Smith theory

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Abstract

We discuss theorems of P. Smith and Floyd connecting the cohomology of a simplicial complex equipped with an action of a finite p-group to the cohomology of its fixed points.

1 Introduction

Let G be a discrete group. A *simplicial G-complex* is a simplicial complex X equipped with an action of G by simplicial automorphisms. Around 1940, Paul Smith proved a number of striking theorems that relate the cohomology of a simplicial G-complex to the cohomology of the subspace X^G of fixed points. See [Sm] for references. In these notes, we discuss and prove one of Smith's main results.

Basic definitions. To state Smith's results, we need some definitions. Fix a prime p and a simplicial complex X.

- The simplicial complex X is *mod-p acyclic* if

$$\mathrm{H}_k(X; \mathbb{F}_p) = \begin{cases} \mathbb{F}_p & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

  Note that this implies that X is nonempty and connected.
- The simplicial complex X is a *mod-p homology n-sphere* if

$$\mathrm{H}_k(X; \mathbb{F}_p) = \begin{cases} \mathbb{F}_p & \text{if } k = 0 \text{ or } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

  Note that we are not requiring that X be a manifold nor that it be n-dimensional.
- If all the mod-p homology groups of X are finite-dimensional and only finitely many of them are nonzero, define

$$\chi_p(X) = \sum_{k=1}^{\infty} (-1)^k \dim \mathrm{H}_k(X; \mathbb{F}_p) \in \mathbb{Z}.$$

  We emphasize that this is an integer.

Smith's theorem. The following theorem summarizes some of Smith's results.

Theorem A (Smith)**.** *Let p be a prime, let G be a finite p-group, and let X be a finite-dimensional simplicial G-complex. The following then hold.*

(a) If all the mod-p homology groups of X are finite dimensional, then the same is true for X^G and $\chi_p(X) \equiv \chi_p(X^G)$ modulo p.
(b) If X is mod-p acyclic, then so is X^G. In particular, X^G is nonempty.
(c) If X is a mod-p homology n-sphere, then X^G is either empty or a mod-p homology m-sphere for some $0 \leq m \leq n$.

Remark 1.1. The condition that X is a simplicial complex can be somewhat relaxed at the cost of using Čech cohomology and imposing certain technical conditions on the spaces involved. See [B] for a discussion, and Remark 4.2 below for some more details on why our proof does not work in general. □

Remark 1.2. No theorem like Theorem A holds for groups that are not finite p-groups. For instance, if G is a finite cyclic group whose order is not a power of a prime, then building on work of Conner–Floyd [CF], Kister [K] constructed a simplicial action of G on a triangulation X of Euclidean space such that $X^G = \emptyset$. The ultimate result in this direction is a remarkable theorem of Oliver [O] that characterizes the possible fixed-point sets for actions of *arbitrary* finite groups on finite-dimensional contractible complexes. □

Compactness and Euler characteristic. If p is a prime, G is a finite p-group, and X is a **compact** simplicial G-complex, then there is the following elementary argument for why $\chi_p(X) \equiv \chi_p(X^G)$ modulo p. Subdividing X appropriately, we can assume that X/G is the simplicial complex whose k-simplices are the G-orbits of simplices of X. In particular, X^G is a subcomplex of X. The proof will be by induction on $|G|$.

The base case is where $|G| = p$, so G is a cyclic group of order p. This implies that for all simplices Δ of X, the G-orbit $G \cdot \Delta$ either consists of a single simplex (so $\Delta \in X^G$) or consists of p distinct simplices. Adding all of these up, we see that in fact

$$\chi(X) + (p-1)\chi(X^G) = p\chi(X/G);$$

here we can talk about χ rather than χ_p since everything is compact. Reducing everything modulo p, we see that $\chi(X) \equiv \chi(X^G)$ modulo p, as desired.

Now assume that $|G| > p$ and that the result is true for all smaller groups. Since G is a finite p-group, it is nilpotent. This implies that there is a nontrivial proper normal subgroup G' of G. The group G/G' acts on $X^{G'}$, and applying our inductive hypothesis twice we see that

$$\chi(X) \equiv \chi(X^{G'}) \equiv \chi\left(\left(X^{G'}\right)^{G/G'}\right) = \chi(X^G)$$

modulo p, as desired.

