Tiling rectangles by squares

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In this note, I will describe two proofs of the following theorem of Dehn [2]. All rectangles and squares in this note lie in the plane and have sides parallel to the x and y axes.

Theorem 0.1. *Let R be a rectangle with side lengths a and b. Then R can be tiled by squares if and only if $a/b \in \mathbb{Q}$.*

If $a/b \in \mathbb{Q}$, then it is trivial to tile R by squares; indeed, scaling R we can ensure that its side length are integers, so it can be tiles by 1×1 squares. What is more interesting is the converse.

1 Proof 1: linear algebra

I learned this proof from [3], which indicates that the author does not know its source. The idea is to give a strange definition of "area" that is always nonnegative for rectangles that are tiled by squares. This "area" depends on a parameter, and we will show that if a rectangle has noncommensurable side lengths, then this parameter can be chosen such that the rectangle's "area" is -1.

Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a $\mathbb{Q}$-linear map. For a rectangle X with sides c and d, define the *ϕ-area* of X to be

$$A_\phi(X) = \phi(c) \cdot \phi(d).$$

More generally, if Y is a region in the plane that can be tiled by rectangles $X_1, \ldots, X_n$ (where the X_i only overlap along their edges, but where an edge of one of the X_i can contain segments of the edges of several of the X_j), then define

$$A_\phi(Y) = A_\phi(X_1) + \cdots + A_\phi(X_n).$$

We now prove that this does not depend on the choice of tiling. Any two tilings have a common subdivision into rectangles whose sides exactly match up (i.e. an edge of one of the X_i either lies in the boundary of Y or is exactly equal to an edge of another X_j). It is thus enough to show that if we take one of our rectangles X_i and decompose it into rectangles whose sides exactly match up, then its ϕ-area is unchanged, which is immediate from the $\mathbb{Q}$-bilinearity of the map $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $(c, d) \mapsto \phi(c) \cdot \phi(d)$.

The key observation now is that if X is a square with side-length c, then

$$A_\phi(X) = \phi(c) \cdot \phi(c) \geq 0.$$

It follows that if Y can be decomposed into squares, then $A_\phi(Y) \geq 0$.

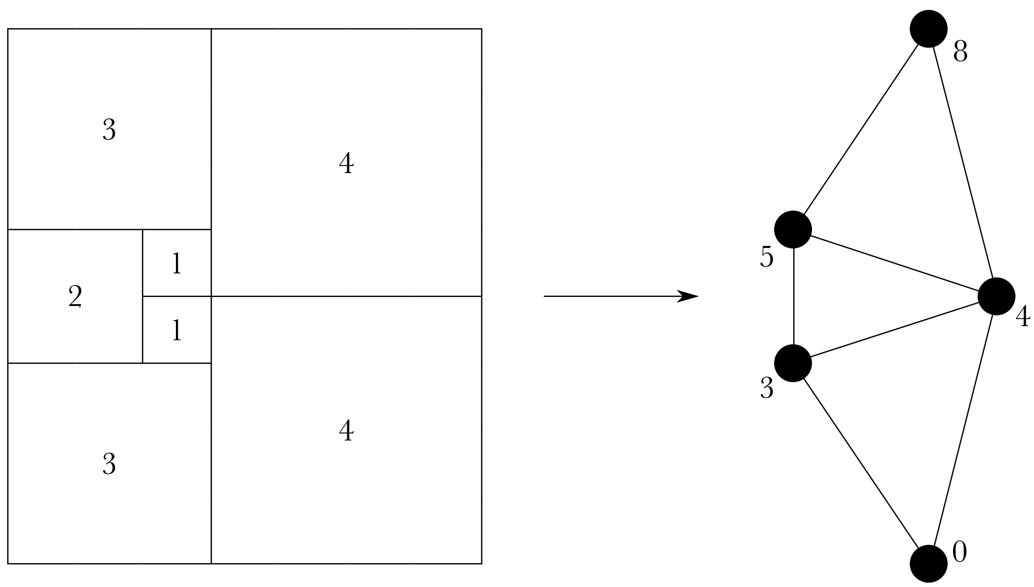
We now turn to our theorem. Assume that R is a rectangle with side lengths a and b, and that $a/b \notin \mathbb{Q}$. It follows that we can find a $\mathbb{Q}$-linear map $\phi\colon \mathbb{R} \to \mathbb{R}$ such that $\phi(a) = 1$ and $\phi(b) = -1$, so $A_\phi(R) = -1$. By what we said above, if R could be decomposed into squares, then we would have $A_\phi(R) \geq 0$, so this must not be possible.

