The word problem for surface groups and hyperbolic geometry

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Abstract

We explain Dehn's solution to the word problem for fundamental groups of surfaces using hyperbolic geometry.

Fix some $g \geq 2$ and let Σ_g be a closed oriented genus g surface. Recall that

$$\pi_1(\Sigma_g) \cong \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \mid [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = 1 \rangle.$$

Dehn [D] gave an elegant algorithm to decide whether or not a word in the generators $S = \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ represents the identity in $\pi_1(\Sigma_g)$. To describe this algorithm, we must introduce some notation. Let $F(S)$ denote the free group on S. For $w \in F(S)$, write $|w|$ for the word length of w. Let $r = [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \in F(S)$. A *shortening relation* in $\pi_1(\Sigma_g)$ is a relation $r_1 = r_2$, where $r_1, r_2 \in F(S)$ satisfy the following:

- $|r_1| > |r_2|$.
- Either $r_1 r_2^{-1}$ or $r_2 r_1^{-1}$ is a cyclic permutation of r, which implies in particular that it is conjugate to r and thus is a relation in $\pi_1(\Sigma_g)$.
- $|r_1| + |r_2| = |r|$. In other words, no cancellation occurs between the r_1 and r_2 pieces of the aforementioned cyclic permutation of r.

The key to Dehn's algorithm is the following theorem.

Theorem 0.1 (Dehn)**.** *Let $w \in F(S)$ be a nontrivial reduced word that represents the identity in $\pi_1(\Sigma_g)$. Then there exists a subword r_1 of w and a shortening relation $r_1 = r_2$ in $\pi_1(\Sigma_g)$.*

Theorem 0.1 leads to the following algorithm. Consider a reduced word $w \in F(S)$.

Step 1. Check if w contains a subword r_1 as in Theorem 0.1. If it does not, then w does not represent the identity in $\pi_1(\Sigma_g)$.

Step 2. Assume now that w does contain such a subword r_1, and let $r_1 = r_2$ be the corresponding shortening relation. Replace the subword r_1 of w with r_2 and freely reduce.

Step 3. If $w = 1$, then w represented the identity in $\pi_1(\Sigma_g)$. If $w \neq 1$, then go back to Step 1.

Since Step 2 shortens w, this algorithm always terminates.

The goal of this note is to give a proof of Theorem 0.1 using hyperbolic geometry that is similar to Dehn's original proof. The idea here has been very influential in geometric group theory and formed part of the inspiration for Gromov's theory of hyperbolic groups; see [C] and [GH].

A regular $4g$-gon. Identify Σ_g in the standard way with a $4g$-gon with sides identified in pairs according to the surface relation $[a_1, b_1] \cdots [a_g, b_g]$ (see Figure 1). Endow Σ_g with a hyperbolic metric by realizing this $4g$-gon as a regular hyperbolic $4g$-gon whose interior angles are all $\frac{1}{2g}\pi$ (this angle is need to ensure that there is precisely 2π worth of angle around the vertex, so no singularity occurs there). The following argument shows that such a hyperbolic $4g$-gon exists. Use the unit disc model for $\mathbb{H}^2$. For $0 < R \leq 1$, let D_R be the hyperbolic $4g$-gon whose vertices are the points $(R\cos(k\pi/2g), R\sin(k\pi/2g)) \in \mathbb{H}^2$ for

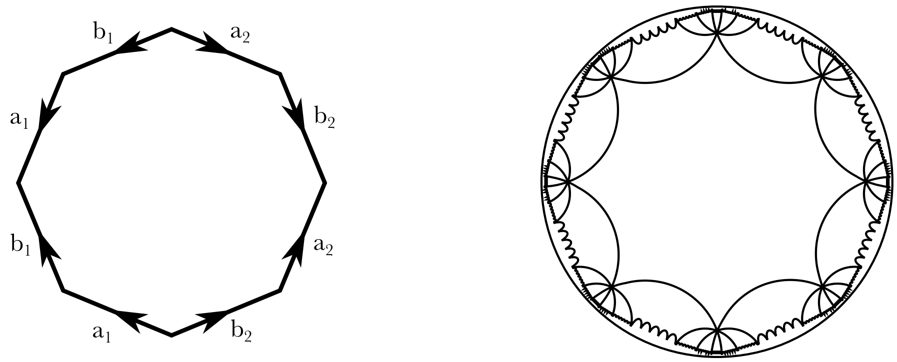

Figure 1: *On the left is a genus 2 surface obtained by identifying sides of an octagon in pairs. On the right is a schematic drawing of part of a tiling of the hyperbolic plane by regular octagons.*

$0 \leq k < 4g$. For $R = 1$, the vertices of D_R are on the boundary at infinity, so the interior angles of D_R are 0. For R very close to 0, the hyperbolic metric on D_R is very close to the Euclidean metric on D_R, so the interior angles are very close to those of a regular Euclidean $4g$-gon, namely $\frac{4g-2}{4g}\pi$. Since $\frac{4g-2}{4g}\pi > \frac{1}{2g}\pi$, the intermediate value theorem says that there is some $0 < R < 1$ such that the interior angles of D_R are precisely $\frac{1}{2g}\pi$.