Finite-dimensionality. The condition that X is finite-dimensional is essential for all three parts of Theorem A. Let G be a finite p-group (in fact, the arguments below will not use the fact that G is finite nor that it is a p-group).

- Let X be the universal cover of a $K(G,1)$. The group G then acts freely on X, so $X^G = \emptyset$, contradicting (b), and $0 = \chi_p(X^G) \neq \chi_p(X) = 1$, contradicting (a).
- Generalizing the usual construction of a $K(G,1)$, more elaborate infinite-dimensional counterexamples to (a) and (b) can be constructed as follows. Let K be *any* simplicial complex. Endow K with the trivial G-action. Form X by first equivariantly attaching 1-cells to K that are permuted freely by G to make it connected, then equivariantly attaching 2-cells to make it simply-connected, etc. The result is a contractible simplicial G-complex X with $X^G = K$.
- With a bit more care, an argument like in the previous bullet point can show that any simplicial complex K can be the fixed-point set of a simplicial action of G on a mod-p homology n-sphere for any n (necessarily infinite-dimensional), contradicting (c).

Remark 1.3. L. Jones [J] proved a beautiful converse to part (b) of Theorem A that says that if K is a finite simplicial complex that is mod-p acyclic and G is a finite p-group, then there exists a finite-dimensional contractible simplicial G-complex X such that $X^G \cong K$. □

Stronger result. In fact, what we will actually prove is the following theorem, which is due to Floyd [F]. See below for how to derive Theorem A from it.

Theorem B (Floyd)**.** *Let G be a finite p-group and let X be a finite-dimensional simplicial G-complex whose mod-p homology groups are all finitely generated. The following then hold.*

(i) For all $n \geq 0$, we have

$$\sum_{k=n}^{\infty} \dim \mathrm{H}_k(X^G; \mathbb{F}_p) \leq \sum_{k=n}^{\infty} \dim \mathrm{H}_k(X; \mathbb{F}_p).$$

In particular, all the mod-p homology groups of X^G are finite-dimensional.

(ii) $\chi_p(X^G) \equiv \chi_p(X)$ modulo p.

Derivation of Smith's theorem. Theorem A can be derived from Theorem B as follows. As in Theorem A, let G be a finite p-group and let X be a finite-dimensional simplicial G-complex.

- Part (a) of Theorem A is contained in the conclusions of Theorem B.
- For part (b) of Theorem A, assume that X is mod-p acyclic. Part (i) of Theorem B then says that
$$\sum_{k=0}^{\infty} \dim \mathrm{H}_k(X^G; \mathbb{F}_p) \leq \sum_{k=0}^{\infty} \dim \mathrm{H}_k(X; \mathbb{F}_p) = 1.$$
This implies that either $X^G = \emptyset$ or that X^G is mod-p acyclic. To rule out $X^G = \emptyset$, we use part (ii) of Theorem B to see that
$$\chi_p(X^G) \equiv \chi_p(X) \equiv 1 \pmod{p}.$$
- For part (c) of Theorem A, assume that X is a mod-p homology n-sphere and that $X^G \neq \emptyset$. We must prove that X^G is a mod-p homology m-sphere for some $0 \leq m \leq n$.

Part (i) of Theorem B says that

$$\sum_{k=0}^{\infty} \dim \mathrm{H}_k(X^G; \mathbb{F}_p) \leq \sum_{k=0}^{\infty} \dim \mathrm{H}_k(X; \mathbb{F}_p) = 2. \tag{1.1}$$

The left hand side of (1.1) is thus either 0 or 1 or 2. The case where it is 0 is precisely the case where $X^G = \emptyset$, which is allowed. If the left-hand side of (1.1) is 1, then necessarily the nonzero homology group is the 0^{th} one, so $\chi_p(X^G) = 1$, which contradicts the conclusion in part (ii) of Theorem B that says that modulo p we have

$$\chi_p(X^G) \equiv \chi_p(X) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Finally, assume that the left hand side of (1.1) is 2, so X^G is a mod-p homology m-sphere for some m. To see that $0 \leq m \leq n$, we apply part (i) of Theorem B again to see that

$$\sum_{k=n+1}^{\infty} \dim \mathrm{H}_k(X^G; \mathbb{F}_p) \leq \sum_{k=n+1}^{\infty} \dim \mathrm{H}_k(X; \mathbb{F}_p) = 0,$$

so $\mathrm{H}_k(X^G; \mathbb{F}_p) = 0$ for all $k \geq n+1$.

