\right)\right). \tag{6.2}$$

Noting that the roles of ρ and $\overline{\rho}$ in Lemma 5.8 can be reversed, we obtain in a similar way that

$$a_k + \dim\left(\mathrm{H}_k\left(X^G\right)\right) \leq \overline{a}_{k+1} + \dim\left(\mathrm{H}_k\left(X\right)\right). \tag{6.3}$$

Let X be N-dimensional. From the definition of the Smith special homology groups, we see that for $k \geq N+1$ we have

$$\mathrm{H}_k^{\rho}(X) = \mathrm{H}_k^{\overline{\rho}}(X) = 0.$$

The case $k = N$ of (6.2) and (6.3) thus reduce to

$$\overline{a}_N + \dim\left(\mathrm{H}_N\left(X^G\right)\right) \leq \dim\left(\mathrm{H}_N\left(X\right)\right) < \infty$$

and

$$a_N + \dim\left(\mathrm{H}_N\left(X^G\right)\right) \leq \dim\left(\mathrm{H}_N\left(X\right)\right) < \infty,$$

so $\overline{a}_N, a_N < \infty$. The case $k = N-1$ of (6.2) and (6.3) then say that

$$\overline{a}_{N-1} + \dim\left(\mathrm{H}_{N-1}\left(X^G\right)\right) \leq a_N + \dim\left(\mathrm{H}_{N-1}\left(X\right)\right) < \infty$$

and

$$a_{N-1} + \dim\left(\mathrm{H}_{N-1}\left(X^G\right)\right) \leq \overline{a}_N + \dim\left(\mathrm{H}_{N-1}\left(X\right)\right) < \infty,$$

so $\overline{a}_{N-1}, a_{N-1} < \infty$. Repeating this argument and working backwards, we see that $\overline{a}_k, a_k < \infty$ for all k, as claimed in the first conclusion of the proposition.

We now turn to the second conclusion. Rearranging (6.2) and (6.3), we see that

$$\dim\left(\mathrm{H}_k\left(X^G\right)\right) \leq \dim\left(\mathrm{H}_k\left(X\right)\right) + \left(a_{k+1} - \overline{a}_k\right). \tag{6.4}$$

and

$$\dim\left(\mathrm{H}_k\left(X^G\right)\right) \leq \dim\left(\mathrm{H}_k\left(X\right)\right) + \left(\overline{a}_{k+1} - a_k\right). \tag{6.5}$$

As we will see, there is a slight difference in the calculation we are about to do depending on whether or not

$$\sum_{k=n}^{N} \dim\left(\mathrm{H}_k\left(X^G\right)\right)$$

has an even or an odd number of terms; to avoid cluttering our notation, we will assume that it has an even number. Now, alternately using (6.4) and (6.5), we see that

$$\sum_{k=n}^{N} \dim\left(\mathrm{H}_k\left(X^G\right)\right) \leq \sum_{k=n}^{N} \dim\left(\mathrm{H}_k\left(X\right)\right) + (a_{n+1} - \overline{a}_n) + (\overline{a}_{n+2} - a_{n+1}) + \cdots + (\overline{a}_{N+1} - a_n).$$

This would end with $a_{N+1} - \overline{a}_n$ if there were an even number of terms in our sum. The above sum telescopes and thus reduces to

$$\sum_{k=n}^{N} \dim\left(\mathrm{H}_k\left(X^G\right)\right) \leq \sum_{k=n}^{N} \dim\left(\mathrm{H}_k\left(X\right)\right) + \left(\overline{a}_{N+1} - \overline{a}_n\right).$$

Since X is N-dimensional, the term $\overline{a}_{N+1}$ is actually 0. We conclude that

$$\sum_{k=n}^{N} \dim\left(\mathrm{H}_k\left(X^G\right)\right) \leq \sum_{k=n}^{N} \dim\left(\mathrm{H}_k\left(X\right)\right) - \overline{a}_n,$$

as desired. □

7 The Euler characteristic

We finally prove part (ii) of Theorem B for the case of cyclic groups of order p.

We start by recalling what we must prove. Fix a prime p. All homology groups in this section have coefficients in $\mathbb{F}_p$. Let G be a cyclic group of order p and let X be a finite-dimensional nicely transitive simplicial G-complex all of whose mod-p homology groups are finite dimensional. In the previous section, we proved that all the mod-p homology groups of X^G are finite-dimensional, so its mod-p Euler characteristic $\chi_p(X^G)$ is well defined. We must prove that $\chi_p(X) \equiv \chi_p(X^G)$ modulo p.

Let $\tau \in \mathbb{F}_p[G]$ be as in §5. For $1 \leq i \leq p$, Proposition 6.1 says that all the Smith special homology groups $\mathrm{H}_k^{\tau^i}(X)$ are finite-dimensional. Moreover, since X is finite-dimensional, only finitely many of them are nonzero. It thus makes sense to define

$$\chi_p^{\tau^i}(X) = \sum_{k=0}^{\infty} (-1)^k \dim \mathrm{H}_k^{\tau^i}(X).$$

Now, the case $\rho = \tau^1$ of Lemma 5.8 gives a long exact sequence

$$\cdots \to \mathrm{H}_k^{\tau^{p-1}}(X) \oplus \mathrm{H}_k(X^G) \to \mathrm{H}_k(X) \to \mathrm{H}_k^{\tau}(X) \to \mathrm{H}_{k-1}^{\tau^{p-1}}(X) \oplus \mathrm{H}_{k-1}(X^G) \to \cdots.$$

Taking Euler characteristics, we deduce that

$$\chi_p(X) = \chi_p^{\tau}(X) + \chi_p^{\tau^{p-1}}(X) + \chi_p(X^G).$$

To prove that $\chi_p(X)$ and $\chi_p(X^G)$ are equal modulo p, it is enough to prove that

$$\chi_p^{\tau}(X) + \chi_p^{\tau^{p-1}}(X) \equiv 0 \pmod{p}. \tag{7.1}$$

For this, we will use the other long exact sequence connecting the Smith special homology groups, namely the one given by Lemma 5.6. For $1 \leq i \leq p-1$, this is of the form

$$\cdots \to \mathrm{H}_k^{\tau^{p-1}}(X) \to \mathrm{H}_k^{\tau^i}(X) \to \mathrm{H}_k^{\tau^{i+1}}(X) \to \mathrm{H}_{k-1}^{\tau^{p-1}}(X) \to \cdots;$$

here we are using Lemma 4.2, which says that $\tau^{p-1} = \sigma$. Since $\tau^p = 0$, this is only interesting when $1 \leq i \leq p-2$. For these values, taking Euler characteristics we see that

$$\chi_p^{\tau^i}(X) = \chi_p^{\tau^{p-1}}(X) + \chi_p^{\tau^{i+1}}(X).$$

This implies that

$$\begin{aligned}\sum_{i=1}^{p-1} \chi_p^{\tau^i}(X) &= \left(\sum_{i=1}^{p-2} \left(\chi_p^{\tau^{p-1}}(X) + \chi_p^{\tau^{i+1}}(X)\right)\right) + \chi_p^{\tau^{p-1}}(X) \\ &= \left(\sum_{i=2}^{p-1} \chi_p^{\tau^i}(X)\right) + (p-1)\,\chi_p^{\tau^{p-1}}(X).\end{aligned}$$

Rearranging, we see that

$$\chi_p^{\tau^1}(X) = (p-1)\,\chi_p^{\tau^{p-1}}(X).$$

The desired equation (7.1) follows.

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