Knot complements and Spanier–Whitehead duality

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Abstract

We explain Spanier–Whitehead duality, which gives a simple geometric explanation for the fact that the homology of a knot complement is independent of the knot (a fact usually derived from Alexander duality).

Let K be a knot in S^3, i.e. the image of a PL embedding of S^1 into S^3. Alexander duality implies that $\mathrm{H}_1(S^3 \setminus K) \cong \mathbb{Z}$; in particular, it does not depend on the knot. Similarly, if X is a compact simplicial complex and $f, g\colon X \to S^n$ are two different embeddings, then Alexander duality implies that the homology groups of $S^n \setminus f(X)$ and $S^n \setminus g(X)$ are the same. Despite this isomorphism of homology groups, the spaces $S^n \setminus f(X)$ and $S^n \setminus g(X)$ need not be homotopy equivalent. However, Spanier–Whitehead [SW] proved that they become homotopy equivalent after suspending sufficiently many times:

Theorem 0.1 (Spanier–Whitehead)**.** *Let X be a compact simplicial complex. Let $f, g\colon X \to S^n$ be two simplicial embeddings. Then for some $M \gg 0$ the M-fold suspensions $\Sigma^M(S^n \setminus f(X))$ and $\Sigma^M(S^n \setminus g(x))$ are homotopy equivalent.*

Since suspending a space simply shifts its homology groups around, this provides a geometric explanation for the equality in homology groups coming from Alexander duality.

Remark 0.2. The natural general context for Theorem 0.1 is the theory of spectra; see [A, §III.5]. However, knowledge of stable homotopy theory is not needed to appreciate its statement or proof. □

Proof of Theorem 0.1. If n is sufficiently large relative to the dimension of X, then f and g are isotopic, so $S^n \setminus f(X)$ is homeomorphic to $S^n \setminus g(X)$. Fixing an embedding $X \hookrightarrow S^n$, it is thus enough to prove that $\Sigma(S^n \setminus X)$ is homotopy equivalent to $S^{n+1} \setminus X$, where X is included in S^{n+1} via the usual inclusion $S^n \hookrightarrow S^{n+1}$.

Let $I = [-1, 1]$ and $\mathring{I} = (-1, 1)$. By definition, $\Sigma(S^n \setminus X)$ is the quotient of $(S^n \setminus X) \times I$ that collapses $(S^n \setminus X) \times \{-1\}$ and $(S^n \setminus X) \times \{1\}$ to points; see Figure 1.a. Since X is a proper subset of S^n, this is the same as the quotient of $(S^n \times I) \setminus (X \times \mathring{I})$ that collapses $S^n \times \{-1\}$ and $S^n \times \{1\}$ to points. In other words, $\Sigma(S^n \setminus X)$ is obtained from $\Sigma S^n = S^{n+1}$ by removing $X \times \mathring{I}$. We will prove that this is homotopy equivalent to the result of removing $X \times \{0\}$ from ΣS^n.

To do this, we will construct homotopy inverse maps between these two spaces. It is easier to describe our maps before taking the quotients, so what we will actually give are maps

$$\phi\colon (S^n \times I) \setminus (X \times \mathring{I}) \to (S^n \times I) \setminus (X \times \{0\})$$

and

$$\psi\colon (S^n \times I) \setminus (X \times \{0\}) \to (S^n \times I) \setminus (X \times \mathring{I})$$

that restrict to the identity on $S^n \times \{-1\}$ and $S^n \times \{1\}$.

