

ABELIAN COVERS OF SURFACES AND THE HOMOLOGY OF THE TORELLI GROUP

DANIEL MINAHAN AND ANDREW PUTMAN

ABSTRACT. We study the first homology group of the mapping class group and Torelli group with coefficients in the first rational homology group of the universal abelian cover of the surface. We prove two contrasting results: for surfaces with one boundary component these twisted homology groups are finite-dimensional, but for surfaces with one puncture they are infinite-dimensional. These results play an important role in a recent paper of the authors calculating the second rational homology group of the Torelli group.

1. INTRODUCTION

Let $\Sigma_{g,p}^b$ be an oriented genus g surface with p punctures and b boundary components.¹ The mapping class group $\text{Mod}_{g,p}^b$ is the group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,p}^b$ that fix each puncture and boundary component pointwise. Assume² that $p + b \leq 1$. By Poincaré duality, the intersection form on $H_1(\Sigma_{g,p}^b) \cong \mathbb{Z}^{2g}$ is a symplectic form. The action of $\text{Mod}_{g,p}^b$ on $H_1(\Sigma_{g,p}^b)$ preserves this form, yielding a surjection $\text{Mod}_{g,p}^b \rightarrow \text{Sp}_{2g}(\mathbb{Z})$ whose kernel $\mathcal{I}_{g,p}^b$ is the Torelli group. This fits into an exact sequence

$$1 \longrightarrow \mathcal{I}_{g,p}^b \longrightarrow \text{Mod}_{g,p}^b \longrightarrow \text{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1.$$

In this paper, we study the first homology of $\text{Mod}_{g,p}^b$ and $\mathcal{I}_{g,p}^b$ with coefficients in the homology of the universal abelian cover of $\Sigma_{g,p}^b$.

1.1. Homology of mapping class group. The mapping class group $\text{Mod}_{g,p}^b$ is of type F_∞ . In other words, it has a classifying space whose k -skeleton is compact for all $k \geq 0$ (see, e.g., [6]). This implies that $\text{Mod}_{g,p}^b$ is finitely presented and all of its homology groups are finitely generated. In fact, for any finitely generated $\text{Mod}_{g,p}^b$ -module V the homology group $H_k(\text{Mod}_{g,p}^b; V)$ is finitely generated. These have been calculated in many cases, at least in the “stable range” when $g \gg k$. See, e.g., [12, 13, 14, 26].

1.2. Low degree homology of Torelli. We return to the case where $p + b \leq 1$. Since $\mathcal{I}_{g,p}^b$ is an infinite-index subgroup of $\text{Mod}_{g,p}^b$, it does not inherit any finiteness properties. In fact, it is known that many of its homology groups are infinitely generated; see [1, 2, 5].

However, it does have some unexpected finiteness properties. Johnson [9] proved that $\mathcal{I}_{g,p}^b$ is finitely generated for $g \geq 3$. He also calculated its first homology group [10, 11]. Over \mathbb{Q} , this has the following simple description: letting $H = H_1(\Sigma_{g,p}^b; \mathbb{Q})$, we have

$$H_1(\mathcal{I}_g^1; \mathbb{Q}) \cong H_1(\mathcal{I}_{g,1}; \mathbb{Q}) \cong \wedge^3 H \quad \text{and} \quad H_1(\mathcal{I}_g; \mathbb{Q}) \cong (\wedge^3 H)/H.$$

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¹We omit p or b if they vanish.

²See [17] for a discussion of the Torelli group on surfaces with multiple boundary components. Our main results are about Mod_g^1 and $\text{Mod}_{g,1}$, so we do not need to go into this here.

The conjugation action of $\text{Mod}_{g,p}^b$ on $\mathcal{I}_{g,p}^b$ induces an action of $\text{Sp}_{2g}(\mathbb{Z})$ on each $H_d(\mathcal{I}_{g,p}^b)$. The above isomorphisms are $\text{Sp}_{2g}(\mathbb{Z})$ -equivariant. They imply that $H_1(\Sigma_{g,p}^b; \mathbb{Q})$ is not just finite-dimensional, but is also an algebraic³ representation of $\text{Sp}_{2g}(\mathbb{Z})$.

Verifying a long-standing folk conjecture, the authors [16] recently calculated $H_2(\mathcal{I}_{g,p}^b; \mathbb{Q})$ for $g \geq 6$. Like the first homology, the second homology is also a finite-dimensional algebraic representation of $\text{Sp}_{2g}(\mathbb{Z})$. The proof in [16] required a result about the first homology of \mathcal{I}_g^1 with certain twisted coefficients that we prove in the present paper (see Theorem A below).

Remark 1.1. It is not known if the integral homology group $H_2(\mathcal{I}_{g,p}^b)$ is finitely generated. \square

1.3. Action on fundamental group. Fix basepoints $*$ on Σ_g^1 and Σ_g , with the basepoint for Σ_g^1 on $\partial\Sigma_g^1$. Define

$$\pi_g^1 = \pi_1(\Sigma_g^1, *) \quad \text{and} \quad \pi_g = \pi_1(\Sigma_g, *).$$

By definition, elements of Mod_g^1 fix $\partial\Sigma_g^1$ pointwise. In particular, they fix the basepoint $*$ $\in \partial\Sigma_g^1$, so we get a well-defined action of Mod_g^1 on π_g^1 . Also, we have $\Sigma_{g,1} \cong \Sigma_g \setminus \{*\}$, so we can regard $\text{Mod}_{g,1}$ as the group of isotopy classes of orientation-preserving diffeomorphisms of Σ_g that fix $*$. We therefore get a well-defined action of $\text{Mod}_{g,1}$ on π_g .

Define⁴

$$\mathcal{C}_g^1 = H_1([\pi_g^1, \pi_g^1]; \mathbb{Q}) \quad \text{and} \quad \mathcal{C}_g = H_1([\pi_g, \pi_g]; \mathbb{Q}).$$

Alternatively, \mathcal{C}_g^1 and \mathcal{C}_g are the first rational homology groups of the universal abelian covers of Σ_g^1 and Σ_g . The action of Mod_g^1 on π_g^1 preserves $[\pi_g^1, \pi_g^1]$, so Mod_g^1 acts on \mathcal{C}_g^1 . Similarly, $\text{Mod}_{g,1}$ acts on \mathcal{C}_g . The vector spaces \mathcal{C}_g^1 and \mathcal{C}_g are infinite-dimensional representations of Mod_g^1 and $\text{Mod}_{g,1}$, and have been intensely studied via the so-called ‘‘Magnus representations’’. See [25] for a survey.

Remark 1.2. Since they do not preserve basepoints, neither Mod_g nor \mathcal{I}_g act on \mathcal{C}_g . \square

1.4. Main theorems. Since \mathcal{C}_g^1 and \mathcal{C}_g are infinite-dimensional, there is no reason to expect that homology with these representations as coefficients has any finiteness properties. However, we will prove:

Theorem A. *For $g \geq 4$, both $H_1(\text{Mod}_g^1; \mathcal{C}_g^1)$ and $H_1(\mathcal{I}_g^1; \mathcal{C}_g^1)$ are finite-dimensional. Moreover, $H_1(\mathcal{I}_g^1; \mathcal{C}_g^1)$ is an algebraic representation of $\text{Sp}_{2g}(\mathbb{Z})$.*

Remark 1.3. Theorem A is what is needed for the authors’ work on the second homology group of the Torelli group in [16]. \square

Typically decorations like boundary components or punctures have only a minor effect on the mapping class group, so Theorem A might lead the reader to expect that $H_1(\text{Mod}_{g,1}; \mathcal{C}_g)$ and $H_1(\mathcal{I}_{g,1}; \mathcal{C}_g)$ are also finite-dimensional. However, to illustrate the subtlety of Theorem A we will prove:

Theorem B. *For $g \geq 4$, both $H_1(\text{Mod}_{g,1}; \mathcal{C}_g)$ and $H_1(\mathcal{I}_{g,1}; \mathcal{C}_g)$ are infinite-dimensional.*

³A representation \mathbf{V} of $\text{Sp}_{2g}(\mathbb{Z})$ over a field \mathbf{k} of characteristic 0 is algebraic if the action of $\text{Sp}_{2g}(\mathbb{Z})$ on \mathbf{V} extends to a polynomial representation of the \mathbf{k} -points $\text{Sp}_{2g}(\mathbf{k})$ of the algebraic group Sp_{2g} . Since $\text{Sp}_{2g}(\mathbb{Z})$ is Zariski dense in $\text{Sp}_{2g}(\mathbf{k})$, such an extension is unique if it exists.

⁴The \mathcal{C} stands for ‘‘commutator subgroup’’.

1.5. Linear congruence subgroups. For $\ell \geq 2$, let $\text{Mod}_{g,p}^b[\ell]$ be the level- ℓ congruence subgroup of $\text{Mod}_{g,p}^b$, i.e., the kernel of the action of $\text{Mod}_{g,p}^b$ on $H_1(\Sigma_{g,p}^b; \mathbb{Z}/\ell)$. This is a sort of “mod- ℓ Torelli group”. In [18], Putman proved versions of Theorems A and B for $\text{Mod}_g^1[\ell]$ and $\text{Mod}_{g,1}[\ell]$, which we now describe.

The group $[\pi_g^1, \pi_g^1]$ is the kernel of the map $\pi_g^1 \rightarrow H_1(\pi_g^1)$, and similarly for $[\pi_g, \pi_g]$. This suggests defining

$$\pi_g^1[\ell] = \ker(\pi_g^1 \rightarrow H_1(\Sigma_g^1; \mathbb{Z}/\ell)) \quad \text{and} \quad \pi_g[\ell] = \ker(\pi_g \rightarrow H_1(\Sigma_{g,1}; \mathbb{Z}/\ell)).$$

We then set $\mathcal{C}_g^1[\ell] = H_1(\pi_g^1[\ell]; \mathbb{Q})$ and $\mathcal{C}_g[\ell] = H_1(\pi_g[\ell]; \mathbb{Q})$. Just like \mathcal{C}_g^1 and \mathcal{C}_g are the first rational homology groups of the universal abelian covers of Σ_g^1 and Σ_g , the vector spaces $\mathcal{C}_g^1[\ell]$ and $\mathcal{C}_g[\ell]$ are the first rational homology groups of the universal mod- ℓ covers of these surfaces.

Since $\mathcal{C}_g^1[\ell]$ and $\mathcal{C}_g[\ell]$ are finite-dimensional, homology with coefficients in them will also be finite-dimensional. What Putman proved in [18] is that for $g \geq 4$, we have

$$(1.1) \quad H_1(\text{Mod}_g^1[\ell]; \mathcal{C}_g^1[\ell]) \cong H_1(\text{Mod}_g^1; \mathcal{C}_g^1[\ell]) = \mathbb{Q}.$$

Moreover, letting $H_{\mathbb{Z}/\ell} = H_1(\Sigma_g; \mathbb{Z}/\ell)$ and $\tau(\ell)$ be the number of positive divisors of ℓ , Putman also proved that for $g \geq 4$ we have

$$H_1(\text{Mod}_{g,1}[\ell]; \mathcal{C}_g[\ell]) \cong \mathbb{Q}[H_{\mathbb{Z}/\ell}] \quad \text{and} \quad H_1(\text{Mod}_{g,1}; \mathcal{C}_g[\ell]) \cong \mathbb{Q}^{\tau(\ell)}.$$

In particular, the dimensions of $H_1(\text{Mod}_{g,1}[\ell]; \mathcal{C}_g[\ell])$ and $H_1(\text{Mod}_{g,1}; \mathcal{C}_g[\ell])$ are much larger than those of $H_1(\text{Mod}_g^1[\ell]; \mathcal{C}_g^1[\ell])$ and $H_1(\text{Mod}_g^1; \mathcal{C}_g^1[\ell])$. These should be seen as analogues of Theorems A and B, and indeed our proofs of these results use some of the ideas from [18].

Remark 1.4. The first isomorphism in (1.1) was extended to higher homology groups in [21], and these higher twisted homology groups were calculated in [26]. \square

1.6. Reidemeister pairing. Much our work is devoted to understanding an equivariant intersection form on our representations. Let $\tilde{\Sigma}_g$ be the universal abelian cover of Σ_g . Letting $H_{\mathbb{Z}} = H_1(\Sigma_g; \mathbb{Z})$, this is a regular $H_{\mathbb{Z}}$ -cover of Σ_g with $\mathcal{C}_g = H_1(\tilde{\Sigma}_g; \mathbb{Q})$. Using the action of $H_{\mathbb{Z}}$ on $H_1(\tilde{\Sigma}_g; \mathbb{Q})$, the algebraic intersection pairing on $\mathcal{C}_g = H_1(\tilde{\Sigma}_g; \mathbb{Q})$ can be enriched to a pairing

$$\mathfrak{r}: \mathcal{C}_g \otimes \mathcal{C}_g \longrightarrow \mathbb{Q}[H_{\mathbb{Z}}]$$

called the Reidemeister pairing. See §4 for the details.

The group $\text{Mod}_{g,1}$ acts on $\mathcal{C}_g \otimes \mathcal{C}_g$ and $\mathbb{Q}[H_{\mathbb{Z}}]$, and the Reidemeister pairing \mathfrak{r} is $\text{Mod}_{g,1}$ -equivariant. In particular, since $\mathcal{I}_{g,1}$ acts trivially on $\mathbb{Q}[H_{\mathbb{Z}}]$ the map \mathfrak{r} provides a large trivial quotient of the $\mathcal{I}_{g,1}$ -representation $\mathcal{C}_g \otimes \mathcal{C}_g$. It turns out that \mathfrak{r} is a connecting homomorphism in a long exact sequence arising when comparing $H_{\bullet}(\mathcal{I}_g^1; \mathcal{C}_g^1)$ and $H_{\bullet}(\mathcal{I}_{g,1}; \mathcal{C}_g)$.