Outline. We will begin in §2 by reducing Theorem B to the special case of a cyclic group of order p. In §3, we will discuss some algebraic properties of the group ring of a cyclic group of order p. Next, in §4 we will introduce Smith's special homology groups. Part (i) of Theorem B is then proved in §5 and part (ii) in §6.

2 An initial reduction

Let us first recall what we must prove. Let p be a prime, let G be a finite p-group, and let X be a finite-dimensional simplicial G-complex whose mod-p homology groups are all finitely generated. Subdividing X appropriately, we can assume that X/G is the simplicial complex whose k-simplices are the G-orbits of k-simplices of X. In particular, X^G is a subcomplex of X. For all $n \geq 0$, we must prove that

$$\sum_{k=n}^{\infty} \dim \mathrm{H}_k(X^G; \mathbb{F}_p) \leq \sum_{k=n}^{\infty} \dim \mathrm{H}_k(X; \mathbb{F}_p).$$

We must also prove that $\chi_p(X^G) \equiv \chi_p(X)$ modulo p.

The first reduction is that it is enough to prove this for $|G| = p$. Indeed, assume that we have done this. We will prove the general case by induction on the order of G, the base case being the case where $|G| = p$. A general p-group is nilpotent. If $|G| > p$, there thus exists a nontrivial proper normal subgroup G' of G. The group G/G' acts on $X^{G'}$ and

$$\left(X^{G'}\right)^{G/G'} = X^G.$$

We can thus apply our inductive hypothesis twice, first to the action of G' on X and then to the action of G/G' on $X^{G'}$. The desired conclusions follow.

It thus remains to prove the theorem in the case where $|G| = p$, i.e. the case where G is a cyclic group of order p.

3 The group ring of a cyclic group

Let p be a prime and let G be a cyclic group of order p. If X is a simplicial G-complex, then the homology groups $\mathrm{H}_k(X; \mathbb{F}_p)$ have a G-action, and thus are modules over the group ring $\mathbb{F}_p[G]$. In this section, we introduce some special features of $\mathbb{F}_p[G]$ that we will use in the next section to study these homology groups.

Let t be the generator of G, so $\mathbb{F}_p[G] \cong \mathbb{F}_p[t]/(t^p - 1)$. Define

$$\tau = 1 - t \in \mathbb{F}_p[G],$$
$$\sigma = 1 + t + t^2 + \cdots + t^{p-1} \in \mathbb{F}_p[G].$$

The following sequence of results relate τ and σ.

Lemma 3.1. *The kernel of the map $\mathbb{F}_p[G] \to \tau \cdot \mathbb{F}_p[G]$ that multiplies elements by τ is 1-dimensional and spanned by σ. In particular, $\sigma \cdot \mathbb{F}_p[G]$ is 1-dimensional.*

Proof. A general element x of $\mathbb{F}_p[G]$ is of the form

$$x = a_0 + a_1 t + a_2 t^2 + \cdots + a_{p-1} t^{p-1}$$

for some $a_i \in \mathbb{F}_p$. We then have

$$\tau x = (a_0 - a_{p-1}) + (a_1 - a_0)t + (a_2 - a_1)t^2 + \cdots + (a_{p-1} - a_{p-2})t^{p-1}.$$

This is zero if and only if all the a_i are equal, i.e. if and only if x is a multiple of σ. $\square$

Lemma 3.2. *$\tau^{p-1} = \sigma$.*

Proof. The binomial theorem says that

$$\tau^{p-1} = (1-t)^{p-1} = \sum_{i=0}^{p-1} \binom{p-1}{i} (-1)^i t^i.$$

We must therefore prove that

$$\binom{p-1}{i} (-1)^i \equiv 1 \pmod{p}$$

for $1 \leq i \leq p-1$. For this, we calculate:

$$\binom{p-1}{i}(-1)^i = \frac{(p-1)(p-2)\cdots(p-i)}{(i)(i-1)\cdots(1)}(-1)^i \equiv (-1)^i \frac{(1)(2)\cdots(i)}{(i)(i-1)\cdots(1)}(-1)^i = 1. \quad \square$$

