The symplectic representation of the mapping class group is surjective

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Abstract

We give an efficient proof that the symplectic representation of the mapping class group is surjective.

Let Σ_g be a closed oriented genus g surface and let Mod_g be its mapping class group, that is, the group of isotopy classes of orientation-preserving diffeomorphisms of Σ_g. The action of Mod_g on $\text{H}_1(\Sigma_g;\mathbb{Z}) \cong \mathbb{Z}^{2g}$ preserves the algebraic intersection pairing $i(\cdot,\cdot)$, which by Poincaré duality is a symplectic form. We thus get a representation $\text{Mod}_g \to \text{Sp}_{2g}(\mathbb{Z})$. In this note, we prove the following theorem.

Theorem 0.1. *The representation* $\text{Mod}_g \to \text{Sp}_{2g}(\mathbb{Z})$ *is surjective.*

Theorem 0.1 was originally proved by Burkhardt in 1890 [B, pp. 209–212], who wrote down mapping classes that map to generators of $\text{Sp}_{2g}(\mathbb{Z})$ that were previously found by Clebsch–Gordan [CG]. The first modern proof is due to Meeks–Patrusky [MePa, Theorem 2], and our proof is a variant of theirs.

We first introduce some notation. A *symplectic basis* for $\text{H}_1(\Sigma_g;\mathbb{Z})$ is an ordered sequence $(a_1, b_1, \ldots, a_g, b_g)$ of elements of $\text{H}_1(\Sigma_g;\mathbb{Z})$ that form a basis for this free abelian group and satisfy

$$i(a_i, b_i) = \delta_{ij} \quad \text{and} \quad i(a_i, a_j) = i(b_i, b_j) = 0$$

for $1 \leq i, j \leq g$, where δ_{ij} is the Dirac delta function. For an oriented closed curve γ on Σ_g, let $[\gamma] \in \text{H}_1(\Sigma_g;\mathbb{Z})$ be the associated homology class. A *geometric realization* of a symplectic basis $(a_1, b_1, \ldots, a_g, b_g)$ is an ordered sequence $(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$ of oriented simple closed curves satisfying the following two conditions:

- $[\alpha_i] = a_i$ and $[\beta_i] = b_i$ for $1 \leq i \leq g$, and
- $\#|\alpha_i \cap \beta_j| = \delta_{ij}$ and $\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = \emptyset$ for $1 \leq i, j \leq g$.

See Figure 1. The main technical result that goes into proving Theorem 0.1 is as follows.

Lemma 0.2. *Every symplectic basis for* $\text{H}_1(\Sigma_g;\mathbb{Z})$ *has a geometric realization.*

Before we prove Lemma 0.2, we will use it to derive Theorem 0.1.

Proof of Theorem 0.1. Consider some $M \in \text{Sp}_{2g}(\mathbb{Z})$. We will produce a mapping class $f \in \text{Mod}_g$ that induces M as follows. Let $(a_1, b_1, \ldots, a_g, b_g)$ be a symplectic basis for $\text{H}_1(\Sigma_g;\mathbb{Z})$. The sequence $(M(a_1), M(b_1), \ldots, M(a_g), M(b_g))$ is also a symplectic basis. Using Lemma 0.2, we can find geometric realizations $(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$ and $(\alpha_1', \beta_1', \ldots, \alpha_g', \beta_g')$ of $(a_1, b_1, \ldots, a_g, b_g)$ and $(M(a_1), M(b_1), \ldots, M(a_g), M(b_g))$. Since the intersection pattern of the α_i and β_i is the same as that of the α_i' and β_i', the standard "change of coordinates" principle from [FMa, Chapter 1.3] implies that we can find some $f \in \text{Mod}_g$ such that $f(\alpha_i) = \alpha_i'$ and $f(\beta_i) = \beta_i'$ for $1 \leq i \leq g$. By construction, the action of the mapping class f on $\text{H}_1(\Sigma_g;\mathbb{Z})$ has the same effect on the symplectic basis $(a_1, b_1, \ldots, a_g, b_g)$ as M, so f induces M, as desired. $\square$