Roughly speaking, the mechanism behind the difference between $H_1(\mathcal{I}_g^1; \mathcal{C}_g^1)$ and $H_1(\mathcal{I}_{g,1}; \mathcal{C}_g)$ is that $H_1(\mathcal{I}_{g,1}; \mathcal{C}_g)$ involves $\text{Im}(\mathfrak{r})$, while $H_1(\mathcal{I}_g^1; \mathcal{C}_g^1)$ involves $\ker(\mathfrak{r})$. The main technical result that goes into our proofs is as follows. Since $\tilde{\mathcal{I}}_{g,1}$ acts trivially on $H_{\mathbb{Z}}$, the Reidemeister pairing factors through the $\mathcal{I}_{g,1}$ -coinvariants of $\mathcal{C}_g \otimes \mathcal{C}_g$:

$$\bar{\mathfrak{r}}: (\mathcal{C}_g \otimes \mathcal{C}_g)_{\mathcal{I}_{g,1}} \rightarrow \mathbb{Q}[H_{\mathbb{Z}}].$$

The action of $\text{Mod}_{g,1}$ on $(\mathcal{C}_g \otimes \mathcal{C}_g)_{\mathcal{I}_{g,1}}$ factors through $\text{Sp}_{2g}(\mathbb{Z})$, and most of this paper is devoted to proving that $\ker(\bar{\mathfrak{r}})$ is a finite-dimensional algebraic representation of $\text{Sp}_{2g}(\mathbb{Z})$. In fact, we identify this representation precisely: letting $H = H_1(\Sigma_g; \mathbb{Q})$, the vector space $\ker(\bar{\mathfrak{r}})$ is the kernel of a contraction

$$\mathfrak{c}: ((\wedge^2 H)/\mathbb{Q})^{\otimes 2} \rightarrow \text{Sym}^2(H).$$

See §18. This same kernel appears in our work on the second homology of Torelli.

1.7. **Outline.** This paper has three parts. In Part 1, we describe the Reidemeister pairing and show how to reduce Theorems A and B to the description of its kernel we described above. We then construct generators for its kernel in Part 2 and relations in Part 3.

1.8. **Notation and conventions.** We will use all the notation discussed above. In particular, H will always mean $H_1(\Sigma_{g,1}; \mathbb{Q}) \cong H_1(\Sigma_g^1; \mathbb{Q}) \cong H_1(\Sigma_g; \mathbb{Q})$ and $H_{\mathbb{Z}}$ will always mean $H_1(\Sigma_{g,1}) \cong H_1(\Sigma_g^1) \cong H_1(\Sigma_g)$. This is a little ambiguous since the genus g does not appear in H or $H_{\mathbb{Z}}$, but the correct g will always be clear from the context. As is traditional when writing about the mapping class group, we will not distinguish between curves and their homotopy classes. For instance, we will talk about simple closed curves in π_g .

1.9. **Standing assumption.** To avoid having to constantly impose hypotheses to rule out low-genus exceptions to our results, we make the following standing assumption:

Assumption 1.5. Throughout the paper, we assume that $g \geq 4$. □

1.10. **Warning.** Though this is in some sense a sequel to [18], our notation is often different from [18]. For instance, in [18] the notation Σ_g^1 means a surface with one puncture and $\Sigma_{g,1}$ a surface with one boundary component, while our convention is the opposite.⁵ We also use various brackets like $\llbracket x, y \rrbracket$ differently than [18]. The paper [18] was written while the second author was a postdoc, and he wishes to apologize for his youthful expository sins.

Part 1. The Reidemeister pairing and the homology of the Torelli group

This part of the paper introduces the Reidemeister and reduces Theorems A and B to a description of its kernel. It has four sections. In §2 – 3, we discuss how Mod_g^1 and $\text{Mod}_{g,1}$ as well as \mathcal{C}_g^1 and \mathcal{C}_g are related. In §4, we give two different characterizations of the Reidemeister pairing. Finally, in §5 we derive our main theorems from the aforementioned description of the kernel of the Reidemeister pairing.

2. BOUNDARY COMPONENTS, PUNCTURES, AND THE BIRMAN EXACT SEQUENCE

This section explores the inter-relationships between the different mapping class groups and representations appearing in our theorems.

2.1. **Boundary component vs puncture: mapping class groups.** The only difference between Mod_g^1 and $\text{Mod}_{g,1}$ is the Dehn twist⁶ about the boundary component in Mod_g^1 :

Lemma 2.1 ([4, Proposition 3.19]). *We have a central extension*

$$1 \longrightarrow \mathbb{Z} \longrightarrow \text{Mod}_g^1 \longrightarrow \text{Mod}_{g,1} \longrightarrow 1,$$

where the map $\text{Mod}_g^1 \rightarrow \text{Mod}_{g,1}$ is induced by gluing a punctured disk to $\partial = \partial\Sigma_g^1$ and the central \mathbb{Z} is generated by T_{∂} .

Since the action of Mod_g^1 on $H_{\mathbb{Z}} = H_1(\Sigma_g^1) \cong H_1(\Sigma_{g,1})$ factors through $\text{Mod}_{g,1}$, this lemma immediately implies a corresponding result for the Torelli group:

Corollary 2.2. *We have a central extension*

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{I}_g^1 \longrightarrow \mathcal{I}_{g,1} \longrightarrow 1,$$

where the map $\mathcal{I}_g^1 \rightarrow \mathcal{I}_{g,1}$ is induced by gluing a punctured disk to $\partial = \partial\Sigma_g^1$ and the central \mathbb{Z} is generated by T_{∂} .

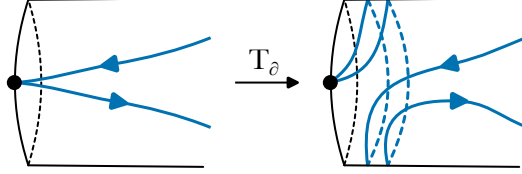
⁵This better aligns our notation with the algebro-geometric notation $\mathcal{M}_{g,p}$ for the moduli space of smooth genus- g curves with p marked points.

⁶For a simple closed curve γ , our convention is that T_{γ} denotes the left-handed Dehn twist about γ .

2.2. Factoring the representation. Using Lemma 2.1, we prove:

Lemma 2.3. *The action of Mod_g^1 on \mathcal{C}_g^1 factors through $\text{Mod}_{g,1}$.*

Proof. Let $\partial = \partial\Sigma_g^1$. By Lemma 2.1, it is enough to prove that T_∂ acts trivially on \mathcal{C}_g^1 . Letting $\delta \in \pi_g^1$ be the loop going around ∂ with Σ_g^1 to its left, for $x \in \pi_g^1$ we have $T_\partial(x) = \delta^{-1}x\delta$:



Since $\delta \in [\pi_g^1, \pi_g^1]$, this implies that T_∂ acts trivially on the abelianization of $[\pi_g^1, \pi_g^1]$, and thus also acts trivially on \mathcal{C}_g^1 . \square

Using this, we will henceforth view \mathcal{C}_g^1 as a representation of both Mod_g^1 and $\text{Mod}_{g,1}$. Via the map $\text{Mod}_g^1 \rightarrow \text{Mod}_{g,1}$, the group Mod_g^1 acts on \mathcal{C}_g , so \mathcal{C}_g is also a representation of both Mod_g^1 and $\text{Mod}_{g,1}$.

2.3. Boundary component vs puncture: representations. The following lemma explains the relationship between the Mod_g^1 -representations \mathcal{C}_g^1 and \mathcal{C}_g .

Lemma 2.4. *We have a short exact sequence*

$$0 \longrightarrow \mathbb{Q}[H_{\mathbb{Z}}] \longrightarrow \mathcal{C}_g^1 \longrightarrow \mathcal{C}_g \longrightarrow 0$$

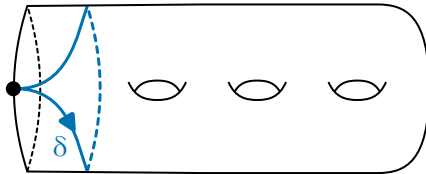
of Mod_g^1 -representations.

Proof. Let $\tilde{\Sigma}_g^1$ and $\tilde{\Sigma}_g$ be the universal abelian covers of Σ_g^1 and Σ_g , respectively, so $\mathcal{C}_g^1 = H_1(\tilde{\Sigma}_g^1; \mathbb{Q})$ and $\mathcal{C}_g = H_1(\tilde{\Sigma}_g; \mathbb{Q})$. To construct $\tilde{\Sigma}_g$ from $\tilde{\Sigma}_g^1$, we glue disks to all components of $\partial\tilde{\Sigma}_g^1$. Let K be the image of $H_1(\partial\tilde{\Sigma}_g^1; \mathbb{Q})$ in $\mathcal{C}_g^1 = H_1(\tilde{\Sigma}_g^1; \mathbb{Q})$, so we have a short exact sequence

$$0 \longrightarrow K \longrightarrow \mathcal{C}_g^1 \longrightarrow \mathcal{C}_g \longrightarrow 0.$$

We must prove that $K \cong \mathbb{Q}[H_{\mathbb{Z}}]$. Let ∂_0 be a component of $\partial\tilde{\Sigma}_g^1$, oriented such that $\tilde{\Sigma}_g^1$ is to its left. The deck group $H_{\mathbb{Z}}$ of the cover $\tilde{\Sigma}_g^1$ acts simply transitively on its boundary components. Since $\tilde{\Sigma}_g^1$ is not compact, the homology classes of its boundary components are linearly independent. The map $\mathbb{Q}[H_{\mathbb{Z}}] \rightarrow K$ taking $h \in H_{\mathbb{Z}}$ to the homology class of $h \cdot \partial_0$ is thus an isomorphism, as desired. \square

Remark 2.5. For later use, we make the above injection $\mathbb{Q}[H_{\mathbb{Z}}] \hookrightarrow \mathcal{C}_g^1$ explicit as follows. For $z \in [\pi_g^1, \pi_g^1]$, let (z) denote the corresponding element of $\mathcal{C}_g^1 = H_1([\pi_g^1, \pi_g^1]; \mathbb{Q})$. Also, for $x \in \pi_g^1$ let \bar{x} denote the corresponding element of $H_{\mathbb{Z}}$. Let $\delta \in [\pi_g^1, \pi_g^1]$ be the following curve:



Then for $x \in \pi_g^1$, the element of \mathcal{C}_g^1 corresponding to $\bar{x} \in H_{\mathbb{Z}}$ is $(x\delta x^{-1})$. \square

2.4. Birman exact sequence. The effect on the mapping class group of adding a puncture is described by the Birman exact sequence:

Theorem 2.6 ([3]). *Let $p, b \geq 0$ and let $x_0 \in \Sigma_{g,p}^b$ be the location of the $(p+1)^{\text{st}}$ puncture in $\Sigma_{g,p+1}^b$. We then have a short exact sequence*

$$1 \longrightarrow \pi_1(\Sigma_{g,p}^b, x_0) \longrightarrow \text{Mod}_{g,p+1}^b \longrightarrow \text{Mod}_{g,p}^b \longrightarrow 1,$$

where the map $\text{Mod}_{g,p+1}^b \rightarrow \text{Mod}_{g,p}^b$ fills in the $(p+1)^{\text{st}}$ puncture.

The normal subgroup $\pi_{g,p}^b = \pi_1(\Sigma_{g,p}^b, x_0)$ of $\text{Mod}_{g,p+1}^b$ is called the *point-pushing subgroup*. Using the action of $\text{Mod}_{g,p+1}^b$ on $\pi_{g,p}^b$, the point-pushing subgroup can be identified with the group of inner automorphisms of $\pi_{g,p}^b$ in the sense that if $\gamma \in \pi_{g,p}^b$ and $f_\gamma \in \text{Mod}_{g,p+1}^b$ is the associated mapping class, then⁷

$$(2.1) \quad f_\gamma(x) = \gamma^{-1}x\gamma \quad \text{for all } x \in \pi_{g,p}^b.$$

This implies in particular that the point-pushing subgroup acts trivially on $H_1(\Sigma_{g,p}^b)$.

The Torelli group $\mathcal{I}_{g,p+1}^b$ is only defined thus far for $(p+1) + b \leq 1$, so as far as Torelli is concerned the above is only relevant for $p = b = 0$. In that case, since $H_1(\Sigma_{g,1}) \cong H_1(\Sigma_g)$ the point-pushing subgroup of $\text{Mod}_{g,1}$ acts trivially on $H_1(\Sigma_{g,1})$ and thus lies in $\mathcal{I}_{g,1}$. Even more is true: the action of $\text{Mod}_{g,1}$ on $H_1(\Sigma_{g,1}) \cong H_1(\Sigma_g)$ factors through Mod_g , so:

Corollary 2.7. *The Birman exact sequence restricts to a short exact sequence*

$$1 \longrightarrow \pi_g \longrightarrow \mathcal{I}_{g,1} \longrightarrow \mathcal{I}_g \longrightarrow 1.$$

2.5. Coinvariants. We will henceforth view \mathcal{C}_g^1 and \mathcal{C}_g as representations of π_g by embedding π_g into $\text{Mod}_{g,1}$ as the point-pushing subgroup and using the actions of $\text{Mod}_{g,1}$ on \mathcal{C}_g^1 and \mathcal{C}_g . By (2.1), this action of π_g on $\mathcal{C}_g = H_1([\pi_g, \pi_g]; \mathbb{Q})$ is the action induced by the conjugation action of π_g on $[\pi_g, \pi_g]$. However, the action of π_g on \mathcal{C}_g^1 is not as easy to describe.

As a first calculation, we determine the coinvariants $(\mathcal{C}_g)_{\pi_g}$. We can view the algebraic intersection form on H as an $\text{Sp}_{2g}(\mathbb{Q})$ -invariant element $\omega \in \wedge^2 H$. If $\{a_1, b_1, \dots, a_g, b_g\}$ is a symplectic basis for H , then

$$\omega = a_1 \wedge b_1 + \dots + a_g \wedge b_g.$$

Henceforth we will view \mathbb{Q} as the trivial subrepresentation of $\wedge^2 H$ spanned by ω , which allows us to talk about $(\wedge^2 H)/\mathbb{Q}$. We then have:

Lemma 2.8. *Letting the notation be as above, we have $(\mathcal{C}_g)_{\pi_g} \cong (\wedge^2 H)/\mathbb{Q}$.*

Proof. Using (2.1), we have

$$(\mathcal{C}_g)_{\pi_g} = H_1([\pi_g, \pi_g]; \mathbb{Q})_{\pi_g} \cong \frac{[\pi_g, \pi_g]}{[\pi_g, [\pi_g, \pi_g]]} \otimes \mathbb{Q}.$$

That this is $(\wedge^2 H)/\mathbb{Q}$ is now classical.⁸ See, for instance, [20, Theorem D]. □

⁷The reader might expect to see $f_\gamma(x) = \gamma x \gamma^{-1}$ since this is what would be needed for $f_{\gamma_1 \gamma_2} = f_{\gamma_1} \circ f_{\gamma_2}$ to hold. This points to a tiny annoying issue: in $\text{Mod}_{g,p+1}^b$, elements are composed right to left like functions, but in $\pi_{g,p}^b$ paths are composed left to right. Strictly speaking, therefore, the map taking $\gamma \in \pi_{g,p}^b$ to $f_\gamma \in \text{Mod}_{g,p+1}^b$ is not an isomorphism but an anti-isomorphism. We could avoid this by changing our conventions about either functions or fundamental groups, but this would lead to endless confusion. This does not affect any of our calculations, and we will not mention it again.