Lemma 3.3. *For all $0 \le i \le p-1$, we have $\sigma \in \tau^i \cdot \mathbb{F}_p[G]$.*

Proof. Lemma 3.2 says that $\sigma = \tau^{p-1} = \tau^i \cdot \tau^{p-1-i}$. $\square$

Lemma 3.4. *For all $0 \le i \le p-1$, we have a short exact sequence*

$$0 \longrightarrow \sigma \cdot \mathbb{F}_p[G] \longrightarrow \tau^i \cdot \mathbb{F}_p[G] \longrightarrow \tau^{i+1} \cdot \mathbb{F}_p[G] \longrightarrow 0.$$

Proof. Immediate from Lemmas 3.1 and 3.3. $\square$

Lemma 3.5. *For all $0 \le i \le p$, the subspace $\tau^i \cdot \mathbb{F}_p[t]$ of $\mathbb{F}_p[t]$ is $(p-i)$-dimensional.*

Proof. The space $\tau^0 \cdot \mathbb{F}_p[t] = \mathbb{F}_p[t]$ is p-dimensional, and Lemma 3.4 implies that for $0 \le i \le p-1$ we have

$$\dim\left(\tau^i \cdot \mathbb{F}_p[G]\right) = \dim\left(\sigma \cdot \mathbb{F}_p[G]\right) + \dim\left(\tau^{i+1} \cdot \mathbb{F}_p[G]\right) = 1 + \dim\left(\tau^{i+1} \cdot \mathbb{F}_p[G]\right).$$

The lemma follows. $\square$

4 The Smith special homology groups

Let p be a prime, let G be a cyclic group of order p, and let X be a simplicial G-complex such that X/G is the simplicial complex whose k-simplices are the G-orbits of k-simplices of X; in particular, X^G is a subcomplex of X. Throughout this section, all chain complexes and homology groups will have $\mathbb{F}_p$-coefficients.

The chain complex $C_\bullet(X)$ is a module over the group ring $\mathbb{F}_p[G]$. Let

$$t, \tau, \sigma \in \mathbb{F}_p[G]$$

be as in §3. Since we are working over $\mathbb{F}_p$, we have

$$\tau^p = (1-t)^p = 1^p - t^p = 1 - 1 = 0.$$

Writing $C_\bullet^{\tau^k}(X)$ for $\tau^k \cdot C_\bullet(X)$, we thus have a filtration

$$C_\bullet(X) = C_\bullet^{\tau^0} \supset C_\bullet^{\tau^1}(X) \supset C_\bullet^{\tau^2}(X) \supset \cdots \supset C_\bullet^{\tau^p}(X) = 0$$

of chain complexes. We make the following definition.

Definition 4.1. For $\rho = \tau^i$ with $0 \le i \le p$, the *Smith special homology groups* of X of index ρ, denoted $\mathrm{H}_\bullet^\rho(X)$, are the homology groups of the complex $C_\bullet^\rho(C)$. $\square$

Remark 4.2. One might wonder why we insist on only defining the Smith special homology groups for simplicial G-complexes. After all, for general spaces it would make perfect sense to mimic the definition of the Smith special homology groups using singular homology. The only serious use we will make of the simplicial complex structure is to deduce that if X is finite-dimensional, then its Smith special homology groups must vanish in sufficiently high degrees (this shows up in the proof of Proposition 5.1 below). This is not at all clear for singular homology. $\square$

The Smith special homology groups of different indices are related by two long exact sequences. The first is the following. In its statement, recall from Lemma 3.2 that $\sigma = \tau^{p-1}$.

Lemma 4.3. *Let G and X be as above. For all $1 \leq i \leq p-1$, we have a long exact sequence*

$$\cdots \to \mathrm{H}_k^{\sigma}(X) \to \mathrm{H}_k^{\tau^i}(X) \to \mathrm{H}_k^{\tau^{i+1}}(X) \to \mathrm{H}_{k-1}^{\sigma}(X) \to \cdots$$

Remark 4.4. Since $\tau^p = 0$, we have $\mathrm{H}_k^{\tau^p}(X) = 0$, so for $i = p-1$ this reduces to the trivial fact that $\mathrm{H}_k^{\sigma}(X) \cong \mathrm{H}_k^{\tau^{p-1}}(X)$. □