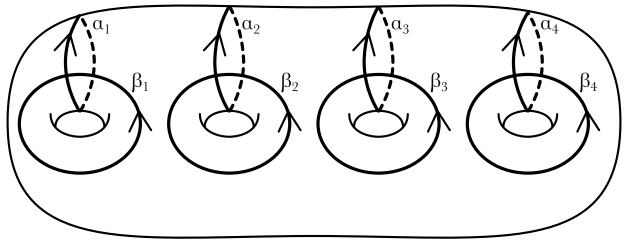

Figure 1: *A geometric realization of a symplectic basis.*

Proof of Lemma 0.2. The proof will be by induction on g. The base case $g = 0$ is trivial, so assume that $g \geq 1$ and that the result is true for all smaller genera. Let $(a_1, b_1, \ldots, a_g, b_g)$ be a symplectic basis for $\mathrm{H}_1(\Sigma_g; \mathbb{Z})$. The heart of our proof is the construction of oriented simple closed curves α_1 and β_1 that intersect once and satisfy $[\alpha_1] = a_1$ and $[\beta_1] = b_1$. Assume that we have constructed α_1 and β_1. Let S be the complement of a regular neighborhood of $\alpha_1 \cup \beta_1$. Thus S is a genus $g-1$ subsurface of Σ_g with one boundary component and the map $\mathrm{H}_1(S; \mathbb{Z}) \to \mathrm{H}_1(\Sigma_g; \mathbb{Z})$ is an injection; identify $\mathrm{H}_1(S; \mathbb{Z})$ with its image in $\mathrm{H}_1(\Sigma_g; \mathbb{Z})$. The subspace $\mathrm{H}_1(S; \mathbb{Z})$ of $\mathrm{H}_1(\Sigma_g; \mathbb{Z})$ is the orthogonal complement of $\langle a_1, b_1 \rangle$ with respect to the algebraic intersection pairing. This orthogonal complement is precisely $\langle a_2, b_2, \ldots, a_g, b_g \rangle$. Let $S' \cong \Sigma_{g-1}$ be the result of gluing a disc D to ∂S. The map $\mathrm{H}_1(S; \mathbb{Z}) \to \mathrm{H}_1(S'; \mathbb{Z})$ is an isomorphism. Let $(a_2', b_2, \ldots, a_g', b_g')$ be the image in $\mathrm{H}_1(S'; \mathbb{Z})$ of the symplectic basis $(a_2, b_2, \ldots, a_g, b_g)$ of $\mathrm{H}_1(S; \mathbb{Z})$. Using our inductive hypothesis, we can find a geometric realization $(\alpha_2', \beta_2', \ldots, \alpha_g', \beta_g')$ for the symplectic basis $(a_2', b_2', \ldots, a_g', b_g')$ of $\mathrm{H}_1(S'; \mathbb{Z})$. Isotoping the α_i' and β_i', we can assume that they are all disjoint from D, and thus are the images of oriented simple closed curves α_i and β_i in S. The sequence $(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g)$ of oriented simple closed curves on Σ_g is the desired geometric realization of the symplectic basis $(a_1, b_1, \ldots, a_g, b_g)$.

It remains to construct α_1 and β_1. Since $i(a_1, b_1) = 1$, the element $a_1 \in \mathrm{H}_1(\Sigma_g; \mathbb{Z})$ is primitive, that is, not equal to a nontrivial multiple of another element. Indeed, if $a_1 = na_1'$ for some $n \in \mathbb{Z}$ and $a_1' \in \mathrm{H}_1(\Sigma_g; \mathbb{Z})$, then $1 = i(a_1, b_1) = i(na_1', b_1) = ni(a_1', b_1)$, so $n = \pm 1$. A classical theorem (see [Pu] for a short proof) then says that there exists an oriented simple closed curve α_1 such that $[\alpha_1] = a_1$. We must construct β_1.

The first step is to construct a closed curve β_1' (not necessarily simple) that intersects α_1 once and satisfies $[\beta_1'] = b_1$. The whole construction is illustrated by Figure 2. Let $X \subset \Sigma_g$ be a one-holed torus containing α_1 and let $Y = \Sigma_g \setminus \mathrm{Int}(X)$, so Y is a genus $g-1$ subsurface with one boundary component. We have a decomposition

$$\mathrm{H}_1(\Sigma_g; \mathbb{Z}) \cong \mathrm{H}_1(X; \mathbb{Z}) \oplus \mathrm{H}_1(Y; \mathbb{Z})$$

that is orthogonal with respect to the algebraic intersection pairing. Let $b_X \in \mathrm{H}_1(X; \mathbb{Z})$ and $b_Y \in \mathrm{H}_1(Y; \mathbb{Z})$ be the projections of $b_1 \in \mathrm{H}_1(\Sigma_g; \mathbb{Z})$ to these two factors, so $b_1 = b_X + b_Y$. Let β_X' be an arbitrary oriented simple closed curve in X that intersects α_1 once with a positive sign. We thus have a basis $\{a_1, [\beta_X']\}$ for $\mathrm{H}_1(X; \mathbb{Z})$, so we can write $b_X = ca_1 + d[\beta_X']$. In fact,

$$1 = i(a_1, b_X) = i(a_1, ca_1 + d[\beta_X']) = d.$$

Letting β_X be the result of Dehn twisting β_X' around α_1 a total of c times, we thus have $[\beta_X] = b_X$. The desired closed curve β_1' can then be obtained by band-summing β_X with an oriented closed curve in Y (not necessarily simple) whose homology class is b_Y.