2 Proof 2: harmonic functions on graphs

We now give an alternate proof that first appeared in [1]. This proof is a little more complicated than the previous one, but it gives more. First, it shows that the squares that appear a tiling of a rectangle whose side lengths are rational all have rational side lengths. Second, it gives a nice way of constructing and studying examples.

Assume that R is a rectangle that can be tiled by squares $S_1, \ldots, S_n$. Scaling everything, we can assume that the length of the vertical side of R is an integer. Our goal is to prove that the length of the horizontal side is a rational number.

Define a graph G in the following way. Let $H \subset R$ be the subspace of R consisting of points that lie on a horizontal edge of some S_i (thus in particular both the top and bottom of R lie in H). The vertices of G are then the connected components σ of H, and two vertices σ and σ' of G are connected by an edge precisely when there exists some S_i whose top is contained in one of σ and σ' and whose bottom is contained in the other. Here is an example:



We now define a function $h\colon V(G) \to \mathbb{R}$ as follows (the vertices of the above example are labeled by the values of this function). Place R in the plane such that its bottom edge lies on the x-axis. For a vertex σ of G, define $h(\sigma)$ to be the height of σ, i.e. the value of the y-coordinate of the points of σ.

There are two special vertices of G, the vertex β corresponding to the bottom edge of R and the vertex τ corresponding to the top edge of R. By construction, we have

$$h(\beta) = 0 \quad \text{and} \quad h(\tau) \in \mathbb{Z}.$$

All the other vertices of G will be called the *interior vertices*. Let $\mathcal{I}$ be the set of interior vertices of G.

We now come to the key observation. Consider some $\sigma \in \mathcal{I}$, and let σ' be a vertex joined to σ by an edge. The value $|h(\sigma) - h(\sigma')|$ is precisely the side-length of the square connecting σ to σ'. The side-lengths of the squares lying above σ must add up to the length ℓ of the segment σ, and similarly for the side-lengths of the squares lying below σ. It follows that

$$
\begin{aligned}
\sum_{\sigma' \text{ adjacent to } \sigma} (h(\sigma) - h(\sigma')) &= \sum_{\sigma' \text{ below } \sigma} (h(\sigma) - h(\sigma')) - \sum_{\sigma' \text{ above } \sigma} (h(\sigma') - h(\sigma)) \\
&= \ell - \ell \\
&= 0.
\end{aligned}
$$

Rearranging this sum, we see that

$$
h(\sigma) = \frac{1}{\deg(\sigma)} \sum_{\sigma' \text{ adjacent to } \sigma} h(\sigma').
$$

In other words, h is a *harmonic function* on G at σ. See [4] for some basic results about harmonic functions on finite graphs.

We now appeal to two facts:

- If G is a finite graph, $W \subset V(G)$ is a set of vertices of G, and $f\colon W \to \mathbb{R}$ is any function, then there exists a *unique* extension $f\colon V(G) \to \mathbb{R}$ that is harmonic on $V(G) \setminus W$ (the "Dirichlet problem" for harmonic functions on graphs). In fact, all we need is uniqueness, which follows from an easy maximum-principle type argument.
- If G is a finite graph, $W \subset V(G)$ is a set of vertices of G, and $f\colon V(G) \to \mathbb{R}$ is a function that is harmonic on $V(G) \setminus W$ and satisfies $F(W) \subset \mathbb{Q}$, then $f(V(G)) \subset \mathbb{Q}$. The reason for this is that the equations defining a function being harmonic on $V(G) \setminus W$ and having the desired values on W are precisely a set of linear equations with rational coefficients. By the previous part, these have a unique real solution, and since their coefficients are rational this must in fact be rational.

The horizontal edge length of R equals the sum of $h(\sigma)$ for the σ that are joined to the bottom vertex β, and is thus a rational number.

References

[1] R. L. Brooks et al., The dissection of rectangles into squares, Duke Math. J. 7 (1940), 312–340.

[2] M. Dehn, Über Zerlegung von Rechtecken in Rechtecke, Math. Ann. 57 (1903), no. 3, 314–332.

[3] J. Matoušek, *Thirty-three miniatures*, Student Mathematical Library, 53, American Mathematical Society, Providence, RI, 2010.

[4] A. Putman, Harmonic functions on finite graphs, informal note.

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