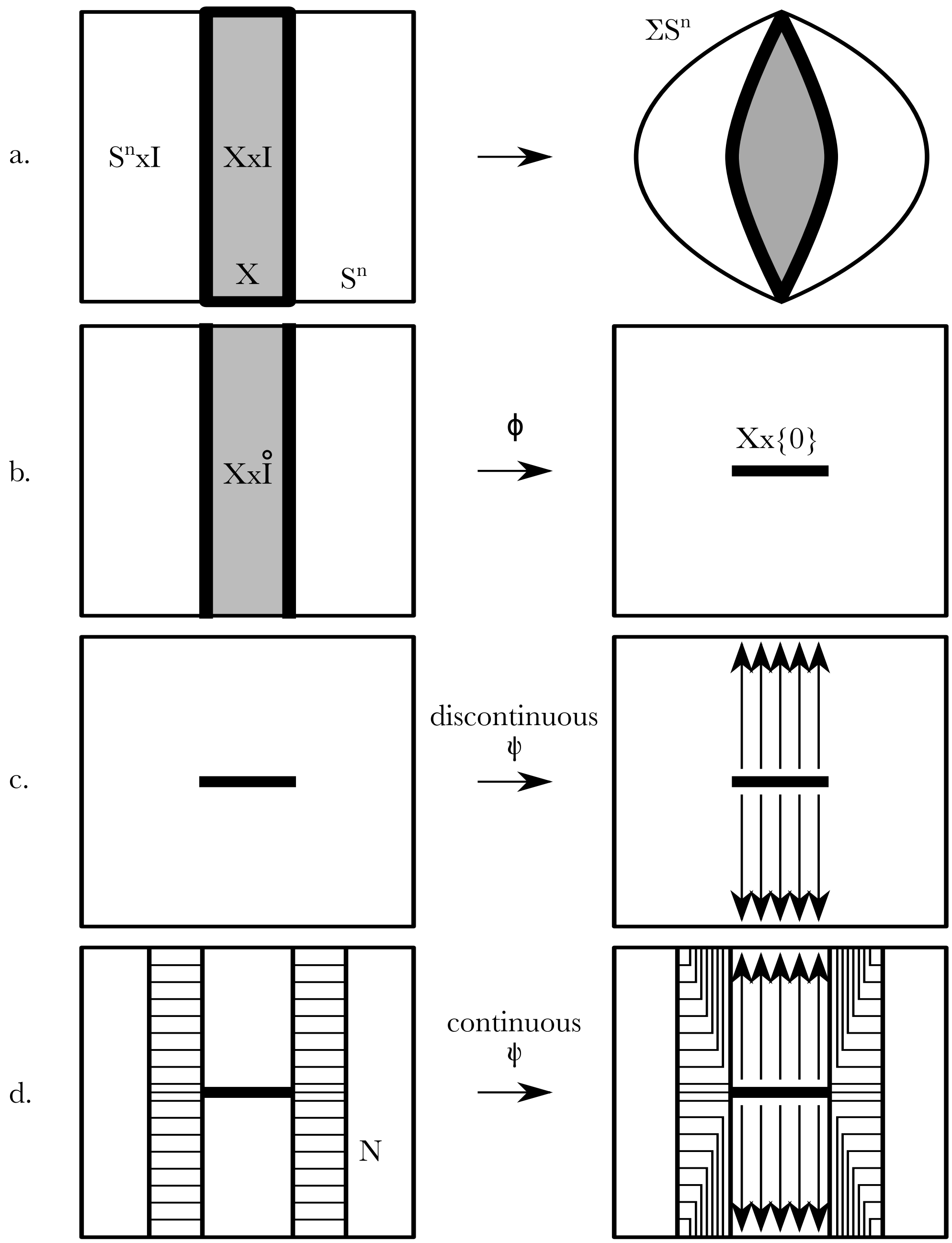

Figure 1: *a. Constructing* $\Sigma(S^n \setminus X)$ *by collapsing the top and bottom of* $(S^n \setminus X) \times I$. *b. The map* ϕ *is the inclusion of* $(S^n \times I) \setminus (X \times \mathring{I})$ *into* $(S^n \times I) \setminus (X \times \{0\})$. *c. The discontinuous wrong definition of* ψ. *d. Fixing this with the regular neighborhood* N *of* X. *The complement* $N \setminus X$ *is foliated by paths ending at points of* X, *and we show* $N \times \mathring{I}$. *The map fixes the paths in* $N \times \{0\}$ *and takes the paths in* $N \times (0,1)$ *(resp.* $N \times (-1,0)$*) to paths as shown that end at at the top (resp. bottom) of the figure.*

The two maps are shown in Figures 1.b and 1.d. The map ϕ is simply the inclusion. As for ψ, what we would like to do is to define it via the formula

$$\psi(p,t) = \begin{cases} (p,1) & \text{if } p \in X \text{ and } t > 0, \\ (p,-1) & \text{if } p \in X \text{ and } t < 0, \\ (p,t) & \text{if } p \notin X \end{cases}$$

for $(p,t) \in (S^n \times I) \setminus (X \times \{0\})$; see Figure 1.c. Unfortunately, this is not continuous. To fix this, as in Figure 1.d we use a regular neighborhood N of X to continuously interpolate between the three pieces of the above formula. □

References

[A] J. F. Adams, *Stable homotopy and generalised homology*, University of Chicago Press, Chicago, IL, 1974.

[SW] E. H. Spanier and J. H. C. Whitehead, Duality in homotopy theory, Mathematika 2 (1955), 56–80.

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