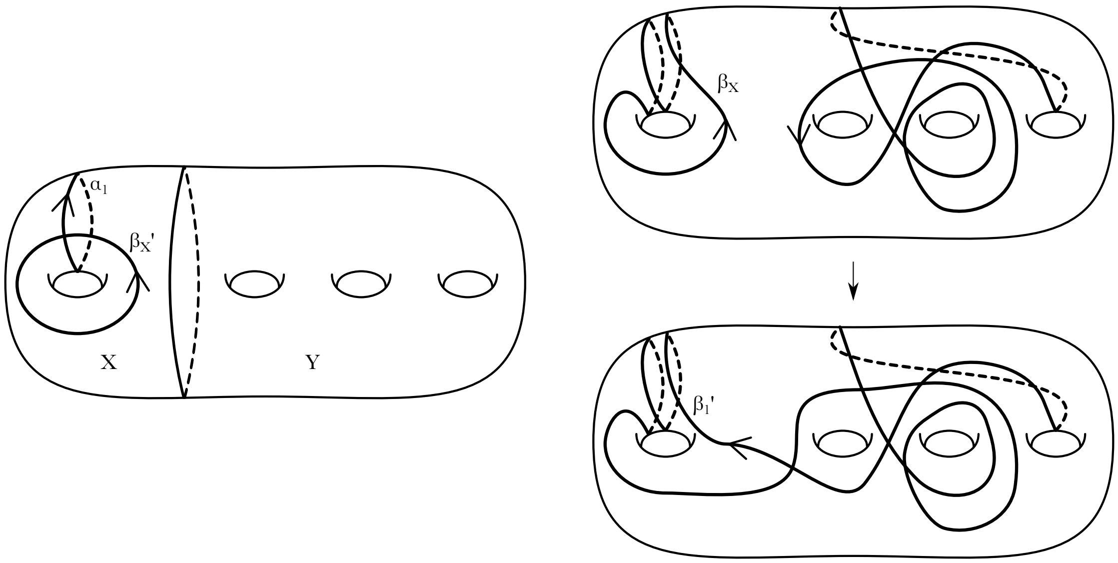

Figure 2: *On the left is X and Y and α_1 and β'_X. On the top right the result β_X of twisting β'_X around α_1 enough times to ensure that $[\beta_X] = b_X$. A not necessarily simple curve in Y realizing b_Y is also depicted. On the bottom right is the result of band-summing the curve in Y into β_X; as is shown here, making sure the orientations match up might require adding another self-intersection.*

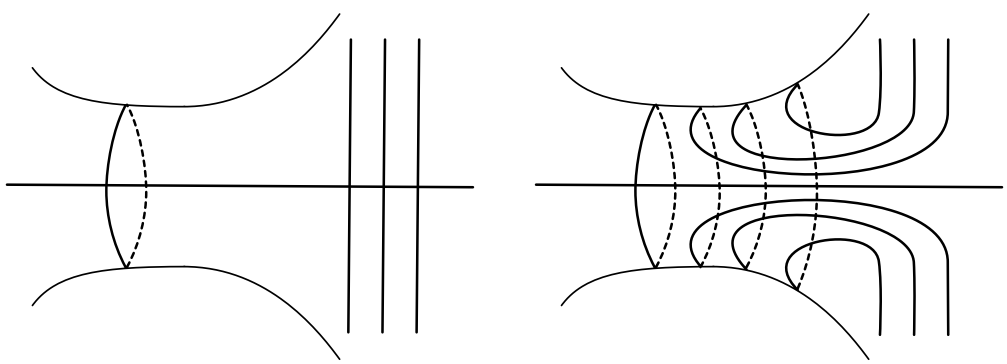

Figure 3: *On the left is the simple closed curve α_1 along with a portion of β'_1 that contains three self-intersections. On the right is the result of "combing" these three self-intersections over α_1.*

The next step is to "comb" all the self-intersections of β'_1 over α_1 as is shown in Figure 3. The result is an oriented simple closed curve β''_1. Every self-intersection we comb over α_1 adds a copy of $\pm a_1$ to $[\beta'_1]$, so we have $[\beta''_1] = b_1 + ea_1$ for some $e \in \mathbb{Z}$. The desired oriented simple closed curve β_1 can now be obtained by Dehn twisting β_1 around α_1 a total of $-e$ times. □

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