⁸In fact, even before tensoring with \mathbb{Q} this is isomorphic to $(\wedge^2 H_{\mathbb{Z}})/\mathbb{Z}$.

Remark 2.9. Since $\text{Mod}_{g,1}/\pi_g \cong \text{Mod}_g$, the coinvariants $(\mathcal{C}_g)_{\pi_g}$ are a representation of Mod_g . The isomorphism in Lemma 2.8 is an isomorphism of Mod_g -representations, where Mod_g acts on $(\wedge^2 H)/\mathbb{Q}$ via $\text{Sp}_{2g}(\mathbb{Z})$. \square

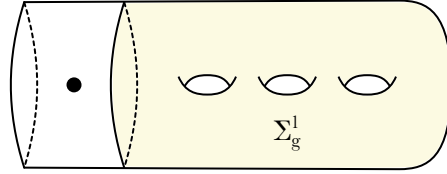
3. SOME DEEPER RELATIONS BETWEEN PUNCTURES AND BOUNDARY COMPONENTS

We continue our study of the interplay between punctures and boundary components.

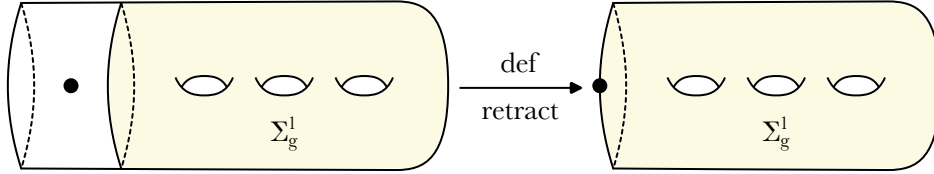
3.1. Splitting Birman exact sequence. Consider the Birman exact sequence for $\text{Mod}_{g,1}^1$:

$$1 \longrightarrow \pi_g^1 \longrightarrow \text{Mod}_{g,1}^1 \longrightarrow \text{Mod}_g^1 \longrightarrow 1.$$

This splits via the map $\text{Mod}_g^1 \rightarrow \text{Mod}_{g,1}^1$ induced by embedding Σ_g^1 as a subsurface of $\Sigma_{g,1}^1$ and extending mapping classes by the identity; see here:

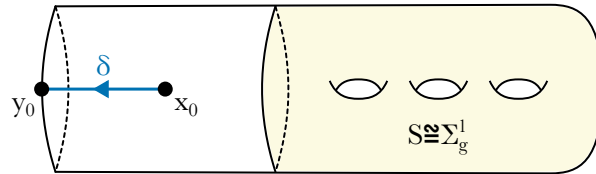


This leads to a semidirect product decomposition $\text{Mod}_{g,1}^1 = \pi_g^1 \rtimes \text{Mod}_g^1$, and in particular an action of Mod_g^1 on π_g^1 . Regarding the puncture of $\Sigma_{g,1}^1$ as a marked point, we can deformation retract $\Sigma_{g,1}^1$ onto Σ_g^1 such that the marked point ends up as a basepoint on $\partial\Sigma_g^1$:



From this, we see that our action of Mod_g^1 on π_g^1 is the natural one arising from placing the basepoint of $\pi_g^1 = \pi_1(\Sigma_g^1)$ on $\partial\Sigma_g^1$.

3.2. Extending Torelli. For this section only, we need a version of the Torelli group on $\Sigma_{g,1}^1$. View the puncture on $\Sigma_{g,1}^1$ as a marked point $x_0 \in \Sigma_g^1$. Let $y_0 \in \partial\Sigma_g^1$ be another point. Let $S \cong \Sigma_g^1$ be a subsurface of Σ_g^1 such that $x_0, \partial\Sigma_g^1 \subset \Sigma_{g,1}^1 \setminus S$ and let $\delta \in H_1(\Sigma_g^1, \{x_0, y_0\})$ be the homology class of an arc connecting x_0 to y_0 in $\Sigma_{g,1}^1 \setminus S$:



We have

$$H_1(\Sigma_g^1, \{x_0, y_0\}) = \mathbb{Z}\langle\delta\rangle \oplus H_1(S) \cong \mathbb{Z} \oplus \mathbb{Z}^{2g}.$$

Let $\mathcal{I}_{g,1}^1$ be the kernel of the action of $\text{Mod}_{g,1}^1$ on $H_1(\Sigma_g^1, \{x_0, y_0\})$. The point-pushing subgroup π_g^1 of $\text{Mod}_{g,1}^1$ does not lie in $\mathcal{I}_{g,1}^1$ since it does not fix δ . In fact:

Theorem 3.1 ([17, Theorem 1.2]). *The intersection of the point-pushing subgroup $\pi_g^1 < \text{Mod}_{g,1}^1$ with $\mathcal{I}_{g,1}^1$ is $[\pi_g^1, \pi_g^1]$, and the Birman exact sequence restricts to a split exact sequence*

$$1 \longrightarrow [\pi_g^1, \pi_g^1] \longrightarrow \mathcal{I}_{g,1}^1 \longrightarrow \mathcal{I}_g^1 \longrightarrow 1.$$

Remark 3.2. The results in [17] are for surfaces with multiple boundary components, not boundary components and punctures. However, the proofs work verbatim to prove the results we discuss above. In fact, even more is true: letting $\{\partial_1, \partial_2\}$ be the boundary components of Σ_g^2 , the group $\mathcal{I}_{g,1}^1$ we discussed above is *isomorphic* to the group $\mathcal{I}(\Sigma_g^2, \{\{\partial_1, \partial_2\}\})$ from [17]. The isomorphism glues a punctured disk to ∂_1 ; since no power of T_{∂_1} lies in $\mathcal{I}(\Sigma_g^2, \{\{\partial_1, \partial_2\}\})$, this does not change the Torelli group (c.f. Lemma 2.1). \square

The conjugation action of $\text{Mod}_{g,1}^1$ on $\mathcal{I}_{g,1}^1$ induces an action of $\text{Mod}_{g,1}^1/\mathcal{I}_{g,1}^1$ on each $H_d(\mathcal{I}_{g,1}^1)$. Using the Birman exact sequences for $\text{Mod}_{g,1}^1$ and $\mathcal{I}_{g,1}^1$, we get an exact sequence

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_g^1/[\pi_g^1, \pi_g^1] & \longrightarrow & \text{Mod}_{g,1}^1/\mathcal{I}_{g,1}^1 & \longrightarrow & \text{Mod}_g^1/\mathcal{I}_g^1 \longrightarrow 1. \\ & & \parallel & & & & \parallel \\ & & H_{\mathbb{Z}} & & & & \text{Sp}_{2g}(\mathbb{Z}) \end{array}$$

The map $\text{Mod}_g^1 \rightarrow \text{Mod}_{g,1}^1$ that splits the Birman exact sequence induces a splitting of this exact sequence, giving an inclusion $\text{Sp}_{2g}(\mathbb{Z}) \hookrightarrow \text{Mod}_{g,1}^1/\mathcal{I}_{g,1}^1$ and thus an action of $\text{Sp}_{2g}(\mathbb{Z})$ on each $H_d(\mathcal{I}_{g,1}^1)$. Putman [19] calculated $H_1(\mathcal{I}_{g,1}^1; \mathbb{Q})$. His calculation implies:⁹

Theorem 3.3 ([19, Theorem B]). *The homology group $H_1(\mathcal{I}_{g,1}^1; \mathbb{Q})$ is a finite-dimensional algebraic representation of $\text{Sp}_{2g}(\mathbb{Z})$.*

Remark 3.4. The same caveat about punctures vs boundary components from Remark 3.2 also applies to [19]. \square

3.3. Capping boundary of groups. Since $\text{Mod}_{g,1}^1/\mathcal{I}_{g,1}^1 \cong \text{Sp}_{2g}(\mathbb{Z})$, the coinvariants $(\mathcal{C}_g^1)_{\mathcal{I}_{g,1}^1}$ are a representation of $\text{Sp}_{2g}(\mathbb{Z})$. We have:

Lemma 3.5. *We have $(\mathcal{C}_g^1)_{\mathcal{I}_{g,1}^1} \cong \wedge^2 H$.*

Proof. Consider the split exact sequence from Theorem 3.1:

$$1 \longrightarrow [\pi_g^1, \pi_g^1] \longrightarrow \mathcal{I}_{g,1}^1 \longrightarrow \mathcal{I}_g^1 \longrightarrow 1.$$

The above short exact sequence yields a five term exact sequence in group homology. Since our short exact sequence splits, the rightmost three terms of the associated five term exact sequence are in fact a short exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & (H_1([\pi_g^1, \pi_g^1]; \mathbb{Q}))_{\mathcal{I}_g^1} & \longrightarrow & H_1(\mathcal{I}_{g,1}^1; \mathbb{Q}) & \longrightarrow & H_1(\mathcal{I}_g^1; \mathbb{Q}) \longrightarrow 0 \\ & & \parallel & & & & \\ & & (\mathcal{C}_g^1)_{\mathcal{I}_{g,1}^1} & & & & \end{array}$$

The result now follows from the calculation of $H_1(\mathcal{I}_{g,1}^1; \mathbb{Q})$ from [19]. \square

This has the following corollary:

Corollary 3.6. *We have a short exact sequence of $\text{Sp}_{2g}(\mathbb{Z})$ -representations*

$$0 \longrightarrow V \longrightarrow H_1(\mathcal{I}_g^1; \mathcal{C}_g^1) \longrightarrow H_1(\mathcal{I}_{g,1}^1; \mathcal{C}_g^1) \longrightarrow 0$$

with V a finite-dimensional algebraic representation of $\text{Sp}_{2g}(\mathbb{Z})$.

⁹Like Johnson's work on the abelianization of Torelli, this theorem requires $g \geq 3$; see Assumption 1.5.

For the proof of Corollary 3.6 and many future proofs as well, we need:

- (♠) the collection of finite-dimensional algebraic representations of $\mathrm{Sp}_{2g}(\mathbb{Z})$ is closed under subquotients, extensions, and tensor products.

Proof of Corollary 3.6. Consider the central extension from Corollary 2.2:

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{I}_g^1 \longrightarrow \mathcal{I}_{g,1} \longrightarrow 1$$

Let $\partial = \partial\Sigma_g^1$. The central \mathbb{Z} is generated by T_∂ , which acts trivially on \mathcal{C}_g^1 . It follows that the associated 5-term exact sequence in group homology contains

$$\begin{array}{ccccccc} \mathrm{H}_1(\mathbb{Z}; \mathcal{C}_g^1)_{\mathcal{I}_{g,1}} & \longrightarrow & \mathrm{H}_1(\mathcal{I}_g^1; \mathcal{C}_g^1) & \longrightarrow & \mathrm{H}_1(\mathcal{I}_{g,1}; \mathcal{C}_g^1) & \longrightarrow & 0 \\ & & \parallel & & & & \\ & & (\mathcal{C}_g^1)_{\mathcal{I}_{g,1}} & & & & \end{array}$$

Lemma 3.5 implies that $(\mathcal{C}_g^1)_{\mathcal{I}_{g,1}}$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$. An application of (♠) now proves the corollary. \square

3.4. Capping boundary of representations. Corollary 3.6 shows how to go from \mathcal{I}_g^1 to $\mathcal{I}_{g,1}$. We now show how to go from \mathcal{C}_g^1 to \mathcal{C}_g :

Lemma 3.7. *We have an exact sequence*

$$0 \longrightarrow W \longrightarrow \mathrm{H}_1(\mathcal{I}_g^1; \mathcal{C}_g^1) \longrightarrow \mathrm{H}_1(\mathcal{I}_g^1; \mathcal{C}_g)$$

with W a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$.

Proof. From the long exact sequence in group homology associated to the short exact sequence

$$0 \longrightarrow \mathbb{Q}[H_{\mathbb{Z}}] \longrightarrow \mathcal{C}_g^1 \longrightarrow \mathcal{C}_g \longrightarrow 0$$

of \mathcal{I}_g^1 -representations from Lemma 2.4, it is enough to prove that the image W of the map $\mathrm{H}_1(\mathcal{I}_g^1; \mathbb{Q}[H_{\mathbb{Z}}]) \rightarrow \mathrm{H}_1(\mathcal{I}_g^1; \mathcal{C}_g^1)$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$.

Now consider the split exact sequence from Theorem 3.1:

$$(3.1) \quad 1 \longrightarrow [\pi_g^1, \pi_g^1] \longrightarrow \mathcal{I}_{g,1}^1 \longrightarrow \mathcal{I}_g^1 \longrightarrow 1.$$

Associated to (3.1) is a Hochschild–Serre spectral sequence with the following properties:

- Since $[\pi_g^1, \pi_g^1]$ is a free group, we have $\mathrm{H}_q([\pi_g^1, \pi_g^1]; \mathbb{Q}) = 0$ for $q \geq 2$ and thus our spectral sequence only has two nonzero rows.
- Since (3.1) splits, all the differentials coming out of the bottom row of our spectral sequence vanish.

Combining these two facts, our spectral sequence collapses on page 2. It therefore breaks up into short exact sequences, one of which is

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{H}_1(\mathcal{I}_g^1; \mathrm{H}_1([\pi_g^1, \pi_g^1]; \mathbb{Q})) & \longrightarrow & \mathrm{H}_2(\mathcal{I}_{g,1}^1; \mathbb{Q}) & \longrightarrow & \mathrm{H}_2(\mathcal{I}_g^1; \mathbb{Q}) \longrightarrow 0. \\ & & \parallel & & & & \\ & & \mathrm{H}_1(\mathcal{I}_g^1; \mathcal{C}_g^1) & & & & \end{array}$$

From this, we get an injection $\mathrm{H}_1(\mathcal{I}_g^1; \mathcal{C}_g^1) \hookrightarrow \mathrm{H}_2(\mathcal{I}_{g,1}^1; \mathbb{Q})$. Let Φ be the composition

$$\mathrm{H}_1(\mathcal{I}_g^1; \mathbb{Q}) \otimes \mathbb{Q}[H_{\mathbb{Z}}] = \mathrm{H}_1(\mathcal{I}_g^1; \mathbb{Q}[H_{\mathbb{Z}}]) \longrightarrow \mathrm{H}_1(\mathcal{I}_g^1; \mathcal{C}_g^1) \hookrightarrow \mathrm{H}_2(\mathcal{I}_{g,1}^1; \mathbb{Q}).$$

To prove that W is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$, it is enough to prove the same statement for $\mathrm{Im}(\Phi)$.