Proof of Lemma 4.3. Lemma 3.2 says that $\sigma = \tau^{p-1}$, which implies that the chain complex $C_\bullet^{\sigma}(X)$ is a subcomplex of $C_\bullet^{\tau^i}(X)$. We will prove that the sequence

$$0 \longrightarrow C_\bullet^{\sigma}(X) \longrightarrow C_\bullet^{\tau^i}(X) \longrightarrow C_\bullet^{\tau^{i+1}}(X) \longrightarrow 0 \tag{4.1}$$

of chain complexes is exact. The desired long exact sequence is the associated long exact sequence in homology.

Fix some $k \geq 0$, let Δ be a k-simplex of X, and let Y_Δ be the subcomplex of X whose k-simplices consist of the G-orbit of Δ. We thus have subspaces

$$C_k^{\sigma}(Y_\Delta) \quad \text{and} \quad C_k^{\tau^i}(Y_\Delta) \quad \text{and} \quad C_k^{\tau^{i+1}}(Y_\Delta)$$

of

$$C_k^{\sigma}(X) \quad \text{and} \quad C_k^{\tau^i}(X) \quad \text{and} \quad C_k^{\tau^{i+1}}(X), \tag{4.2}$$

respectively. The vector spaces (4.2) are direct sums of such subspaces, and the maps in (4.1) preserve this direct sum decomposition. We conclude that it is enough to prove that the sequence

$$0 \longrightarrow C_\bullet^{\sigma}(Y_\Delta) \longrightarrow C_\bullet^{\tau^i}(Y_\Delta) \longrightarrow C_\bullet^{\tau^{i+1}}(Y_\Delta) \longrightarrow 0 \tag{4.3}$$

is exact.

There are two cases. The first is where Δ is a simplex of X^G, so $Y_\Delta = (Y_\Delta)^G = \Delta$. Since $\tau = 1 - t$ kills $C_k(X^G)$, all the terms of (4.3) are 0 and its exactness is trivial.

The second case is where Δ is not a simplex of X^G. In this case, since G is a cyclic group of order p, the G-stabilizer of Δ is trivial. This implies that as a $\mathbb{F}_p[G]$-module we have

$$C_k(Y_\Delta) \cong \mathbb{F}_p[G].$$

The sequence (4.3) is thus of the form

$$0 \longrightarrow \sigma \cdot \mathbb{F}_p[G] \longrightarrow \tau^i \cdot \mathbb{F}_p[G] \longrightarrow \tau^{i+1} \cdot \mathbb{F}_p[G] \longrightarrow 0.$$

The exactness of this is precisely Lemma 3.4. □

The second long exact sequence relating the different Smith special homology groups is as follows. The reader should note the appearance of X^G; this exact sequence will be the key to relating the topology of X and X^G.

Lemma 4.5. *Let G and X be as above. Fix some $\rho = \tau^i$ with $1 \leq i \leq p-1$, and let $\overline{\rho} = \tau^{p-i}$. We then have a long exact sequence*

$$\cdots \to \mathrm{H}_k^{\overline{\rho}}(X) \oplus \mathrm{H}_k(X^G) \to \mathrm{H}_k(X) \to \mathrm{H}_k^{\rho}(X) \to \mathrm{H}_{k-1}^{\overline{\rho}}(X) \oplus \mathrm{H}_{k-1}(X^G) \to \cdots$$

Proof. Let

$$\iota\colon C_\bullet^{\overline{\rho}}(X) \oplus C_\bullet(X^G) \to C_\bullet(X)$$

be the chain complex map induced by the inclusions of the two factors and let

$$\pi\colon C_\bullet(X) \to C_\bullet^{\rho}(X)$$

be the chain complex map that multiplies chains by ρ. We will prove that the sequence

$$0 \longrightarrow C_\bullet^{\overline{\rho}}(X) \oplus C_\bullet(X^G) \xrightarrow{\iota} C_\bullet(X) \xrightarrow{\pi} C_\bullet^{\rho}(X) \longrightarrow 0 \tag{4.4}$$

of chain complexes is exact. The desired long exact sequence is the associated long exact sequence in homology.