The corresponding tiling of hyperbolic space. The identification of Σ_g with a $4g$-gon whose sides are identified in pairs leads to a CW-complex structure on Σ_g with a single zero cell $*$, a single two-cell (the interior of the $4g$-gon), and $2g$ one-cells (the loops corresponding the boundary edges of the $4g$-gon). Using the hyperbolic metric on Σ_g from the previous paragraph, we can identify the universal cover of Σ_g with $\mathbb{H}^2$. Endow $\mathbb{H}^2$ with the CW-complex structure obtained by pulling back the one on Σ_g. Each two-cell of this CW-complex structure is a regular $4g$-gon, so we obtain a tiling of $\mathbb{H}^2$ by regular $4g$-gons with $4g$ tiles arranged around each vertex (see Figure 1).

Reformulation of theorem. Fixing a base vertex $\widetilde{*}$ in $\mathbb{H}^2$, the one-skeleton of our CW-complex structure on $\mathbb{H}^2$ (i.e. the edges in our tiling) can be identified with the Cayley graph of $\pi_1(\Sigma_g)$ with respect to the generating set $S = \{a_1, b_1, \ldots, a_g, b_g\}$. A reduced word in $F(S)$ corresponds to a edge-path in this Cayley graph starting at $\widetilde{*}$ that never backtracks (i.e. that never goes along an edge and then backwards along the same edge). The reduced word represents the identity in $\pi_1(\Sigma_g)$ if and only if the corresponding path is a loop. The following assertion is therefore equivalent to Theorem 0.1:

(†) Every non-backtracking edge loop based at $\widetilde{*}$ in the 1-skeleton of $\mathbb{H}^2$ traverses more than half of the boundary of one of the $4g$-gons in the tiling.

The structure of the tiling. Inductively define polygonal subspaces

$$X_1 \subset X_2 \subset X_3 \subset \cdots \subset \mathbb{H}^2$$

as follows. Let X_1 be the tile that contains $\widetilde{*}$ and whose boundary corresponds to the surface relation $[a_1, b_1] \cdots [a_g, b_g]$. Next, if X_{n-1} has been constructed, let X_n be the union of X_{n-1} and all tiles that intersect X_{n-1}. These new tiles share either an edge or a vertex with a tile in X_{n-1}. As is shown in Figure 2, the polygon X_n can be built from X_{n-1} in two stages:

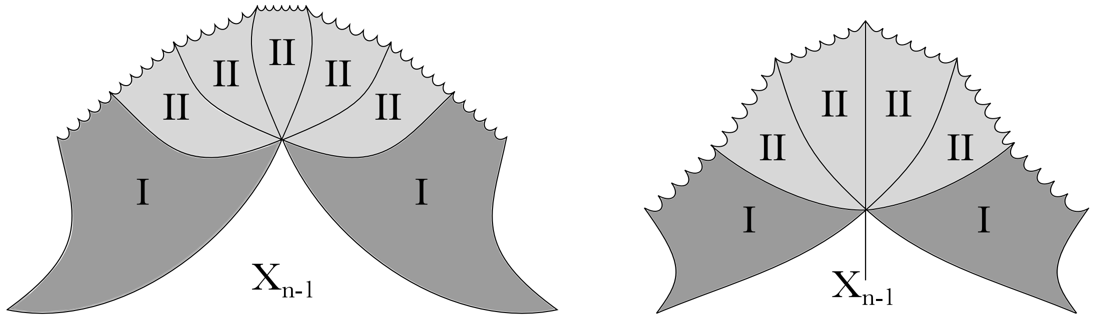

Figure 2: *In genus 2, the two possibilities for how we add type I and type II outermost tiles around a vertex of ∂X_{n-1} to form X_n*

1. First, add a tile adjacent to each edge of ∂X_{n-1}. Call these the *type I outermost tiles* of X_n.
2. Next, consider a vertex of ∂X_{n-1}. This vertex lies in either one or two tiles of X_{n-1}. Add enough tiles to fill in the space between the two type I outermost tiles of X_n that we have just added. Since there are $4g$ tiles around each vertex, we add $4g-3$ tiles if our vertex lies in one tile of X_{n-1} and $4g-4$ tiles if our vertex lies in two tiles of X_{n-1}. Call these new tiles the *type II outermost tiles* of X_n.

From the above description, it is clear that ∂X_n is a simple polygonal loop. It has alternating sections where it first traverses part of the boundary of a type I outermost tile and then tranverses parts of the boundaries of several type II outermost tiles (either $4g-3$ or $4g-4$ of them). Call the portions of ∂X_n that are contained in a single outermost tile of X_n the *segments* of ∂X_n. Here is the key observation:

($*$) Each segment of ∂X_n traverses more than half of the boundary of one of the tiles.

To see this, observe that if the segment in question is the intersection of ∂X_n with an outermost tile of type I, then the segment contains all but 3 edges of the outermost tile, so it has length $4g-3>2g$. If instead it is the intersection of ∂X_n with an outermost tile of type II, then the segment contains all but 2 edges of the outermost tile, so it has length $4g-2>2g$.

Completing the proof. We now verify (†) as follows. Let γ be a non-backtracking edge loop based at $\widetilde{*}$ in the 1-skeleton of $\mathbb{H}^2$. Let $n \geq 1$ be the smallest integer such that $\gamma \subset X_n$. It follows that γ traverses part of ∂X_n. The portion of ∂X_n that is traversed by γ must be a union of segments, so by ($*$) the path γ must traverse more than half of the boundary of one of the tiles, as desired.

References

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