We introduce some notation:

- For subgroups $G_1, G_2 < \mathcal{I}_{g,1}^1$ with $[G_1, G_2] = 1$, let $\mathrm{H}_1(G_1; \mathbb{Q}) \otimes \mathrm{H}_1(G_2; \mathbb{Q})$ denote the corresponding subgroup of $\mathrm{H}_2(G_1 \times G_2; \mathbb{Q})$ and let $\overline{\mathrm{H}_1(G_1; \mathbb{Q}) \otimes \mathrm{H}_1(G_2; \mathbb{Q})}$ be its image under the map $\mathrm{H}_2(G_1 \times G_2; \mathbb{Q}) \rightarrow \mathrm{H}_2(\mathcal{I}_{g,1}^1; \mathbb{Q})$.
- For a subgroup $G < \mathcal{I}_{g,1}^1$ and an element $x \in \mathcal{I}_{g,1}^1$ with $[G, x] = 1$, let $[x] \in \mathrm{H}_1(\langle x \rangle; \mathbb{Q})$ be the corresponding element, let $\mathrm{H}_1(G; \mathbb{Q}) \otimes [x]$ be the corresponding subgroup of $\mathrm{H}_2(G \times \langle x \rangle; \mathbb{Q})$, and let $\overline{\mathrm{H}_1(G; \mathbb{Q}) \otimes [x]}$ be its image in $\overline{\mathrm{H}_1(G; \mathbb{Q}) \otimes \mathrm{H}_1(\langle x \rangle; \mathbb{Q})}$.
- For $x, y \in \mathcal{I}_{g,1}^1$ with $[x, y] = 1$, let $[x] \otimes [y] \in \mathrm{H}_1(\langle x \rangle \times \langle y \rangle; \mathbb{Q})$ be the corresponding element and let $\overline{[x] \otimes [y]}$ be its image in $\overline{\mathrm{H}_1(\langle x \rangle; \mathbb{Q}) \otimes \mathrm{H}_1(\langle y \rangle; \mathbb{Q})}$.

Now let $\partial = \partial\Sigma_{g,1}^1$, so $T_\partial \in \mathcal{I}_{g,1}^1$. Since T_∂ is a central element, we can define

$$U = \overline{[T_\partial] \otimes \mathrm{H}_1(\mathcal{I}_{g,1}^1; \mathbb{Q})} \subset \mathrm{H}_2(\mathcal{I}_{g,1}^1; \mathbb{Q}).$$

This is a quotient of $\mathrm{H}_1(\mathcal{I}_{g,1}^1; \mathbb{Q})$, which by Theorem 3.3 is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$. By (\spadesuit) , we have that U is also a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$. Again using (\spadesuit) , to prove that $\mathrm{Im}(\Phi)$ a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$, it is enough to show:

Claim. $\mathrm{Im}(\Phi) < U$.

Consider $f \in \mathcal{I}_g^1$ and $h \in H_{\mathbb{Z}}$. Let $[f] \in \mathrm{H}_1(\mathcal{I}_g^1; \mathbb{Q})$ be the corresponding element. We must prove that $\Phi([f] \otimes h) \in U$. The conjugation action of $\mathrm{Mod}_{g,1}^1$ induces an action of $\mathrm{Mod}_{g,1}^1$ on $\mathrm{H}_2(\mathcal{I}_{g,1}^1; \mathbb{Q})$. Since T_∂ is central, this action preserves U . The group $\mathrm{Mod}_{g,1}^1$ also acts on

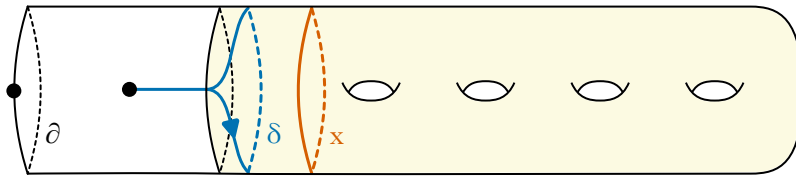
$$\mathbb{Q}[H_{\mathbb{Z}}] < \mathcal{C}_g^1 = \mathrm{H}_1([\pi_g^1, \pi_g^1]; \mathbb{Q})$$

via its action on $[\pi_g^1, \pi_g^1]$ and on $\mathrm{H}_1(\mathcal{I}_g^1; \mathbb{Q})$ via its projection $\mathrm{Mod}_{g,1}^1 \rightarrow \mathrm{Mod}_g^1$. With respect to these actions, the map

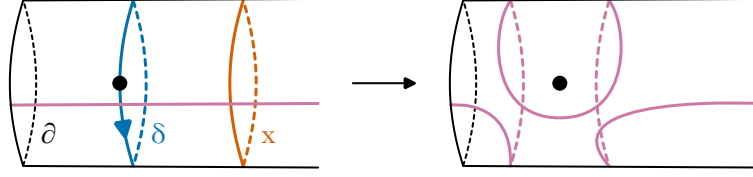
$$\Phi: \mathrm{H}_1(\mathcal{I}_g^1; \mathbb{Q}) \otimes \mathbb{Q}[H_{\mathbb{Z}}] \rightarrow \mathrm{H}_2(\mathcal{I}_{g,1}^1; \mathbb{Q})$$

is $\mathrm{Mod}_{g,1}^1$ -equivariant. To check that $\Phi([f] \otimes h) \in U$, we can therefore first apply any element of $\mathrm{Mod}_{g,1}^1$ we wish to $[f] \otimes h$.

The point-pushing subgroup $\pi_g^1 < \mathrm{Mod}_{g,1}^1$ acts on $\mathbb{Q}[H_{\mathbb{Z}}]$ via its projection $\pi_g^1 \rightarrow H_{\mathbb{Z}}$. Since this projection is surjective, we see that π_g^1 acts transitively on $H_{\mathbb{Z}}$. In light of Remark 2.5, applying an appropriate element of $\mathrm{Mod}_{g,1}^1$ we can thus assume that $h \in \mathcal{C}_g^1$ is the homology class of the element $\delta \in [\pi_g^1, \pi_g^1]$ shown here:



Here the subsurface Σ_g^1 is shaded and x is a loop parallel to $\partial\Sigma_{g,1}^1$. The loops δ and x are homotopic to the following configuration, which illustrates the element of the point-pushing subgroup of $\mathcal{I}_{g,1}^1$ corresponding to δ :

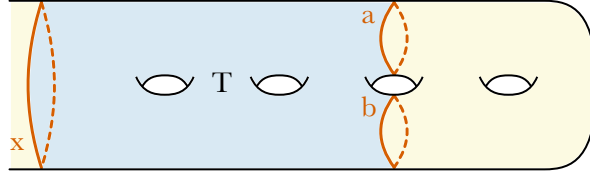


As is clear from this figure, this element is $T_x T_\partial^{-1}$. Since T_x is in the center of Mod_g^1 , embedding \mathcal{I}_g^1 into $\mathcal{I}_{g,1}^1$ using the subsurface inclusion we indicated above identifies $f \in \mathcal{I}_g^1$ with an element of $\mathcal{I}_{g,1}^1$ that commutes with both T_x and T_∂ . Tracing through the definitions of all of our maps, we have

$$\Phi([f] \otimes h) = \overline{[f] \otimes [T_x T_\partial^{-1}]} = \overline{[f] \otimes [T_x]} - \overline{[f] \otimes [T_\partial]}.$$

Since $\overline{[f] \otimes [T_\partial]} \in U$, it is enough to prove that $\overline{[f] \otimes [T_x]} = 0$.

The group \mathcal{I}_g^1 is generated by genus-1 bounding pair maps [7], i.e., maps $T_a T_b^{-1}$ such that a and b are disjoint curves such that $a \cup b$ separates Σ_g^1 into two subsurfaces, one of which is homeomorphic to Σ_1^2 . It is thus enough to prove that $\overline{[f] \otimes [T_x]} = 0$ for $f = T_a T_b^{-1}$ a genus-1 bounding pair map. Let $T \cong \Sigma_{g-2}^3$ be the subsurface bounded by $x \cup a \cup b$:



Let $\mathcal{I}_{g,1}^1(T)$ be the subgroup of $\mathcal{I}_{g,1}^1$ consisting of mapping classes supported on T . We have

$$\overline{[f] \otimes [T_x]} \in \overline{[f] \otimes H_1(\mathcal{I}_{g,1}^1(T); \mathbb{Q})}.$$

Since $g \geq 4$ (see Assumption 1.5), the surface T has genus at least 2. It then follows from [19, Lemma 6.2] that T_x maps to 0 in $H_1(\mathcal{I}_{g,1}^1(T); \mathbb{Q})$. This implies that $\overline{[f] \otimes [T_x]} = 0$, as desired. \square

4. REIDEMEISTER PAIRING

We now turn to the Reidemeister pairing on $\mathcal{C}_g = H_1([\mathcal{C}_g, \mathcal{C}_g]; \mathbb{Q})$.

4.1. Definition of pairing. Let $\tilde{\Sigma}_g \rightarrow \Sigma_g$ be the universal abelian cover of Σ_g . On $\mathcal{C}_g = H_1(\tilde{\Sigma}_g; \mathbb{Q})$, we have the algebraic intersection form $\iota: \mathcal{C}_g \times \mathcal{C}_g \rightarrow \mathbb{Q}$. We also have the action of the deck group $H_{\mathbb{Z}}$ on $\tilde{\Sigma}_g$ and hence on \mathcal{C}_g . The *Reidemeister pairing* [23, 24] is the linear map $\mathfrak{r}: \mathcal{C}_g^{\otimes 2} \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$ defined via the formula¹⁰

$$\mathfrak{r}(x \otimes y) = \sum_{h \in H_{\mathbb{Z}}} \iota(h \cdot x, y) h \quad \text{for } x, y \in \mathcal{C}_g.$$

4.2. Connecting homomorphism. The reason the Reidemeister pairing is relevant to our work is the following. Consider the short exact sequence of representations from Lemma 2.4:

$$0 \longrightarrow \mathbb{Q}[H_{\mathbb{Z}}] \longrightarrow \mathcal{C}_g^1 \longrightarrow \mathcal{C}_g \longrightarrow 0.$$

These are representations of $\text{Mod}_{g,1}$, but we will view them as representations of the subgroup $[\pi_g, \pi_g]$ of the point-pushing subgroup $\pi_g < \text{Mod}_{g,1}$. The group $[\pi_g, \pi_g]$ acts trivially on

¹⁰Since x and y are supported on compact subsurfaces of $\tilde{\Sigma}_g$ and $H_{\mathbb{Z}}$ acts properly discontinuously on $\tilde{\Sigma}_g$, all but finitely many terms in this sum vanish.

$\mathbb{Q}[H_{\mathbb{Z}}]$ and \mathcal{C}_g , but it does not act trivially on \mathcal{C}_g^1 . Consider the long exact sequence in group homology:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_1([\pi_g, \pi_g]; \mathcal{C}_g^1) & \longrightarrow & H_1([\pi_g, \pi_g]; \mathcal{C}_g) & \xrightarrow{\tau'} & H_0([\pi_g, \pi_g]; \mathbb{Q}[H_{\mathbb{Z}}]) & \longrightarrow & \cdots \\ & & & & \parallel & & \parallel & & \\ & & & & \mathcal{C}_g^{\otimes 2} & & \mathbb{Q}[H_{\mathbb{Z}}] & & \end{array}$$

As the following shows, the connecting homomorphism $\tau': \mathcal{C}_g^{\otimes 2} \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$ equals the Reidemeister pairing:

Lemma 4.1. *Let the notation be as above. Then $\tau' = \tau$.*

Proof. This can be proved exactly like [18, Lemma 5.2]. See [21, Lemma 7.1] for an alternate exposition of the argument. \square

5. PROOFS OF MAIN THEOREMS

We now come to the proofs of our main theorems. The key to them is the following. Since $\mathcal{I}_{g,1}$ acts trivially on $H_{\mathbb{Z}}$, the Reidemeister pairing $\tau: \mathcal{C}_g^{\otimes 2} \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$ factors through a map $\bar{\tau}: (\mathcal{C}_g^{\otimes 2})_{\mathcal{I}_{g,1}} \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$ of $\mathrm{Sp}_{2g}(\mathbb{Z})$ -representations that we will call the *coinvariant Reidemeister pairing*. We then have:

Theorem 5.1. *Let $\bar{\tau}: (\mathcal{C}_g^{\otimes 2})_{\mathcal{I}_{g,1}} \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$ be the coinvariant Reidemeister pairing. Then both $\ker(\bar{\tau})$ and $\mathrm{coker}(\bar{\tau}) = \mathrm{coker}(\tau)$ are finite-dimensional. Moreover, $\ker(\bar{\tau})$ is an algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$.*

Remark 5.2. With a bit more work, one can show that $\mathrm{Im}(\bar{\tau}) = \mathrm{Im}(\tau)$ is as follows. Let $\epsilon: \mathbb{Q}[H_{\mathbb{Z}}] \rightarrow \mathbb{Q}$ be the augmentation and let $\mathfrak{h}: \mathbb{Q}[H_{\mathbb{Z}}] \rightarrow H$ be the map coming from the inclusion $H_{\mathbb{Z}} \hookrightarrow H$. Then $\mathrm{Im}(\bar{\tau})$ is the kernel of the surjective map $\epsilon \times \mathfrak{h}: \mathbb{Q}[H_{\mathbb{Z}}] \rightarrow \mathbb{Q} \times H$. Consequently, $\mathrm{coker}(\bar{\tau}) \cong \mathbb{Q} \oplus H$ is also an algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$. \square

The rest of this paper will be devoted to proving Theorem 5.1: in Part 2 we calculate $\mathrm{Im}(\bar{\tau})$ and find generators for $\ker(\bar{\tau})$, and in Part 3 we find some relations in $\ker(\bar{\tau})$ and use these relations to prove that $\ker(\bar{\tau})$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$. For now, we assume the truth of Theorem 5.1 and show how to prove our main results.

