Periodic billiard paths on smooth tables

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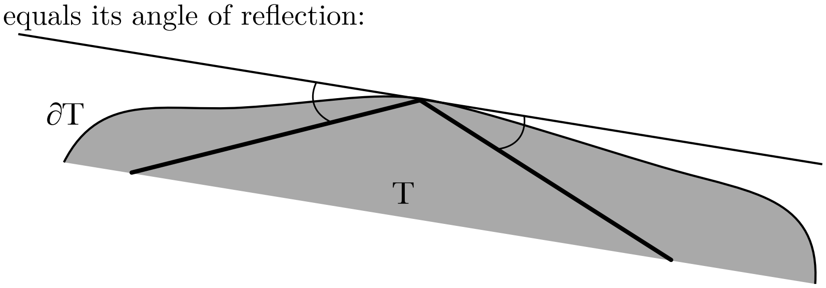
Abstract

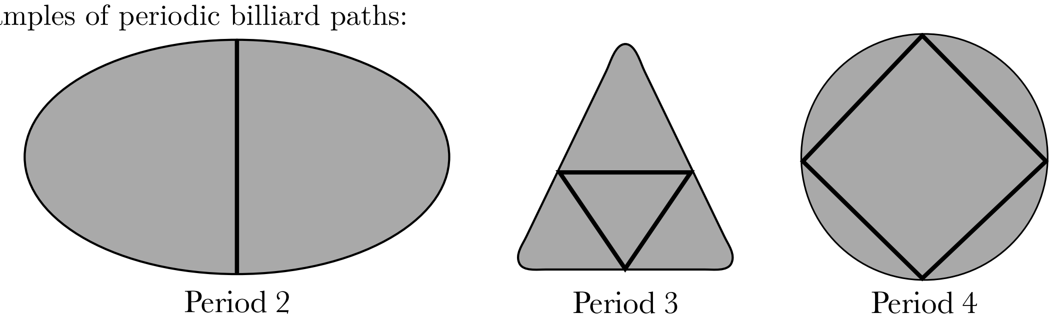
We explain a theorem of Birkhoff that says that a smooth convex billiard table always has periodic billiard paths of any given prime period.

1 Introduction

Let T be a compact region in $\mathbb{R}^2$ with smooth boundary ∂T. A *billiard path* in T is a bi-infinite polygonal path in T which changes direction only at points of ∂T, where its angle of incidence equals its angle of reflection:



The billiard path is *periodic* if it eventually closes up and starts repeating, in which case its *period* is the number of line segments making up this repeating pattern. Here are a few examples of periodic billiard paths:



The goal of this note is to explain the following theorem of Birkhoff [B]:

Theorem 1.1 (Birkhoff)**.** *Let T be a compact convex region in $\mathbb{R}^2$ with smooth boundary and let p be a prime. Then T has a periodic billiard path of period p.*

One can also consider billiard paths in compact regions of $\mathbb{R}^2$ whose boundaries are only piecewise smooth, in which case a billiard path is not allowed to hit one of the corners. In contrast to the elementary proof of Theorem 1.1, it is not known whether or not every convex polygon in $\mathbb{R}^2$ has a periodic billiard path. Masur [M] proved that periodic billiard paths exist on polygons whose angles are rational multiples of π. This question is otherwise wide open even for triangles, where the best current result is a theorem of Schwartz [S] that says that such paths exist for triangles whose angles are all less that 100°.

2 Billiards in an ellipse

For the proof of Theorem 1.1, we will first need to understand some classical facts about billiards in an ellipse. Recall that an ellipse E has two points f_1 and f_2 called its *foci* such that for some $c > 0$ we have

$$E = \{x \in \mathbb{R}^2 \mid |x - f_1| + |x - f_2| \leq c\}.$$

The main result we will need about billiards in an ellipse is as follows.