Fix some $k \geq 0$, let Δ be a k-simplex of X, and let Y_Δ be the subcomplex of X whose k-simplices consist of the G-orbit of Δ. We thus have subspaces

$$C_k^{\overline{\rho}}(Y_\Delta) \oplus C_k\left((Y_\Delta)^G\right) \quad \text{and} \quad C_k(Y_\Delta) \quad \text{and} \quad C_k^{\rho}(Y_\Delta)$$

of

$$C_k^{\overline{\rho}}(X) \oplus C_k(X^G) \quad \text{and} \quad C_k(X) \quad \text{and} \quad C_k^{\rho}(X), \tag{4.5}$$

respectively. The vector spaces (4.5) are direct sums of such subspaces, and the maps ι and π preserve this direct sum decomposition. We conclude that it is enough to prove that the sequence

$$0 \longrightarrow C_k^{\overline{\rho}}(Y_\Delta) \oplus C_k\left((Y_\Delta)^G\right) \xrightarrow{\iota} C_k(Y_\Delta) \xrightarrow{\pi} C_k^{\rho}(Y_\Delta) \longrightarrow 0 \tag{4.6}$$

is exact.

There are two cases. The first is where Δ is a simplex of X^G, so $Y_\Delta = (Y_\Delta)^G = \Delta$. Since both $\rho = \tau^i$ and $\overline{\rho} = \tau^{p-i}$ kill $C_k(X^G)$, the sequence (4.6) is of the form

$$0 \longrightarrow 0 \oplus \mathbb{F}_p \xrightarrow{\cong} \mathbb{F}_p \longrightarrow 0 \longrightarrow 0.$$

This is evidently exact.

The second case is where Δ is not a simplex of X^G. In this case, since G is a cyclic group of order p, the G-stabilizer of Δ is trivial. This implies that $(Y_\Delta)^G$ has no k-simplices, so

$$C_k\left((Y_\Delta)^G\right) = 0.$$

Moreover, as a $\mathbb{F}_p[G]$-module we have

$$C_k(Y_\Delta) \cong \mathbb{F}_p[G].$$

The sequence (4.6) is thus of the form

$$0 \longrightarrow \overline{\rho} \cdot \mathbb{F}_p[G] \longrightarrow \mathbb{F}_p[G] \longrightarrow \rho \cdot \mathbb{F}_p[G] \longrightarrow 0. \tag{4.7}$$

The leftmost map here is clearly injective. Similarly, the rightmost map is clearly surjective. Finally, Lemma 3.5 says that

$$\dim\left(\rho \cdot \mathbb{F}_p[G]\right) = \dim\left(\tau^i \cdot \mathbb{F}_p[G]\right) = p - i$$

and

$$\dim\left(\overline{\rho} \cdot \mathbb{F}_p[G]\right) = \dim\left(\tau^{p-i} \cdot \mathbb{F}_p[G]\right) = i.$$

Since these add up to $\dim \mathbb{F}_p[G] = p$, the exactness of (4.7) follows. □

5 The inequality

Let p be a prime. In this section, all homology groups will have $\mathbb{F}_p$ coefficients. Our goal in this section is to use the Smith special homology groups to prove part (i) of Theorem B for the case of cyclic groups of order p, which we observed in §2 implies the general case. In fact, we will prove the following stronger proposition, whose additional conclusions will be important later when we study Euler characteristics. Let $\tau \in \mathbb{F}_p[G]$ be as in §4.

Proposition 5.1. *Let G be a cyclic group of order p and let X be a finite-dimensional simplicial G-complex such that the quotient X/G is the simplicial complex whose k-simplices are the G-orbits of simplices of X. Assume that all the mod-p homology groups of X are finite-dimensional. For some $1 \leq i \leq p-1$, set $\rho = \tau^i$. The following then hold.*

- *All the Smith special homology groups $\mathrm{H}_k^{\rho}(X)$ are finite-dimensional.*
- *For all $n \geq 0$, we have*

$$\sum_{k=n}^{\infty} \dim \mathrm{H}_k(X^G) \leq \left(\sum_{k=n}^{\infty} \dim \mathrm{H}_k\left(X\right)\right) - \dim \mathrm{H}_n^{\rho}(X).$$

In particular, all the $\mathrm{H}_k(X^G)$ are finite-dimensional.