5.1. First main theorem. The first of these main theorems is:¹¹

Theorem A. *For $g \geq 4$, both $H_1(\mathrm{Mod}_g^1; \mathcal{C}_g^1)$ and $H_1(\mathcal{I}_g^1; \mathcal{C}_g^1)$ are finite-dimensional. Moreover, $H_1(\mathcal{I}_g^1; \mathcal{C}_g^1)$ is an algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$.*

Proof of Theorem A for the Torelli group, assuming Theorem 5.1. Lemma 2.4 gives an exact sequence of representations

$$0 \longrightarrow \mathbb{Q}[H_{\mathbb{Z}}] \longrightarrow \mathcal{C}_g^1 \longrightarrow \mathcal{C}_g \longrightarrow 0.$$

The associated long exact sequence in \mathcal{I}_g^1 -homology contains

$$\begin{array}{ccccc} H_1(\mathcal{I}_g^1; \mathcal{C}_g^1) & \xrightarrow{\iota} & H_1(\mathcal{I}_g^1; \mathcal{C}_g) & \xrightarrow{\mathbf{b}} & H_0(\mathcal{I}_g^1; \mathbb{Q}[H_{\mathbb{Z}}]) \\ & & & & \parallel \\ & & & & \mathbb{Q}[H_{\mathbb{Z}}] \end{array}$$

¹¹This is copied from the introduction before we imposed our genus assumption, but we remind the reader that we are assuming throughout the paper that $g \geq 4$ (see Assumption 1.5).

Lemma 3.7 says that $\ker(\iota)$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$. We have $\mathrm{Im}(\iota) = \ker(\mathbf{b})$, so using (\spadesuit) we deduce that it is enough to prove that the kernel of the connecting homomorphism \mathbf{b} is also a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$.

There is a similar connecting homomorphism $\mathbf{d}: \mathrm{H}_1(\mathcal{I}_{g,1}; \mathcal{C}_g) \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$. Regarding π_g as the point-pushing subgroup of $\mathcal{I}_{g,1}$, we also have connecting homomorphisms $\boldsymbol{\delta}: \mathrm{H}_1(\pi_g; \mathcal{C}_g) \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$ and $\mathfrak{r}: \mathrm{H}_1([\pi_g, \pi_g]; \mathcal{C}_g) \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$. Identifying $\mathrm{H}_1([\pi_g, \pi_g]; \mathcal{C}_g)$ with $\mathcal{C}_g^{\otimes 2}$, Lemma 4.1 says that \mathfrak{r} is the Reidemeister pairing. These factor through maps $\bar{\boldsymbol{\delta}}: \mathrm{H}_1(\pi_g; \mathcal{C}_g)_{\mathcal{I}_{g,1}} \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$ and $\bar{\mathfrak{r}}: \mathrm{H}_1([\pi_g, \pi_g]; \mathcal{C}_g)_{\mathcal{I}_{g,1}} \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$, with $\bar{\mathfrak{r}}$ the coinvariant Reidemeister pairing. These all fit into a commutative diagram

$$\begin{array}{ccccccc} \mathrm{H}_1([\pi_g, \pi_g]; \mathcal{C}_g)_{\mathcal{I}_{g,1}} & \longrightarrow & \mathrm{H}_1(\pi_g; \mathcal{C}_g)_{\mathcal{I}_{g,1}} & \longrightarrow & \mathrm{H}_1(\mathcal{I}_{g,1}; \mathcal{C}_g) & \longleftarrow & \mathrm{H}_1(\mathcal{I}_g^1; \mathcal{C}_g) \\ \downarrow \bar{\mathfrak{r}} & & \downarrow \bar{\boldsymbol{\delta}} & & \downarrow \mathbf{d} & & \downarrow \mathbf{b} \\ \mathbb{Q}[H_{\mathbb{Z}}] & \xlongequal{\quad} & \mathbb{Q}[H_{\mathbb{Z}}] & \xlongequal{\quad} & \mathbb{Q}[H_{\mathbb{Z}}] & \xlongequal{\quad} & \mathbb{Q}[H_{\mathbb{Z}}] \end{array}$$

From this, we get maps

$$\ker(\bar{\mathfrak{r}}) \longrightarrow \ker(\bar{\boldsymbol{\delta}}) \longrightarrow \ker(\mathbf{d}) \longleftarrow \ker(\mathbf{b})$$

Our goal is to prove that $\ker(\mathbf{b})$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$, and by Theorem 5.1 we know that this holds for $\ker(\bar{\mathfrak{r}})$. We work from left to right:

Claim 1. *We have that $\ker(\bar{\boldsymbol{\delta}})$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$.*

Since $\ker(\bar{\mathfrak{r}})$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$, by (\spadesuit) it is enough to prove that this also holds for the cokernel of the map $\ker(\bar{\mathfrak{r}}) \rightarrow \ker(\bar{\boldsymbol{\delta}})$. Consider the short exact sequence

$$1 \longrightarrow [\pi_g, \pi_g] \longrightarrow \pi_g \longrightarrow H_{\mathbb{Z}} \longrightarrow 1.$$

Since $[\pi_g, \pi_g]$ acts trivially on \mathcal{C}_g , it follows from the 5-term exact sequence in group homology with coefficients in \mathcal{C}_g that we have an exact sequence¹²

$$\mathrm{H}_1([\pi_g, \pi_g]; \mathcal{C}_g) \rightarrow \mathrm{H}_1(\pi_g; \mathcal{C}_g) \rightarrow \mathrm{H}_1(H_{\mathbb{Z}}; \mathcal{C}_g) \rightarrow 0.$$

It follows from [20, Theorem D] that $\mathrm{H}_1(H_{\mathbb{Z}}; \mathcal{C}_g) \cong \wedge^3 H$, which is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$. Since taking coinvariants is right-exact, the above remains exact if we take $\mathcal{I}_{g,1}$ -coinvariants. Do this and add $\bar{\mathfrak{r}}$ and $\bar{\boldsymbol{\delta}}$:

$$\begin{array}{ccccccc} \mathrm{H}_1([\pi_g, \pi_g]; \mathcal{C}_g)_{\mathcal{I}_{g,1}} & \longrightarrow & \mathrm{H}_1(\pi_g; \mathcal{C}_g)_{\mathcal{I}_{g,1}} & \longrightarrow & \mathrm{H}_1(H_{\mathbb{Z}}; \mathcal{C}_g)_{\mathcal{I}_{g,1}} & \longrightarrow & 0 \\ \downarrow \bar{\mathfrak{r}} & & \downarrow \bar{\boldsymbol{\delta}} & & \parallel & & \\ \mathbb{Q}[H_{\mathbb{Z}}] & \xlongequal{\quad} & \mathbb{Q}[H_{\mathbb{Z}}] & & \wedge^3 H & & \end{array}$$

This is a commutative diagram of $\mathrm{Mod}_{g,1}$ -representations. Let U be the image of $\ker(\bar{\boldsymbol{\delta}})$ in $\wedge^3 H$. By (\spadesuit) , U is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$. Examining the above diagram, we see that its top row restricts to an exact sequence

$$\ker(\bar{\mathfrak{r}}) \rightarrow \ker(\bar{\boldsymbol{\delta}}) \rightarrow U \rightarrow 0.$$

In other words, the cokernel of the map $\ker(\bar{\mathfrak{r}}) \rightarrow \ker(\bar{\boldsymbol{\delta}})$ is isomorphic to U , which is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$, as desired.

¹²The usual 5-term exact sequence has $\mathrm{H}_1([\pi_g, \pi_g]; \mathcal{C}_g)_{H_{\mathbb{Z}}}$; however, taking these coinvariants is only needed if you want to continue it to the left.

Claim 2. *We have that $\ker(\mathbf{d})$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$.*

Since $\ker(\bar{\mathbf{d}})$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$, by (\spadesuit) it is enough to prove that this also holds for the cokernel of the map $\ker(\bar{\mathbf{d}}) \rightarrow \ker(\mathbf{d})$. Consider the Birman exact sequence from Corollary 2.7:

$$1 \longrightarrow \pi_g \longrightarrow \mathcal{I}_{g,1} \longrightarrow \mathcal{I}_g \longrightarrow 1.$$

Just like in Claim 1, we can derive from this a commutative diagram¹³

$$\begin{array}{ccccccc} \mathrm{H}_1(\pi_g; \mathcal{C}_g)_{\mathcal{I}_{g,1}} & \longrightarrow & \mathrm{H}_1(\mathcal{I}_{g,1}; \mathcal{C}_g) & \longrightarrow & \mathrm{H}_1(\mathcal{I}_g; (\mathcal{C}_g)_{\pi_g}) & \longrightarrow & 0. \\ \downarrow \bar{\mathbf{d}} & & \downarrow \mathbf{d} & & & & \\ \mathbb{Q}[H_{\mathbb{Z}}] & \xlongequal{\quad} & \mathbb{Q}[H_{\mathbb{Z}}] & & & & \end{array}$$

whose first row is exact. Lemma 2.8 says that $(\mathcal{C}_g)_{\pi_g} \cong (\wedge^2 H)/\mathbb{Q}$, so

$$\mathrm{H}_1(\mathcal{I}_g; (\mathcal{C}_g)_{\pi_g}) \cong \mathrm{H}_1(\mathcal{I}_g; (\wedge^2 H)/\mathbb{Q}) \cong \mathrm{H}_1(\mathcal{I}_g; \mathbb{Q}) \otimes ((\wedge^2 H)/\mathbb{Q}).$$

Johnson [11] proved that $\mathrm{H}_1(\mathcal{I}_g; \mathbb{Q})$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$, so by (\spadesuit) we deduce that $\mathrm{H}_1(\mathcal{I}_g; (\mathcal{C}_g)_{\pi_g})$ is as well. An argument identical to the one used in Claim 1 now shows that the cokernel of the map $\ker(\bar{\mathbf{d}}) \rightarrow \ker(\mathbf{d})$ is isomorphic to a subrepresentation of $\mathrm{H}_1(\mathcal{I}_g; (\mathcal{C}_g)_{\pi_g})$, and thus by (\spadesuit) is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$, as desired.

Claim 3. *We have that $\ker(\mathbf{b})$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$.*

Since $\ker(\mathbf{d})$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$, by (\spadesuit) it is enough to prove that this also holds for the kernel of the map $\ker(\mathbf{b}) \rightarrow \ker(\mathbf{d})$. To help us manipulate this kernel, we call this map $\phi: \ker(\mathbf{b}) \rightarrow \ker(\mathbf{d})$.

Recall that our connecting homomorphisms \mathbf{b} and \mathbf{d} form part of the long exact sequences in homology associated to the short exact sequence

$$0 \longrightarrow \mathbb{Q}[H_{\mathbb{Z}}] \longrightarrow \mathcal{C}_g^1 \longrightarrow \mathcal{C}_g \longrightarrow 0$$

of representations from Lemma 2.4. They thus fit into a commutative diagram

$$\begin{array}{ccccccc} \mathrm{H}_1(\mathcal{I}_g^1; \mathbb{Q}[H_{\mathbb{Z}}]) & \longrightarrow & \mathrm{H}_1(\mathcal{I}_g^1; \mathcal{C}_g^1) & \longrightarrow & \mathrm{H}_1(\mathcal{I}_g^1; \mathcal{C}_g) & \xrightarrow{\mathbf{b}} & \mathbb{Q}[H_{\mathbb{Z}}] \\ \downarrow \Psi & & \downarrow \Phi & & \downarrow & & \parallel \\ \mathrm{H}_1(\mathcal{I}_{g,1}; \mathbb{Q}[H_{\mathbb{Z}}]) & \longrightarrow & \mathrm{H}_1(\mathcal{I}_{g,1}; \mathcal{C}_g^1) & \longrightarrow & \mathrm{H}_1(\mathcal{I}_{g,1}; \mathcal{C}_g) & \xrightarrow{\mathbf{d}} & \mathbb{Q}[H_{\mathbb{Z}}] \end{array}$$

with exact rows. This gives a commutative diagram

$$\begin{array}{ccccccc} \mathrm{H}_1(\mathcal{I}_g^1; \mathbb{Q}[H_{\mathbb{Z}}]) & \longrightarrow & \mathrm{H}_1(\mathcal{I}_g^1; \mathcal{C}_g^1) & \longrightarrow & \ker(\mathbf{b}) & \longrightarrow & 0 \\ \downarrow \Psi & & \downarrow \Phi & & \downarrow \phi & & \\ \mathrm{H}_1(\mathcal{I}_{g,1}; \mathbb{Q}[H_{\mathbb{Z}}]) & \longrightarrow & \mathrm{H}_1(\mathcal{I}_{g,1}; \mathcal{C}_g^1) & \longrightarrow & \ker(\mathbf{d}) & \longrightarrow & 0 \end{array}$$

with exact rows.

We claim that the map Ψ in this diagram is an isomorphism. Indeed, both \mathcal{I}_g^1 and $\mathcal{I}_{g,1}$ act trivially on $\mathbb{Q}[H_{\mathbb{Z}}]$, so

$$\mathrm{H}_1(\mathcal{I}_g^1; \mathbb{Q}[H_{\mathbb{Z}}]) = \mathrm{H}_1(\mathcal{I}_g^1; \mathbb{Q}) \otimes \mathbb{Q}[H_{\mathbb{Z}}] \quad \text{and} \quad \mathrm{H}_1(\mathcal{I}_{g,1}; \mathbb{Q}[H_{\mathbb{Z}}]) = \mathrm{H}_1(\mathcal{I}_{g,1}; \mathbb{Q}) \otimes \mathbb{Q}[H_{\mathbb{Z}}].$$

¹³The reader might expect to see the coinvariants $\mathrm{H}_1(\mathcal{I}_{g,1}; \mathcal{C}_g)_{\mathcal{I}_{g,1}}$ here, but since $\mathcal{I}_{g,1}$ acts trivially on $\mathrm{H}_1(\mathcal{I}_{g,1}; \mathcal{C}_g)$ we have $\mathrm{H}_1(\mathcal{I}_{g,1}; \mathcal{C}_g)_{\mathcal{I}_{g,1}} = \mathrm{H}_1(\mathcal{I}_{g,1}; \mathcal{C}_g)$, so the coinvariants are not needed.

That Ψ is an isomorphism is thus equivalent to the fact that the map $H_1(\mathcal{I}_g^1; \mathbb{Q}) \rightarrow H_1(\mathcal{I}_{g,1}; \mathbb{Q})$ is an isomorphism, which follows from Johnson's computation of the first homology of the Torelli group [11].

Since Ψ is an isomorphism,¹⁴ a diagram chase shows that the map $\ker(\Phi) \rightarrow \ker(\phi)$ is a surjection. Our goal is to prove that $\ker(\phi)$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$, so by (\spadesuit) it is enough to prove this for $\ker(\Phi)$. This is exactly the content of Corollary 3.6, so we are done. \square

The above only proved part of Theorem A. It remains to prove that $H_1(\mathrm{Mod}_g^1; \mathcal{C}_g^1)$ is finite-dimensional.

