Existence and uniqueness for integral curves

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Abstract

We explain why integral curves to vector fields exist if the vector fields are continuous and are unique if the vector fields are locally Lipshitz.

Let $\vec{\mathfrak{v}}: U \to \mathbb{R}^n$ be a vector field defined on an open set $U \subset \mathbb{R}^n$. A *integral curve* of $\vec{\mathfrak{v}}$ is a differentiable function $\gamma: I \to U$ for some interval $I \subset \mathbb{R}$ such that $\gamma'(t) = \vec{\mathfrak{v}}(\gamma(t))$ for all $t \in I$. The Peano existence theorem says that if $\vec{\mathfrak{v}}$ is continuous, then these exist with any specified initial point:

Theorem A (Peano existence theorem). *Let $\vec{\mathfrak{v}}: U \to \mathbb{R}^n$ be a continuous vector field on an open set $U \subset \mathbb{R}^n$ and let $p_0 \in U$. Then for some $\epsilon > 0$ there exists an integral curve $\gamma: [0,\epsilon) \to U$ of $\vec{\mathfrak{v}}$ such that $\gamma(0) = p_0$.*

Remark 0.1. One might also want an integral curve $\gamma: (-\epsilon,\epsilon) \to U$ with $\gamma(0) = p_0$. This can be obtained by applying Theorem A to obtain integral curves $\gamma_1: [0,\epsilon_1) \to U$ for $\vec{\mathfrak{v}}$ and $\gamma_2: [0,\epsilon_2) \to U$ for $-\vec{\mathfrak{v}}$ with $\gamma_1(0) = \gamma_2(0) = p_0$. Setting $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, the desired integral curve $\gamma: (-\epsilon,\epsilon) \to U$ can then be defined via the formula

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{if } 0 \leq t < \epsilon, \\ \gamma_2(-t) & \text{if } -\epsilon < t \leq 0. \end{cases} \qquad \square$$

Such integral curves need not be unique; indeed, consider the follow example:

Example 0.2. Let $\vec{\mathfrak{v}}: \mathbb{R}^1 \to \mathbb{R}^1$ be the vector field $\vec{\mathfrak{v}}(x) = \sqrt{|x|}$. We then have two integral curves to $\vec{\mathfrak{v}}$ starting at 0:

- The constant curve $\gamma: [0,\infty) \to \mathbb{R}^1$ defined by $\gamma(x) = 0$.
- The curve $\gamma: [0,\infty) \to \mathbb{R}^1$ defined by $\gamma(x) = x^2/4$. $\square$

However, the Picard–Lindelöf theorem says that integral curves are unique if $\vec{\mathfrak{v}}$ is smooth. In fact, even weaker conditions on $\vec{\mathfrak{v}}$ are sufficient. Say that $\vec{\mathfrak{v}}: U \to \mathbb{R}^n$ is *locally Lipshitz* if for all $p_0 \in U$, there exists some $L > 0$ and some neighborhood $U' \subset U$ of p_0 such that the restriction of $\vec{\mathfrak{v}}$ to U' is L-Lipshitz, i.e. such that

$$\|\vec{\mathfrak{v}}(x) - \vec{\mathfrak{v}}(y)\| \leq L\|x - y\| \qquad (x, y \in U').$$

Clearly smooth vector fields are locally Lipshitz, and we have the following:

Theorem B (Picard–Lindelöf theorem). *Let $\vec{\mathfrak{v}}: U \to \mathbb{R}^n$ be a locally Lipshitz vector field on an open set $U \subset \mathbb{R}^n$ and let $\gamma_1, \gamma_2: [0,\epsilon) \to U$ be two integral curves of $\vec{\mathfrak{v}}$ with $\gamma_1(0) = \gamma_2(0)$. Then $\gamma_1 = \gamma_2$.*

In most textbooks, existence and uniqueness of integral curves to locally Lipshitz vector fields are proved simultaneously by recasting the problem as a solution to an integral (rather than differential) equation and then showing that the solution to the integral equation is a fixed point for a contracting map on a space of functions. This has the following downsides:

- It doesn't give the most general existence theorem.
- The solution is indirect and not very geometric.