Proof. Set $\overline{\rho} = \tau^{p-i}$. For all k, Lemma 4.5 gives a long exact sequence which contains the segment

$$\mathrm{H}_{k+1}^{\rho}(X) \longrightarrow \mathrm{H}_k^{\overline{\rho}}(X) \oplus \mathrm{H}_k(X^G) \longrightarrow \mathrm{H}_k(X). \tag{5.1}$$

Setting

$$a_i = \dim\left(\mathrm{H}_i^{\rho}\left(X\right)\right) \quad \text{and} \quad \overline{a}_i = \dim\left(\mathrm{H}_i^{\overline{\rho}}\left(X\right)\right),$$

we deduce from (5.1) that

$$\overline{a}_k + \dim\left(\mathrm{H}_k\left(X^G\right)\right) \leq a_{k+1} + \dim\left(\mathrm{H}_k\left(X\right)\right). \tag{5.2}$$

Noting that the roles of ρ and $\overline{\rho}$ in Lemma 4.5 can be reversed, we obtain in a similar way that

$$a_k + \dim\left(\mathrm{H}_k\left(X^G\right)\right) \leq \overline{a}_{k+1} + \dim\left(\mathrm{H}_k\left(X\right)\right). \tag{5.3}$$

Let X be N-dimensional. From the definition of the Smith special homology groups, we see that for $k \geq N+1$ we have

$$\mathrm{H}_k^{\rho}(X) = \mathrm{H}_k^{\overline{\rho}}(X) = 0.$$

The case $k = N$ of (5.2) and (5.3) thus reduce to

$$\overline{a}_N + \dim\left(\mathrm{H}_N\left(X^G\right)\right) \le \dim\left(\mathrm{H}_N\left(X\right)\right) < \infty$$

and

$$a_N + \dim\left(\mathrm{H}_N\left(X^G\right)\right) \le \dim\left(\mathrm{H}_N\left(X\right)\right) < \infty,$$

so $\overline{a}_N, a_N < \infty$. The case $k = N-1$ of (5.2) and (5.3) then say that

$$\overline{a}_{N-1} + \dim\left(\mathrm{H}_{N-1}\left(X^G\right)\right) \le a_N + \dim\left(\mathrm{H}_{N-1}\left(X\right)\right) < \infty$$

and

$$a_{N-1} + \dim\left(\mathrm{H}_{N-1}\left(X^G\right)\right) \le \overline{a}_N + \dim\left(\mathrm{H}_{N-1}\left(X\right)\right) < \infty,$$

so $\overline{a}_{N-1}, a_{N-1} < \infty$. Repeating this argument and working backwards, we see that $\overline{a}_k, a_k < \infty$ for all k, as claimed in the first conclusion of the proposition.

We now turn to the second conclusion. Rearranging (5.2) and (5.3), we see that

$$\dim\left(\mathrm{H}_k\left(X^G\right)\right) \le \dim\left(\mathrm{H}_k\left(X\right)\right) + \left(a_{k+1} - \overline{a}_k\right). \tag{5.4}$$

and

$$\dim\left(\mathrm{H}_k\left(X^G\right)\right) \le \dim\left(\mathrm{H}_k\left(X\right)\right) + \left(\overline{a}_{k+1} - a_k\right). \tag{5.5}$$

As we will see, there is a slight difference in the calculation we are about to do depending on whether or not

$$\sum_{k=n}^{N} \dim\left(\mathrm{H}_k\left(X^G\right)\right)$$

has an even or an odd number of terms; to avoid cluttering our notation, we will assume that it has an even number. Now, alternately using (5.4) and (5.5), we see that

$$\sum_{k=n}^{N} \dim\left(\mathrm{H}_k\left(X^G\right)\right) \le \sum_{k=n}^{N} \dim\left(\mathrm{H}_k\left(X\right)\right) + (a_{n+1} - \overline{a}_n) + (\overline{a}_{n+2} - a_{n+1}) + \cdots + (\overline{a}_{N+1} - a_n).$$

This would end with $a_{N+1} - \overline{a}_n$ if there were an even number of terms in our sum. The above sum telescopes and thus reduces to

$$\sum_{k=n}^{N} \dim\left(\mathrm{H}_k\left(X^G\right)\right) \le \sum_{k=n}^{N} \dim\left(\mathrm{H}_k\left(X\right)\right) + (\overline{a}_{N+1} - \overline{a}_n).$$

Since X is N-dimensional, the term $\overline{a}_{N+1}$ is actually 0. We conclude that

$$\sum_{k=n}^{N} \dim\left(\mathrm{H}_k\left(X^G\right)\right) \le \sum_{k=n}^{N} \dim\left(\mathrm{H}_k\left(X\right)\right) - \overline{a}_n,$$

as desired. $\square$

6 The Euler characteristic

We finally prove part (ii) of Theorem B for the case of cyclic groups of order p.