Proof of Theorem A for the mapping class group, assuming Theorem 5.1. Consider the exact sequence

$$1 \longrightarrow \mathcal{I}_g^1 \longrightarrow \mathrm{Mod}_g^1 \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1.$$

The associated 5-term exact sequence with coefficients in \mathcal{C}_g^1 contains

$$H_1(\mathcal{I}_g^1; \mathcal{C}_g^1)_{\mathrm{Sp}_{2g}(\mathbb{Z})} \longrightarrow H_1(\mathrm{Mod}_g^1; \mathcal{C}_g^1) \longrightarrow H_1(\mathrm{Sp}_{2g}(\mathbb{Z}); (\mathcal{C}_g^1)_{\mathcal{I}_g^1}) \longrightarrow 0.$$

We proved above that $H_1(\mathcal{I}_g^1; \mathcal{C}_g^1)$ is finite-dimensional, so $H_1(\mathcal{I}_g^1; \mathcal{C}_g^1)_{\mathrm{Sp}_{2g}(\mathbb{Z})}$ is also finite-dimensional. Also, Lemma 3.5 says that $(\mathcal{C}_g^1)_{\mathcal{I}_g^1}$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$. Since $\mathrm{Sp}_{2g}(\mathbb{Z})$ is finitely generated, it follows that $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}); (\mathcal{C}_g^1)_{\mathcal{I}_g^1})$ is finite-dimensional. Plugging all of this into the above exact sequence, we conclude that $H_1(\mathrm{Mod}_g^1; \mathcal{C}_g^1)$ is finite-dimensional, as desired. \square

5.2. Second main theorem. Our second main theorem is:¹⁵

Theorem B. *For $g \geq 4$, both $H_1(\mathrm{Mod}_{g,1}; \mathcal{C}_g)$ and $H_1(\mathcal{I}_{g,1}; \mathcal{C}_g)$ are infinite-dimensional.*

Proof of Theorem B for the Torelli group, assuming Theorem 5.1. Consider the short exact sequence of representations from Lemma 2.4:

$$0 \longrightarrow \mathbb{Q}[H_{\mathbb{Z}}] \longrightarrow \mathcal{C}_g^1 \longrightarrow \mathcal{C}_g \longrightarrow 0.$$

There is an associated long exact sequence in $\mathcal{I}_{g,1}$ -homology that contains the connecting homomorphism $\mathbf{d}: H_1(\mathcal{I}_{g,1}; \mathcal{C}_g) \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$. It is enough to prove that the image of \mathbf{d} is infinite-dimensional. Let π_g be the point-pushing subgroup of $\mathcal{I}_{g,1}$. By looking at the associated long exact sequence in $[\pi_g, \pi_g]$ -homology, we get a connecting homomorphism $\mathbf{r}: H_1([\pi_g, \pi_g]; \mathcal{C}_g) \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$ fitting into a commutative diagram

$$(5.1) \quad \begin{array}{ccc} H_1([\pi_g, \pi_g]; \mathcal{C}_g) & \xrightarrow{\mathbf{r}} & \mathbb{Q}[H_{\mathbb{Z}}] \\ \downarrow & & \parallel \\ H_1(\mathcal{I}_{g,1}; \mathcal{C}_g) & \xrightarrow{\mathbf{d}} & \mathbb{Q}[H_{\mathbb{Z}}] \end{array}$$

Identifying $H_1([\pi_g, \pi_g]; \mathcal{C}_g)$ with $\mathcal{C}_g^{\otimes 2}$, Lemma 4.1 says that \mathbf{r} is the Reidemeister pairing. Theorem 5.1 then implies that $\mathrm{Im}(\mathbf{r})$ is infinite-dimensional, so $\mathrm{Im}(\mathbf{d})$ is also infinite-dimensional. \square

The above only proved part of Theorem B. It remains to prove that $H_1(\mathrm{Mod}_{g,1}; \mathcal{C}_g)$ is infinite-dimensional:

¹⁴Actually, all that is needed is that it is a surjection.

¹⁵Just like above, this is copied from the introduction before we imposed our genus assumption, but we remind the reader that we are assuming throughout the paper that $g \geq 4$ (see Assumption 1.5).

Proof of Theorem B for the mapping class group, assuming Theorem 5.1. We have a short exact sequence

$$1 \longrightarrow \mathcal{I}_{g,1} \longrightarrow \text{Mod}_{g,1} \longrightarrow \text{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1.$$

The associated 5-term exact sequence in homology with coefficients in \mathcal{C}_g contains the segment

$$\mathrm{H}_2(\text{Sp}_{2g}(\mathbb{Z}); (\mathcal{C}_g)_{\mathcal{I}_{g,1}}) \rightarrow \mathrm{H}_1(\mathcal{I}_{g,1}; \mathcal{C}_g)_{\text{Sp}_{2g}(\mathbb{Z})} \rightarrow \mathrm{H}_1(\text{Mod}_{g,1}; \mathcal{C}_g).$$

Since $\mathcal{I}_{g,1}$ contains the point-pushing subgroup π_g , Lemma 2.8 implies that $(\mathcal{C}_g)_{\mathcal{I}_{g,1}} \cong (\wedge^2 H)/\mathbb{Q}$. It follows that

$$\mathrm{H}_2(\text{Sp}_{2g}(\mathbb{Z}); (\mathcal{C}_g)_{\mathcal{I}_{g,1}}) \cong \mathrm{H}_2(\text{Sp}_{2g}(\mathbb{Z}); (\wedge^2 H)/\mathbb{Q})$$

is finite-dimensional. We deduce that it is enough to prove that the $\text{Sp}_{2g}(\mathbb{Z})$ -coinvariants of $\mathrm{H}_1(\mathcal{I}_{g,1}; \mathcal{C}_g)$ are infinite-dimensional. Let $\mathbf{d}: \mathrm{H}_1(\mathcal{I}_{g,1}; \mathcal{C}_g) \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$ be the connecting homomorphism discussed in the previous proof. Since taking coinvariants is right exact, we have a surjection $\mathrm{H}_1(\mathcal{I}_{g,1}; \mathcal{C}_g)_{\text{Sp}_{2g}(\mathbb{Z})} \rightarrow \text{Im}(\mathbf{d})_{\text{Sp}_{2g}(\mathbb{Z})}$. It is thus enough to prove that $\text{Im}(\mathbf{d})_{\text{Sp}_{2g}(\mathbb{Z})}$ is infinite-dimensional.

Consider the short exact sequence of representations

$$0 \longrightarrow \text{Im}(\mathbf{d}) \longrightarrow \mathbb{Q}[H_{\mathbb{Z}}] \longrightarrow \text{coker}(\mathbf{d}) \longrightarrow 0.$$

The associated long exact sequence in $\text{Sp}_{2g}(\mathbb{Z})$ -homology contains

$$\begin{array}{ccccccc} \mathrm{H}_1(\text{Sp}_{2g}(\mathbb{Z}); \text{coker}(\mathbf{d})) & \rightarrow & \mathrm{H}_0(\text{Sp}_{2g}(\mathbb{Z}); \text{Im}(\mathbf{d})) & \rightarrow & \mathrm{H}_0(\text{Sp}_{2g}(\mathbb{Z}); \mathbb{Q}[H_{\mathbb{Z}}]) & \rightarrow & \mathrm{H}_0(\text{Sp}_{2g}(\mathbb{Z}); \text{coker}(\mathbf{d})) \\ & & \parallel & & \parallel & & \parallel \\ & & \text{Im}(\mathbf{d})_{\text{Sp}_{2g}(\mathbb{Z})} & & \mathbb{Q}[H_{\mathbb{Z}}]_{\text{Sp}_{2g}(\mathbb{Z})} & & \text{coker}(\mathbf{d})_{\text{Sp}_{2g}(\mathbb{Z})} \end{array}$$

Using the commutative diagram (5.1), Theorem 5.1 implies that $\text{coker}(\mathbf{d})$ is finite-dimensional. Since $\text{Sp}_{2g}(\mathbb{Z})$ is finitely generated, we see that $\mathrm{H}_1(\text{Sp}_{2g}(\mathbb{Z}); \text{coker}(\mathbf{d}))$ and $\text{coker}(\mathbf{d})_{\text{Sp}_{2g}(\mathbb{Z})}$ are finite-dimensional.

It follows that $\text{Im}(\mathbf{d})_{\text{Sp}_{2g}(\mathbb{Z})}$ is infinite-dimensional if and only if $\mathbb{Q}[H_{\mathbb{Z}}]_{\text{Sp}_{2g}(\mathbb{Z})}$ is infinite-dimensional, so we only need to prove the latter fact. But this is easy: the dimension of $\mathbb{Q}[H_{\mathbb{Z}}]_{\text{Sp}_{2g}(\mathbb{Z})}$ is the cardinality of the set of $\text{Sp}_{2g}(\mathbb{Z})$ -orbits in $H_{\mathbb{Z}}$, and there are infinitely many orbits. Indeed, if $v \in H_{\mathbb{Z}}$ is primitive,¹⁶ then $\{\ell \cdot v \mid \ell \geq 0\}$ is a complete set of orbit representatives. \square

Part 2. Generators for the kernel of the coinvariant Reidemeister pairing

Let $\bar{\tau}$ be the coinvariant Reidemeister pairing. Our remaining task is to prove Theorem 5.1, which says that $\ker(\bar{\tau})$ is a finite-dimensional algebraic representation of $\text{Sp}_{2g}(\mathbb{Z})$ and that $\text{coker}(\bar{\tau})$ is finite-dimensional. In this part of the paper, we calculate $\text{Im}(\bar{\tau})$ and find generators for $\ker(\bar{\tau})$. We will then find some relations in $\ker(\bar{\tau})$ in Part 3 and use these generators and relations to complete the proof of Theorem 5.1. We will outline this part more in the introductory §6 below.

6. INTRODUCTION TO PART 2

This section fixes some notation and does some preliminary calculations, and then outlines what we will do in the rest of Part 2.

¹⁶That is, not divisible by any integers other than ± 1 .

6.1. Conjugation and commutator conventions. Let G be a group. We have been viewing the conjugation action of G on itself as a left action. For $x, y \in G$, we therefore write ${}^y x$ for $yx y^{-1}$. With this notation, we have ${}^z({}^y x) = {}^{zy} x$. For $x, y \in G$ we also write $[x, y] = xyx^{-1}y^{-1}$.

6.2. Notation for group ring. For $h \in H_{\mathbb{Z}}$, let $\{h\}$ denote the corresponding element of $\mathbb{Q}[H_{\mathbb{Z}}]$. Though \mathbb{Z} acts on both $H_{\mathbb{Z}}$ and $\mathbb{Q}[H_{\mathbb{Z}}]$, for $n \in \mathbb{Z}$ the elements $n\{h\}$ and $\{nh\}$ are not the same. For $x \in \pi_g$, let $\bar{x} \in H_{\mathbb{Z}}$ be its image, so $\{\bar{x}\} \in \mathbb{Q}[H_{\mathbb{Z}}]$.

6.3. Notation for \mathcal{C}_g . For $x, y \in \pi_g$, let (x, y) be the element of $\mathcal{C}_g = H_1([\pi_g, \pi_g]; \mathbb{Q})$ corresponding to $[x, y]$. Similarly, for $z \in [\pi_g, \pi_g]$ let (z) be the corresponding element of \mathcal{C}_g . The group π_g has a left action by conjugation on $[\pi_g, \pi_g]$. This descends to a left action of $H_{\mathbb{Z}}$ on \mathcal{C}_g . Since $H_{\mathbb{Z}}$ is abelian, it is harmless¹⁷ to write this with superscripts: for $c \in \mathcal{C}_g$ and $h \in H_{\mathbb{Z}}$, we denote the image of c under the action of h by c^h . For $x, y, z \in \pi_g$, it follows that $(x, y)^{\bar{z}}$ is the element of \mathcal{C}_g corresponding to ${}^z[x, y]$.

6.4. Commutator identities. Commutator identities give identities between the elements $(x, y)^h$. The ones we need are:

Lemma 6.1 (Commutator identities). *Let $x, y, z \in \pi_g$ and $h \in H_{\mathbb{Z}}$. The following hold:*

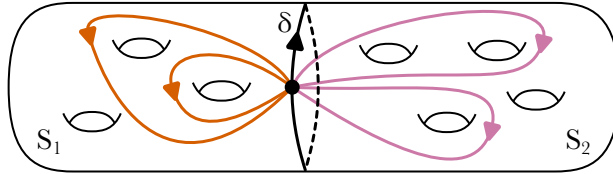
- $(y, x)^h = -(x, y)^h$
- $(xy, z)^h = (x, z)^h + (y, z)^{h+\bar{x}}$
- $(x^{-1}, y)^h = -(x, y)^{h-\bar{x}}$.

Proof. The first follows from the commutator identity $[y, x] = [x, y]^{-1}$. The second follows from the commutator identity $[xy, z] = {}^x[y, z][x, z]$. The third follows from the commutator identity $([x^{-1}, y])({}^{x^{-1}}[x, y]) = 1$. \square

6.5. Separation properties of curves. Say that a collection of elements of π_g are *almost disjoint* if they can be realized so as to only intersect at the basepoint. Also, $\delta \in \pi_g$ is said to *separate* a subset $C_1 \subset \pi_g$ from a subset $C_2 \subset \pi_g$ if:

- δ is a simple closed separating curve¹⁸ that separates Σ_g into subsurfaces S_1 and S_2 , ordered such that S_1 is to the left of δ and S_2 to the right of δ ; and
- for $i = 1, 2$, each curve in C_i can be realized so as to lie in S_i .

See here, where the curves in C_1 and C_2 are in two different colors:



Note that this is *not* symmetric; in fact, if δ separates C_1 from C_2 , then δ^{-1} separates C_2 from C_1 . We also allow δ to be an element of C_1 or C_2 (or both!). For instance, if $\delta \in \pi_g$ is a simple closed separating curve and $\gamma \in \pi_g$ is almost disjoint from δ , then δ separates $\{\delta\}$ from $\{\gamma\}$. In fact, in this case δ even separates $\{\delta\}$ from $\{\delta, \gamma\}$.

For subsets $C_1 \subset \pi_g$ and $C_2 \subset \pi_g$, we say that C_1 and C_2 are *separated* if there exists a $\delta \in \pi_g$ that separates C_1 from C_2 . This implies that each curve in C_1 is almost disjoint from

¹⁷The point here is that since $H_{\mathbb{Z}}$ is abelian, even though this is a left action for $c \in \mathcal{C}_g$ and $h_1, h_2 \in H_{\mathbb{Z}}$ we have $(c^{h_1})^{h_2} = c^{h_1+h_2}$.

¹⁸This implies that $\delta \in [\pi_g, \pi_g]$. Also, like we described in §1.8 when we say that δ is a simple closed separating curve we mean that it can be realized by such a curve.

each curve in C_2 . We will abuse notation in the obvious way and talk about a single $\gamma \in \pi_g$ and a subset $C \subset \pi_g$ being separated, etc.