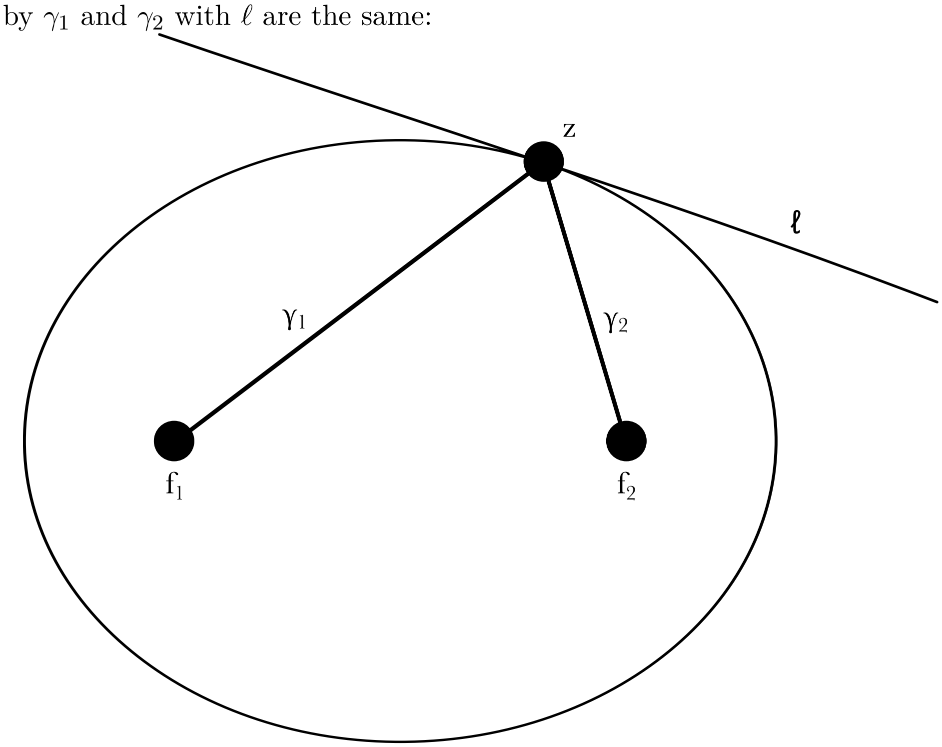
Theorem 2.1. *Let E be an ellipse with foci f_1 and f_2. Then any billiard path in E that passes through f_1 bounces off ∂E once and then passes through f_2.*

Theorem 2.1 implies that all billiard paths in an ellipse that go through one of the foci keep going back and forth between the two foci. This fact has been used in architecture to construct "whispering galleries": since sound waves travel like billiard paths, in an elliptical room if a person whispers at one of the foci then their voice will be concentrated and easily audible at the other one.

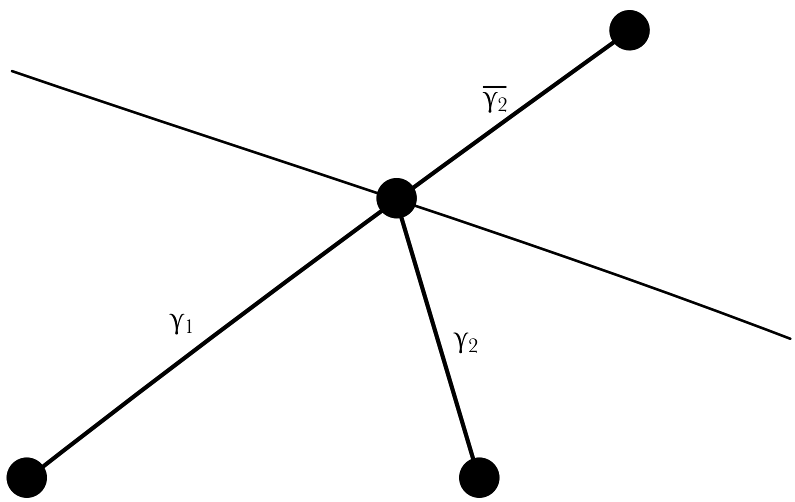
Proof of Theorem 2.1. Write

$$E = \{x \in \mathbb{R}^2 \mid |x - f_1| + |x - f_2| \leq c\}.$$

Let $z \in \partial E$ be any point, let ℓ be the tangent line through E at z, and let γ_1 and γ_2 be the line segments connecting f_1 and f_2 to z, respectively. Our goal is to prove that the angles formed by γ_1 and γ_2 with ℓ are the same:



The points on the outside of E are precisely those points x such that $|x - f_1| + |x - f_2| > c$. Since ℓ intersects E at z and otherwise lies outside of E, the point z is the unique point of ℓ that minimizes the sum of the distances from f_1 and f_2. Let $\overline{f}_2$ and $\overline{\gamma}_2$ be the reflections of f_2 and γ_2 across ℓ:



The point z is the unique point of ℓ that minimizes the sum of the distances from f_1 and $\overline{f}_2$. Reflecting on the above picture, we see that the fact that a straight line minimizes the length of a path connecting two points implies that the union of γ_1 and $\overline{\gamma}_2$ must be a straight line. The desired conclusion about the angles follows. □

3 Proof of Theorem 1.1

Let us first recall the statement. Let T be a compact convex region in $\mathbb{R}^2$ with smooth boundary and let p be a prime. Our goal is to prove that T has a periodic billiard path of period p.