We start by recalling what we must prove. Fix a prime p. All homology groups in this section have coefficients in $\mathbb{F}_p$. Let G be a cyclic group of order p and let X be a finite-dimensional simplicial G-complex all of whose mod-p homology groups are finite dimensional. Subdividing, we can assume that X/G is the simplicial complex whose k-simplices are the G-orbits of the simplices of X; in particular, X^G is a subcomplex of X. In the previous section, we proved that all the mod-p homology groups of X^G are finite-dimensional, so its mod-p Euler characteristic $\chi_p(X^G)$ is well defined. We must prove that $\chi_p(X) \equiv \chi_p(X^G)$ modulo p.

Let $\tau \in \mathbb{F}_p[G]$ be as in §4. For $1 \leq i \leq p$, Proposition 5.1 says that all the Smith special homology groups $\mathrm{H}_k^{\tau^i}(X)$ are finite-dimensional. Moreover, since X is finite-dimensional, only finitely many of them are nonzero. It thus makes sense to define

$$\chi_p^{\tau^i}(X) = \sum_{k=0}^{\infty} (-1)^k \dim \mathrm{H}_k^{\tau^i}(X).$$

Now, the case $\rho = \tau^1$ of Lemma 4.5 gives a long exact sequence

$$\cdots \to \mathrm{H}_k^{\tau^{p-1}}(X) \oplus \mathrm{H}_k(X^G) \to \mathrm{H}_k(X) \to \mathrm{H}_k^{\tau}(X) \to \mathrm{H}_{k-1}^{\tau^{p-1}}(X) \oplus \mathrm{H}_{k-1}(X^G) \to \cdots.$$

Taking Euler characteristics, we deduce that

$$\chi_p(X) = \chi_p^{\tau}(X) + \chi_p^{\tau^{p-1}}(X) + \chi_p(X^G).$$

To prove that $\chi_p(X)$ and $\chi_p(X^G)$ are equal modulo p, it is enough to prove that

$$\chi_p^{\tau}(X) + \chi_p^{\tau^{p-1}}(X) \equiv 0 \pmod{p}. \tag{6.1}$$

For this, we will use the other long exact sequence connecting the Smith special homology groups, namely the one given by Lemma 4.3. For $1 \leq i \leq p-1$, this is of the form

$$\cdots \to \mathrm{H}_k^{\tau^{p-1}}(X) \to \mathrm{H}_k^{\tau^i}(X) \to \mathrm{H}_k^{\tau^{i+1}}(X) \to \mathrm{H}_{k-1}^{\tau^{p-1}}(X) \to \cdots;$$

here we are using Lemma 3.2, which says that $\tau^{p-1} = \sigma$. Since $\tau^p = 0$, this is only interesting when $1 \leq i \leq p-2$. For these values, taking Euler characteristics we see that

$$\chi_p^{\tau^i}(X) = \chi_p^{\tau^{p-1}}(X) + \chi_p^{\tau^{i+1}}(X).$$

This implies that

$$\begin{aligned} \sum_{i=1}^{p-1} \chi_p^{\tau^i}(X) &= \left(\sum_{i=1}^{p-2} \left(\chi_p^{\tau^{p-1}}(X) + \chi_p^{\tau^{i+1}}(X) \right) \right) + \chi_p^{\tau^{p-1}}(X) \\ &= \left(\sum_{i=2}^{p-1} \chi_p^{\tau^i}(X) \right) + (p-1)\, \chi_p^{\tau^{p-1}}(X). \end{aligned}$$

Rearranging, we see that

$$\chi_p^{\tau^1}(X) = (p-1)\, \chi_p^{\tau^{p-1}}(X).$$

The desired equation (6.1) follows.

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