6.6. Key Reidemeister pairing calculation. Recall that the Reidemeister pairing is the map $\mathfrak{r}: \mathcal{C}_g^{\otimes 2} \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$ defined as follows. Let $\tilde{\Sigma}_g \rightarrow \Sigma_g$ be the universal abelian cover of Σ_g , so $H_{\mathbb{Z}}$ is the deck group of $\tilde{\Sigma}_g$ and $\mathcal{C}_g = H_1(\tilde{\Sigma}_g; \mathbb{Q})$. Let ι be the algebraic intersection pairing on $\mathcal{C}_g = H_1(\tilde{\Sigma}_g; \mathbb{Q})$. Then

$$\mathfrak{r}(x \otimes y) = \sum_{h \in H_{\mathbb{Z}}} \iota(h \cdot x, y) h \quad \text{for } x, y \in \mathcal{C}_g.$$

Our results about the Reidemeister pairing are based on the following calculation:

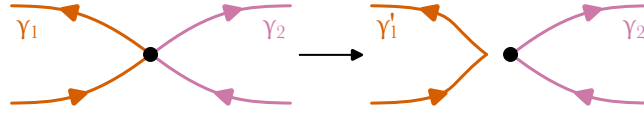
Lemma 6.2. *Let $\mathfrak{r}: \mathcal{C}_g^{\otimes 2} \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$ be the Reidemeister pairing. Then:*

- (a) *If $\gamma_1, \gamma_2 \in [\pi_g, \pi_g]$ are almost disjoint and $h_1, h_2 \in H_{\mathbb{Z}}$, then $\mathfrak{r}((\gamma_1)^{h_1} \otimes (\gamma_2)^{h_2}) = 0$.*
- (b) *If $\delta \in [\pi_g, \pi_g]$ separates $\eta \in \pi_g$ from $\lambda \in \pi_g$, then for arbitrary $h \in H_{\mathbb{Z}}$ we have $\mathfrak{r}((\delta) \otimes (\eta, \lambda)^h) = \{h\} - \{h + \bar{\eta}\} - \{h + \bar{\lambda}\} + \{h + \bar{\eta} + \bar{\lambda}\}$.*

Proof. Let $\rho: \tilde{\Sigma}_g \rightarrow \Sigma_g$ be the universal abelian cover of Σ_g and let ι be the algebraic intersection pairing on $\tilde{\Sigma}_g$. The cover $\tilde{\Sigma}_g$ has a basepoint, and for $x \in [\pi_g, \pi_g]$, the corresponding element $(x) \in \mathcal{C}_g$ is the homology class of the closed curve obtained by lifting the based curve x to $\tilde{\Sigma}_g$ starting at the basepoint of $\tilde{\Sigma}_g$. We prove the two parts separately:

Claim 1. *If $\gamma_1, \gamma_2 \in [\pi_g, \pi_g]$ are almost disjoint and $h_1, h_2 \in H_{\mathbb{Z}}$, then $\mathfrak{r}((\gamma_1)^{h_1} \otimes (\gamma_2)^{h_2}) = 0$.*

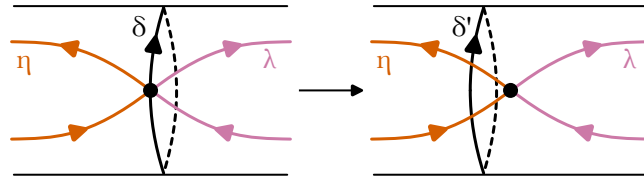
Since γ_1 and γ_2 lie in $[\pi_g, \pi_g]$, their algebraic intersection number on Σ_g is 0. Their single intersection at the basepoint is thus not a transverse intersection, so γ_1 can be freely homotoped to a curve γ'_1 that is disjoint from γ_2 as follows:



Let $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be the lifts of γ_1 and γ_2 to $\tilde{\Sigma}_g$. Lifting the homotopy between γ_1 and γ'_1 to $\tilde{\Sigma}_g$ starting at $\tilde{\gamma}_1$, we get a lift $\tilde{\gamma}'_1$ of γ'_1 that is homotopic to $\tilde{\gamma}_1$. Since γ'_1 and γ_2 are disjoint, for $k_1, k_2 \in H_{\mathbb{Z}}$ the curves $k_1 \cdot \tilde{\gamma}'_1$ and $k_2 \cdot \tilde{\gamma}_2$ are also disjoint. This implies that $\iota(k_1 \cdot [\tilde{\gamma}'_1], k_2 \cdot [\tilde{\gamma}_2]) = \iota(k_1 \cdot [\tilde{\gamma}_1], k_2 \cdot [\tilde{\gamma}_2]) = 0$. We conclude that $\mathfrak{r}((\gamma_1)^{h_1} \otimes (\gamma_2)^{h_2}) = \sum_{k \in H_{\mathbb{Z}}} \iota((k + h_1) \cdot [\tilde{\gamma}_1], h_2 \cdot [\tilde{\gamma}_2]) k = 0$.

Claim 2. *If $\delta \in [\pi_g, \pi_g]$ separates $\eta \in \pi_g$ from $\lambda \in \pi_g$, then for arbitrary $h \in H_{\mathbb{Z}}$ we have $\mathfrak{r}((\delta) \otimes (\eta, \lambda)^h) = \{h\} - \{h + \bar{\eta}\} - \{h + \bar{\lambda}\} + \{h + \bar{\eta} + \bar{\lambda}\}$.*

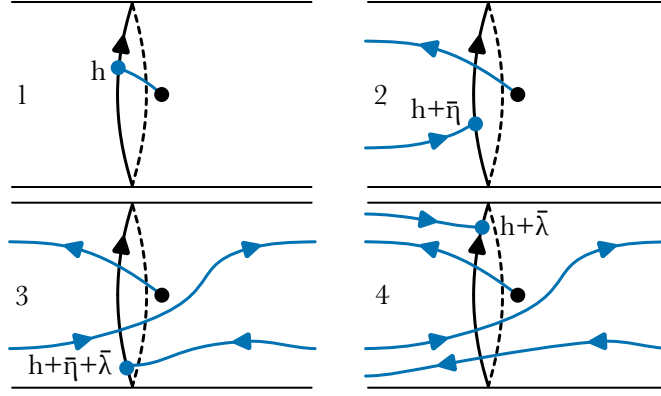
Let δ' be the curve obtained by homotoping δ like this:



Let $\tilde{\delta}$ be the lift of δ to $\tilde{\Sigma}_g$. Lifting the homotopy between δ and δ' to $\tilde{\Sigma}_g$ starting at $\tilde{\delta}$, we get a lift $\tilde{\delta}'$ of δ' that is homotopic to $\tilde{\delta}$. We then have

$$\mathfrak{r}((\delta) \otimes (\eta, \lambda)^h) = \sum_{k \in H_{\mathbb{Z}}} \iota(k \cdot [\delta], (\eta, \lambda)^h) = \sum_{k \in H_{\mathbb{Z}}} \iota(k \cdot [\tilde{\delta}'], (\eta, \lambda)^h) k.$$

Let $\tilde{*}$ be the basepoint of $\tilde{\Sigma}_g$. The homology class $(\eta, \lambda)^h$ is the homology class of the curve obtained by lifting $[\eta, \lambda] = \eta\lambda\eta^{-1}\lambda^{-1}$ to $\tilde{\Sigma}_g$ starting at $h\cdot\tilde{*}$. The curve $[\eta, \lambda]$ intersects δ' four times, so this lift will intersect four different curves of the form $k\cdot\tilde{\delta}'$. See here, which shows the initial segments of $[\eta, \lambda] \in \pi_g$ whose lifts end in those four translates of $\tilde{\delta}'$ (the numbering is the order in which those intersections appear in $[\eta, \lambda]$) and where the end of the lift is labeled with the $k \in H_{\mathbb{Z}}$ such that the lift terminates in $k\cdot\tilde{\delta}'$:



We have perturbed some parts of these initial segments to make the picture easier to read. Examining the signs of those intersections, those labeled by h and $h + \bar{\eta} + \bar{\lambda}$ have a positive sign and those labeled by $h + \bar{\eta}$ and $h + \bar{\lambda}$ have a negative sign. We conclude that

$$\tau((\delta) \otimes (\eta, \lambda)^h) = \{h\} - \{h + \bar{\eta}\} - \{h + \bar{\lambda}\} + \{h + \bar{\eta} + \bar{\lambda}\}. \quad \square$$

6.7. Notation for coinvariant quotient. For $\kappa \in \mathcal{C}_g^{\otimes 2}$, we denote by $\underline{\kappa}$ the associated element of $(\mathcal{C}_g^{\otimes 2})_{\mathcal{I}_{g,1}}$. For example, for $c_1, c_2 \in \mathcal{C}_g$ the image in $(\mathcal{C}_g^{\otimes 2})_{\mathcal{I}_{g,1}}$ of $c_1 \otimes c_2 \in \mathcal{C}_g^{\otimes 2}$ is written $\underline{c_1 \otimes c_2}$. Since the point-pushing subgroup π_g of $\mathcal{I}_{g,1}$ acts on π_g by inner automorphisms, for $h \in H_{\mathbb{Z}}$ the elements $c_1 \otimes c_2$ and $c_1^h \otimes c_2^h$ differ by an element of $\mathcal{I}_{g,1}$ and hence $\underline{c_1 \otimes c_2} = \underline{c_1^h \otimes c_2^h}$. Equivalently, we have $\underline{c_1^h \otimes c_2} = \underline{c_1 \otimes c_2^{-h}}$.

6.8. Main result of Part 2. Let $\bar{\tau}: (\mathcal{C}_g^{\otimes 2})_{\mathcal{I}_{g,1}} \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$ be the coinvariant Reidemeister pairing. Lemma 6.2.(a) implies that $\ker(\bar{\tau})$ contains all elements of the form $(\gamma_1)^{h_1} \otimes (\gamma_2)^{h_2}$ with $\gamma_1, \gamma_2 \in [\pi_g, \pi_g]$ almost disjoint and $h_1, h_2 \in H_{\mathbb{Z}}$. We will prove that these generate the kernel. In fact, we only need elements where γ_1 and γ_2 are separated:

Theorem 6.3. *Let $\bar{\tau}: (\mathcal{C}_g^{\otimes 2})_{\mathcal{I}_{g,1}} \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$ be the coinvariant Reidemeister pairing. Then:*

- *the cokernel of $\bar{\tau}$ is finite-dimensional; and*
- *the kernel of $\bar{\tau}$ is generated by the set of elements of the form $(\gamma_1)^{h_1} \otimes (\gamma_2)^{h_2}$ with $\gamma_1 \in [\pi_g, \pi_g]$ and $\gamma_2 \in [\pi_g, \pi_g]$ separated and $h_1, h_2 \in H_{\mathbb{Z}}$.*

This will be the main theorem of this part of the paper. Recall that our yet-unproven Theorem 5.1 says that $\text{coker}(\bar{\tau})$ is finite-dimensional and that $\ker(\bar{\tau})$ is a finite-dimensional algebraic representation of $\text{Sp}_{2g}(\mathbb{Z})$. The first conclusion of Theorem 6.3 gives the first part of this, and in Part 3 we will use the generators for $\ker(\bar{\tau})$ given by Theorem 6.3 to prove the second part, i.e., that $\ker(\bar{\tau})$ is a finite-dimensional algebraic representation of $\text{Sp}_{2g}(\mathbb{Z})$.

6.9. Outline of Part 2. Before outlining our proof of Theorem 6.3, we rephrase it. Make the following definition:

Definition 6.4. Let \mathcal{Q}_g be the quotient of $(\mathcal{C}_g^{\otimes 2})_{\mathcal{I}_{g,1}}$ by the span of the elements of the form $(\gamma_1)^{h_1} \otimes (\gamma_2)^{h_2}$ with $\gamma_1 \in [\pi_g, \pi_g]$ and $\gamma_2 \in [\pi_g, \pi_g]$ separated and $h_1, h_2 \in H_{\mathbb{Z}}$. \square

By Lemma 6.2.(a), the coinvariant Reidemeister pairing factors through a map $\mathfrak{q}: \mathcal{Q}_g \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$ that we will call the *quotiented Reidemeister pairing*. Theorem 6.3 is equivalent to:

Theorem 6.5. *Let $\mathfrak{q}: \mathcal{Q}_g \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$ be the quotiented Reidemeister pairing. Then \mathfrak{q} is an injection whose image has finite codimension.*

The rest of Part 2 is devoted to the proof of Theorem 6.5. The outline is as follows. In §7, we introduce generators $X(h, x, y)$ for \mathcal{Q}_g . In §8, we construct some relations between the $X(h, x, y)$. In §9, we prove that \mathcal{Q}_g is generated by a certain subset of the $X(h, x, y)$. Finally, in §10 – §13 we prove that these generators go to linearly independent elements of $\mathbb{Q}[H_{\mathbb{Z}}]$ that span a subspace of finite codimension.

7. GENERATORS FOR THE QUOTIENT

This section constructs generators $X(h, x, y)$ for \mathcal{Q}_g .