In this note, we explain how to directly prove Theorems A and B. The proof of existence we give shows that integral curves can be constructed as limits of perhaps the most naive possible approximation (in the numerical analysis literature, the approximation scheme we use is called the *forward Euler method*).

Proof of Theorem A. Let $\vec{\mathfrak{v}}\colon U \to \mathbb{R}^n$ be a continuous vector field on an open set $U \subset \mathbb{R}^n$ and let $p_0 \in U$. Our goal is to find an integral curve of $\vec{\mathfrak{v}}$ starting at p_0. To simplify our notation, we start by making several transformations to this data:

- Translating everything, we can assume that $p_0 = 0$.
- Shrinking U, we can assume that U is an open disc around 0 and that $\vec{\mathfrak{v}}$ extends to the closure D that disc.
- Composing everything with a dilatation, we can assume that in fact D is the disc of radius 1.
- Finally, multiplying $\vec{\mathfrak{v}}$ by an appropriate positive constant we can assume that $\|\vec{\mathfrak{v}}(x)\| \leq 1$ for all $x \in D$. We remark that this global rescaling merely rescales the time parameter in our integral curves.

Having done all this, what we will prove is that there exists an integral curve $\gamma\colon [0,1] \to D$ for $\vec{\mathfrak{v}}$ with $\gamma(0) = 0$.

We will construct γ as a limit of curves $\gamma_n\colon [0,1] \to D$ as follows. Consider some $n \geq 1$. Define γ_n to be the following piecewise-linear curve. Divide the interval $[0,1]$ into n intervals $[k/n,(k+1)/n]$ for $0 \leq k < n$. The curve γ_n starts at 0, then on $[0/n, 1/n]$ follows the straight line in the direction of $\vec{\mathfrak{v}}(0)$, then on $[1/n, 2/n]$ follows the straight line in the direction of $\vec{\mathfrak{v}}(\gamma_n(1/n))$, etc. In formulas, γ_n is defined via $\gamma_n(0) = 0$ and

$$\gamma_n(t) = \gamma_n(k/n) + (t - k/n)\vec{\mathfrak{v}}(\gamma_n(k/n)) \quad \text{for } 0 \leq k < n \text{ and } t \in [k/n, (k+1)/n].$$

The fact that $\|\vec{\mathfrak{v}}(x)\| \leq 1$ for all $x \in D$ implies that $\gamma_n(t) \in D$ for all $t \in [0,1]$. It also implies that the γ_n are equicontinuous, so by the Arzela–Ascoli Theorem we can pass to a subsequence and ensure that the γ_n converge uniformly to a continuous function $\gamma\colon [0,1] \to D$.

We want to prove that γ is an integral curve to $\vec{\mathfrak{v}}$. One mildly confusing thing here is that the approximations γ_n are only piecewise differentiable, and in fact as n increases the points where γ_n is not differentiable become dense in $[0,1]$. However, the following claim shows that in some sense the lack of differentiability at those points becomes milder and milder.

Claim 1. *For all $\epsilon > 0$, there exists some $\delta > 0$ and some $N \geq 1$ such that for $s,t \in [0,1]$ with $0 < |s-t| < \delta$ and for $n \geq N$ we have*

$$\|\frac{\gamma_n(s) - \gamma_n(t)}{s-t} - \vec{\mathfrak{v}}(\gamma_n(t))\| < \epsilon.$$

Proof. Let us first contemplate the quantity we must bound. Fix some $n \geq 1$ and some $t, s \in [0,1]$. Set $u_0 = \min\{t, s\}$, and after doing this swap s and t if necessary to ensure that $t \leq s$. This swap does not change the indicated difference quotient, and what we want to do is to find some $\delta > 0$ and $N \geq 1$ such that if $n \geq N$ and $0 < |s - t| < \delta$ then

$$\|\frac{\gamma_n(s) - \gamma_n(t)}{s - t} - \vec{\mathfrak{v}}(\gamma_n(u_0))\| < \epsilon.$$