What we will prove is that the longest possible closed path consisting of p straight line segments connecting points of ∂T is a billiard path. More precisely, define the function

$$L\colon (\partial T)^p \longrightarrow \mathbb{R}$$

via the formula

$$L(x_1, \ldots, x_p) = |x_2 - x_1| + \cdots + |x_p - x_{p-1}| + |x_1 - x_p|.$$

The function L is continuous and its domain is compact, so it achieves a global maximum at a point $(x_1, \ldots, x_p) \in (\partial T)^p$. Let B be the closed path that starts at x_1, then goes in a straight line to x_2, then goes in a straight line to x_3, etc., and ends by going in a straight line from x_p to x_1. We will prove that B is a billiard path if we extend it to repeat infinitely often; since p is prime, the period of the resulting path is p (instead of merely a divisor of p).

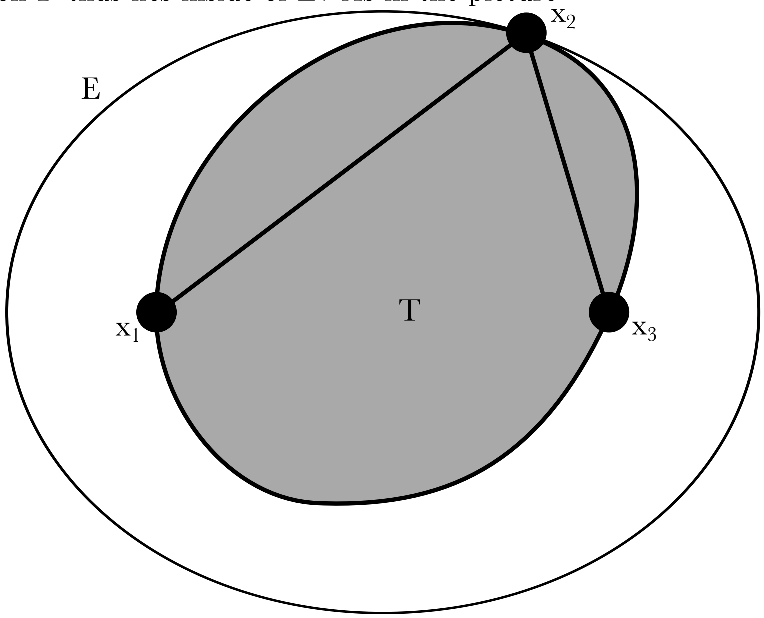
We must prove that the angle of incidence equals the angle of reflection at each x_i. Cyclically permuting the the x_i, it is enough to do this at x_2. To keep our notation from getting out of hand, in the case $p = 2$ let $x_3 = x_1$. Define

$$c = |x_2 - x_1| + |x_3 - x_2|$$

and let E be the ellipse

$$E = \{x \mid |x - x_1| + |x - x_3| \leq c\},$$

so E has foci x_1 and x_3 and $x_2 \in \partial E$. The points on the outside of E are precisely those $x \in \mathbb{R}^2$ such that $|x - x_1| + |x - x_3| > c$. If there was a point $x_2' \in \partial T$ outside of E, then we could replace x_2 with x_2' and increase L, contradicting the maximality of $L(x_1, \ldots, x_p)$. The whole region T thus lies inside of E. As in the picture



this implies that the tangent lines to ∂T and ∂E at x_2 are the same. Theorem 2.1 now implies that the angles of incidence and reflection at x_2 are indeed the same, as desired.

References

[B] G. D. Birkhoff, On the periodic motions of dynamical systems, Acta Math. 50 (1927), no. 1, 359–379.

[M] H. Masur, Closed trajectories for quadratic differentials with an application to billiards, Duke Math. J. 53 (1986), no. 2, 307–314.

[S] R. E. Schwartz, Obtuse triangular billiards. II. One hundred degrees worth of periodic trajectories, Experiment. Math. 18 (2009), no. 2, 137–171.

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