Since γ_n is a piecewise-smooth curve, we have

$$\frac{\gamma_n(s) - \gamma_n(t)}{s - t} = \frac{1}{s - t} \int_t^s \gamma_n'(u)\, \mathrm{d}u\,.$$

This is precisely the average value of the vector $\gamma_n'(u)$ as u ranges over $[t, s]$. We want this average value to be within distance ϵ of $\vec{\mathfrak{v}}(\gamma_n(u_0))$. By construction, for $u \in [t, s]$ we have $\gamma_n'(u) \in \vec{\mathfrak{v}}(\gamma_n([t - 1/n, s]))$. Since $\vec{\mathfrak{v}}(\gamma_n(u_0)) \in \vec{\mathfrak{v}}(\gamma_n([t - 1/n, s]))$ and the average of a vector-valued function must lie in the convex hull of its image, we see that what we really want is for the diameter of $\vec{\mathfrak{v}}(\gamma_n([t - 1/n, s]))$ to be at most ϵ.

Since $\|\vec{\mathfrak{v}}(x)\| \leq 1$ for all $x \in D$, the curve γ_n travels at most at unit speed. It follows that the diameter of $\gamma_n([t - 1/n, s]) \subset D$ is at most $s - t + 1/n$. We deduce that it is enough to prove that there exists some $\delta > 0$ and some $N \geq 1$ such that for $n \geq N$ and $x, y \in D$ with $\|x - y| < \delta + 1/n$, we have $\|\vec{\mathfrak{v}}(x) - \vec{\mathfrak{v}}(y)| < \epsilon$. This is immediate from the fact that $\vec{\mathfrak{v}}$ is uniformly continuous on the closed unit disc D. Indeed, uniform continuity implies that there exists some $\Delta > 0$ such that for $x, y \in D$ with $\|x - y\| < \Delta$, we have $\|\vec{\mathfrak{v}}(x) - \vec{\mathfrak{v}}(y)| < \epsilon$. We can then simply take $\delta = \Delta/2$ and N large enough such that $1/N < \Delta/2$. $\square$

We now prove that γ_n is an integral curve for $\vec{\mathfrak{v}}$. Consider some $t \in [0,1]$ and some $\epsilon > 0$. We must prove that there exists some $\delta > 0$ such that for $s \in [0,1]$ with $0 < |s - t| < \delta$, we have

$$\|\frac{\gamma(s) - \gamma(t)}{s - t} - \vec{\mathfrak{v}}(\gamma(t))\| < \epsilon.$$

For all $n \geq 1$, we have

$$\begin{aligned}\|\frac{\gamma(s) - \gamma(t)}{s - t} - \vec{\mathfrak{v}}(\gamma(t))\| \leq \|&\frac{\gamma_n(s) - \gamma_n(t)}{s - t} - \vec{\mathfrak{v}}(\gamma_n(t))\| \\ &+ \|\frac{\gamma_n(s) - \gamma(s)}{s - t}\| + \|\frac{\gamma_n(t) - \gamma(t)}{s - t}\| + \|\vec{\mathfrak{v}}(\gamma_n(t)) - \vec{\mathfrak{v}}(\gamma(t))\|\end{aligned} \tag{0.1}$$

Claim 1 implies that we can find some $\delta > 0$ and some $N \geq 1$ such that for $n \geq N$ and $s \in [0,1]$ with $0 < |s - t| < \delta$, we have

$$\|\frac{\gamma_n(s) - \gamma_n(t)}{s - t} - \vec{\mathfrak{v}}(\gamma_n(t))\| < \frac{\epsilon}{4}. \tag{0.2}$$

Now fix some $s \in [0,1]$ with $0 < |s - t| < \delta$. Since the γ_n converge uniformly to γ and $\vec{\mathfrak{v}}$ is continuous, we can find some $M \geq 1$ such that for $n \geq M$ we have

$$\|\frac{\gamma_n(s) - \gamma(s)}{s - t}\| < \frac{\epsilon}{4} \quad \text{and} \quad \|\frac{\gamma_n(t) - \gamma(t)}{s - t}\| < \frac{\epsilon}{4} \quad \text{and} \quad \|\vec{\mathfrak{v}}(\gamma_n(t)) - \vec{\mathfrak{v}}(\gamma(t))\| < \frac{\epsilon}{4}. \tag{0.3}$$

Choosing some $n \geq N, M$, we can plug (0.2) and (0.3) into (0.1) and deduce that

$$\|\frac{\gamma(s)-\gamma(t)}{s-t} - \vec{\mathfrak{v}}(\gamma(t))\| < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon,$$

as desired. □

Proof of Theorem B. Let $\vec{\mathfrak{v}} \colon U \to \mathbb{R}^n$ be a locally Lipshitz vector field on an open set $U \subset \mathbb{R}^n$ and let $\gamma_1, \gamma_2 \colon [0, \epsilon) \to U$ be two integral curves of $\vec{\mathfrak{v}}$ with $\gamma_1(0) = \gamma_2(0)$. Our goal is to prove that $\gamma_1 = \gamma_2$. Assume otherwise. Letting

$$S = \sup\{s \in [0, \epsilon) \mid \gamma_1 \text{ and } \gamma_2 \text{ are equal on } [0, s]\},$$

the γ_i are equal on $[0, S] \subsetneq [0, \epsilon)$. Replacing γ_i with the result of reparameterizing the curve $\gamma_i|_{[S,\epsilon)} \colon [S, \epsilon) \to U$ so that it is defined on $[0, \epsilon - S)$, we can assume that $\gamma_1(0) = \gamma_2(0)$ but that the γ_i do not agree on any positive-length interval $[0, \epsilon']$.

Set $p_0 = \gamma_1(0) = \gamma_2(0)$. Let $L > 0$ and let $U' \subset U$ be a neighborhood of p_0 such that $\vec{\mathfrak{v}}$ is L-Lipshitz on U'. Decreasing ϵ if necessary, we can assume that the images of the γ_i lie in U', that the γ_i extend to the closed interval $[0, \epsilon]$, and that $\epsilon L < 1$. Define

$$R = \max\{\|\gamma_1(t) - \gamma_2(t)\| \mid t \in [0, \epsilon]\},$$

and let $t_0 \in [0, \epsilon]$ be such that $\|\gamma_1(t_0) - \gamma_2(t_0)\| = R$.

It follows from the fundamental theorem of calculus that for all $t \in [0, \epsilon)$, we have

$$\gamma_i(t) = p_0 + \int_0^t \gamma_i'(s) \, \mathrm{ds} = p_0 + \int_0^t \vec{\mathfrak{v}}(\gamma_i(s)) \, \mathrm{ds}\,.$$

We thus have

$$\begin{aligned} R = \|\gamma_1(t_0) - \gamma_2(t_0)\| &= \| \int_0^{t_0} \left(\vec{\mathfrak{v}}\left(\gamma_1\left(s\right)\right) - \vec{\mathfrak{v}}\left(\gamma_2\left(s\right)\right)\right) \mathrm{ds} \,\| \\ &\leq \int_0^{t_0} \|\vec{\mathfrak{v}}\left(\gamma_1\left(s\right)\right) - \vec{\mathfrak{v}}\left(\gamma_2\left(s\right)\right)\| \, \mathrm{ds} \\ &\leq L \int_0^{t_0} \|\gamma_1(s) - \gamma_2(s)\| \, \mathrm{ds} \\ &\leq t_0 L R. \end{aligned}$$

Since $t_0 L \leq \epsilon L < 1$, this implies that $R = 0$, so the γ_i agree on $[0, \epsilon]$, contrary to our assumption. □