7.1. Fixing the first curve, I. Recall that $\mathcal{C}_g = H_1([\pi_g, \pi_g]; \mathbb{Q})$. The group \mathcal{Q}_g is a quotient of $(\mathcal{C}_g^{\otimes 2})_{\mathcal{I}_{g,1}}$, which is itself a quotient of $\mathcal{C}_g^{\otimes 2}$. For $\delta \in [\pi_g, \pi_g]$, let $\mathcal{Q}_g[\delta]$ be the image of the map $\mathcal{C}_g \rightarrow \mathcal{Q}_g$ taking $c \in \mathcal{C}_g$ to $\langle \delta \rangle \otimes c$. We then have:

Lemma 7.1. *Let $S \subset [\pi_g, \pi_g]$ be such that $[\pi_g, \pi_g]$ is π_g -normally generated by S . Then \mathcal{Q}_g is spanned by the set of all $\mathcal{Q}_g[\delta]$ with $\delta \in S$.*

Proof. Since $[\pi_g, \pi_g]$ is π_g -normally generated by S , it follows that \mathcal{Q}_g is spanned by

$$\bigcup_{\delta \in S} \bigcup_{x \in \pi_g} \mathcal{Q}_g[x\delta x^{-1}].$$

The point-pushing subgroup π_g of $\mathcal{I}_{g,1}$ acts on π_g by conjugation, so since $\mathcal{I}_{g,1}$ acts trivially on \mathcal{Q}_g we have

$$\mathcal{Q}_g[x\delta x^{-1}] = \mathcal{Q}_g[\delta] \quad \text{for all } x \in \pi_g \text{ and } \delta \in [\pi_g, \pi_g].$$

The lemma follows. \square

7.2. Fixing the first curve, II. The set S we will use in Lemma 7.1 will consist of simple closed separating curves $\delta \in [\pi_g, \pi_g]$. For such δ , the subspace $\mathcal{Q}_g[\delta]$ is spanned by the following elements:

Lemma 7.2. *Let $\delta \in [\pi_g, \pi_g]$ be a simple closed separating curve. Then $\mathcal{Q}_g[\delta]$ is spanned by elements of the form $\langle \delta \rangle \otimes \langle \eta, \lambda \rangle^h$ where δ separates $\eta \in \pi_g$ from $\lambda \in \pi_g$ and $h \in H_{\mathbb{Z}}$.*

Proof. Let S and T be the subsurfaces to the left and right of δ , respectively. The group π_g is generated by $\pi_1(S)$ and $\pi_1(T)$, and thus $[\pi_g, \pi_g]$ is the subgroup of π_g normally generated by $[\pi_1(S), \pi_1(S)]$ and $[\pi_1(T), \pi_1(T)]$ and $[\pi_1(S), \pi_1(T)]$. It follows that $\mathcal{Q}_g[\delta]$ is generated by the following three types of elements:

- Elements of the form $\langle \delta \rangle \otimes \langle \eta_1, \eta_2 \rangle^h$ with $\eta_1, \eta_2 \in \pi_1(S)$ and $h \in H_{\mathbb{Z}}$. Since δ and $[\eta_1, \eta_2]$ are separated, it follows that such $\langle \delta \rangle \otimes \langle \eta_1, \eta_2 \rangle^h$ are 0.
- Elements of the form $\langle \delta \rangle \otimes \langle \lambda_1, \lambda_2 \rangle^h$ with $\lambda_1, \lambda_2 \in \pi_1(T)$ and $h \in H_{\mathbb{Z}}$. Since δ and $[\lambda_1, \lambda_2]$ are separated, it follows that such $\langle \delta \rangle \otimes \langle \lambda_1, \lambda_2 \rangle^h$ are 0.
- Elements of the form $\langle \delta \rangle \otimes \langle \eta, \lambda \rangle^h$ with $\eta \in \pi_1(S)$ and $\lambda \in \pi_1(T)$ and $h \in H_{\mathbb{Z}}$. \square

7.3. **Vanishing.** We now prove certain classes in \mathcal{Q}_g vanish:

Lemma 7.3. *Let $\delta \in [\pi_g, \pi_g]$ be a simple closed separating curve that separates $\mu \in \pi_g$ from $\lambda \in \pi_g$ and let $h \in H_{\mathbb{Z}}$ be arbitrary. Assume that $\bar{\mu} = 0$ or $\bar{\lambda} = 0$. Then $\underline{(\delta)} \otimes \underline{(\mu, \lambda)}^h = 0$.*

Proof. The proofs for $\bar{\mu} = 0$ and $\bar{\lambda} = 0$ are similar, so we will give the details for the case where $\bar{\mu} = 0$. Since $\bar{\mu} = 0$, we have $\mu \in [\pi_g, \pi_g]$ and thus $\langle \mu \rangle$ is well-defined. We have

$$\langle \mu, \lambda \rangle^h = \langle \mu \lambda \mu^{-1} \lambda^{-1} \rangle^h = \langle \mu \rangle - \langle \mu \rangle^{h+\lambda}.$$

This implies that

$$\underline{(\delta)} \otimes \underline{(\mu, \lambda)}^h = \underline{(\delta)} \otimes \underline{(\mu)}^h - \underline{(\delta)} \otimes \underline{(\mu)}^{h+\lambda}.$$

The curves δ and μ are separated, so by Lemma 6.2 both of these terms vanish. The lemma follows. \square

7.4. **Only homology matters.** Recall that our goal (Theorem 6.3) is to prove that the quotiented Reidemeister pairing $\mathfrak{q}: \mathcal{Q}_g \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$ is an injection whose image has finite codimension. Let $\delta \in [\pi_g, \pi_g]$ be a simple closed separating curve that separates $\eta \in \pi_g$ from $\lambda \in \pi_g$ and let $h \in H_{\mathbb{Z}}$, so $\underline{(\delta)} \otimes \underline{(\eta, \lambda)}^h$ is one of the generators for $\mathcal{Q}_g[\delta]$ given by Lemma 7.2. Lemma 6.2 implies that

$$\mathfrak{q}(\underline{(\delta)} \otimes \underline{(\eta, \lambda)}^h) = \{h\} - \{h + \bar{\eta}\} - \{h + \bar{\lambda}\} + \{h + \bar{\eta} + \bar{\lambda}\}.$$

This only depends on $\bar{\eta}$ and $\bar{\lambda}$ and h , so we expect that $\underline{(\delta)} \otimes \underline{(\eta, \lambda)}^h \in \mathcal{Q}_g$ only depends on $\bar{\eta}$ and $\bar{\lambda}$ and h . The following shows that this expectation holds:

Lemma 7.4. *For $i = 1, 2$, let $\delta_i \in [\pi_g, \pi_g]$ be a simple closed separating curve that separates $\eta_i \in \pi_g$ from $\lambda_i \in \pi_g$. Assume that $\bar{\eta}_1 = \bar{\eta}_2$ and $\bar{\lambda}_1 = \bar{\lambda}_2$. Let $h \in H_{\mathbb{Z}}$. Then $\underline{(\delta_1)} \otimes \underline{(\eta_1, \lambda_1)}^h = \underline{(\delta_2)} \otimes \underline{(\eta_2, \lambda_2)}^h$.*

Proof. Set $x = \bar{\eta}_1 = \bar{\eta}_2$ and $y = \bar{\lambda}_1 = \bar{\lambda}_2$. We start by proving a special case of the lemma:

Claim 1. *If $\delta_1 = \delta_2$, then $\underline{(\delta_1)} \otimes \underline{(\eta_1, \lambda_1)}^h = \underline{(\delta_2)} \otimes \underline{(\eta_2, \lambda_2)}^h$.*

Proof of claim. Let $\delta = \delta_1 = \delta_2$. We will prove that $\underline{(\delta)} \otimes \underline{(\eta_1, \lambda_1)}^h = \underline{(\delta)} \otimes \underline{(\eta_2, \lambda_1)}^h$. The proof that this then equals $\underline{(\delta)} \otimes \underline{(\eta_2, \lambda_2)}^h$ is identical. Recall that $x = \bar{\eta}_1 = \bar{\eta}_2$ and $y = \bar{\lambda}_1 = \bar{\lambda}_2$. Using our commutator identities (Lemma 6.1), we have

$$\langle \eta_1, \lambda_1 \rangle^h - \langle \eta_2, \lambda_1 \rangle^h = \langle \eta_1, \lambda_1 \rangle^h + \langle \eta_2^{-1}, \lambda_1 \rangle^{h+x} = \langle \eta_1 \eta_2^{-1}, \lambda_1 \rangle^h.$$

We have $\eta_1 \eta_2^{-1} \in [\pi_g, \pi_g]$, so

$$\langle \eta_1 \eta_2^{-1}, \lambda_1 \rangle^h = \langle (\eta_1 \eta_2^{-1}) \lambda_1 (\eta_1 \eta_2^{-1})^{-1} \lambda_1^{-1} \rangle^h = \langle \eta_1 \eta_2^{-1} \rangle^h - \langle \eta_1 \eta_2^{-1} \rangle^{h+y}.$$

The curves δ and $\eta_1 \eta_2^{-1}$ are separated, so we conclude that

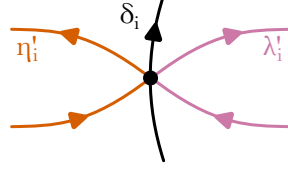
$$\begin{aligned} \underline{(\delta)} \otimes \underline{(\eta_1, \lambda_1)}^h - \underline{(\delta)} \otimes \underline{(\eta_2, \lambda_1)}^h &= \underline{(\delta)} \otimes \underline{(\eta_1 \eta_2^{-1})}^h - \underline{(\eta_1 \eta_2^{-1})}^{h+y} \\ &= \underline{(\delta)} \otimes \underline{(\eta_1 \eta_2^{-1})}^h - \underline{(\delta)} \otimes \underline{(\eta_1 \eta_2^{-1})}^{h+y} \\ &= 0 - 0 = 0. \end{aligned} \quad \square$$

We now turn to the general case. If either $x = 0$ or $y = 0$, then we are done by Lemma 7.3, so assume that neither are 0. Write $x = kx'$ and $y = \ell y'$ with $k, \ell \geq 1$ and $x', y' \in H_{\mathbb{Z}}$ primitive. Since x' and y' are primitive, for $i = 1, 2$ we can choose the following (see [22]):

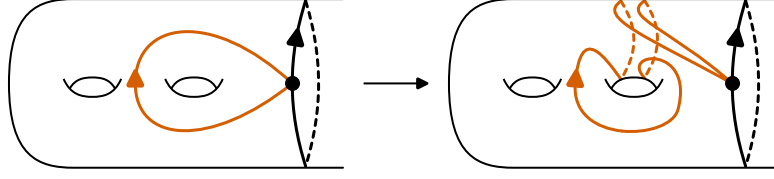
- in the component containing η_i of Σ_g cut open along δ_i , a nonseparating simple closed curve $\eta'_i \in \pi_g$ with $\bar{\eta}'_i = x'$; and

- in the component containing λ_i of Σ_g cut open along δ_i , a nonseparating simple closed curve $\lambda'_i \in \pi_g$ with $\bar{\lambda}'_i = y'$.

By freely homotoping η'_i and λ'_i to other based curves, we can assume that a regular neighborhood of the basepoint looks like this (where the key property is the cyclic order in which the curves enter and leave the basepoint):



For instance, such a homotopy might move η'_i as follows:



For $i = 1, 2$, the homology classes of η_i and $(\eta'_i)^k$ are the same, and also the homology classes of λ_i and $(\lambda'_i)^\ell$ are the same. Using Claim 1, we can therefore assume without loss of generality that $\eta_i = (\eta'_i)^k$ and $\lambda_i = (\lambda'_i)^\ell$.

Farb–Margalit’s “change of coordinates principle” [4, §1.3] implies that there exists some $f \in \text{Mod}_{g,1}$ with $f(\eta'_1) = \eta'_2$ and $f(\lambda'_1) = \lambda'_2$. In fact, using the argument from the proof of [17, Lemma 6.2] we can actually find such an f in $\mathcal{I}_{g,1}$. Since $\mathcal{I}_{g,1}$ acts trivially on \mathcal{Q}_g , we thus have

$$\begin{aligned} \underline{\langle \delta_1 \rangle \otimes \langle \eta_1, \lambda_1 \rangle^h} &= \underline{\langle \delta_1 \rangle \otimes \langle (\eta'_1)^k, (\lambda'_1)^\ell \rangle^h} \\ &= \underline{\langle f(\delta_1) \rangle \otimes \langle (\eta'_2)^k, (\lambda'_2)^\ell \rangle^h} \\ &= \underline{\langle f(\delta_1) \rangle \otimes \langle \eta_2, \lambda_2 \rangle^h}. \end{aligned}$$

We can therefore assume without loss of generality that $\eta_1 = \eta_2$ and $\lambda_1 = \lambda_2$. When doing this, we replace δ_1 with $f(\delta_1)$. We have therefore reduced the proof to:

Claim 2. *Assume that*

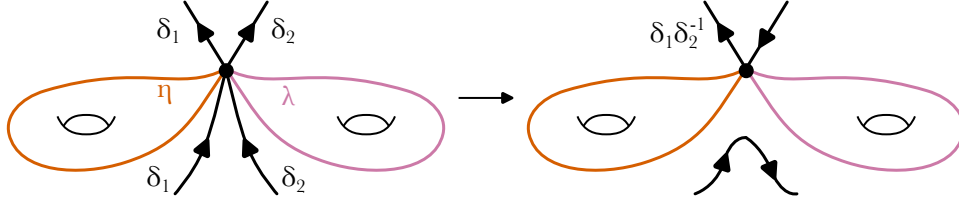
- $\eta = (\eta')^k$ with $\eta' \in \pi_g$ a nonseparating simple closed curve; and
- $\lambda = (\lambda')^\ell$ with $\lambda' \in \pi_g$ a nonseparating simple closed curve; and
- for $i = 1, 2$, the simple closed separating curve $\delta_i \in [\pi_g, \pi_g]$ separates η from λ .

Then $\underline{\langle \delta_1 \rangle \otimes \langle \eta, \lambda \rangle^h} = \underline{\langle \delta_2 \rangle \otimes \langle \eta, \lambda \rangle^h}$.

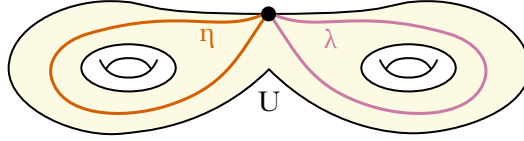
Proof of claim. We have

$$\underline{\langle \delta_1 \rangle \otimes \langle \eta, \lambda \rangle^h} - \underline{\langle \delta_2 \rangle \otimes \langle \eta, \lambda \rangle^h} = \underline{\langle \delta_1 \delta_2^{-1} \rangle \otimes \langle \eta, \lambda \rangle^h}.$$

We want to prove this vanishes. To do this, it is enough to prove that $\delta_1 \delta_2^{-1}$ can be written as a product of elements of $[\pi_g, \pi_g]$ that are separated from $[\eta, \lambda] \in [\pi_g, \pi_g]$. The loops in question look like those in the following figure:



We have not drawn the rest of the δ_i since while individually they are simple closed curves, they potentially intersect each other, and the loop $\delta_1 \delta_2^{-1}$ potentially has self-intersections. Let U be a 3-holed sphere embedded in Σ_g such that one of the boundary components of U contains the basepoint, the loops η and λ lie in U , and $\delta_1 \delta_2^{-1}$ only intersects U in the basepoint:



Set $S = \Sigma_g \setminus \text{Int}(U)$, so $S \cong \Sigma_{g-2}^3$. Regard $\pi_1(S)$ as a subgroup of π_g , so $\delta_1 \delta_2^{-1} \in \pi_1(S)$. Since the map $H_1(S) \rightarrow H_1(\Sigma_g)$ is injective, the intersection of $[\pi_g, \pi_g]$ with $\pi_1(S)$ is $[\pi_1(S), \pi_1(S)]$. It follows that $\delta_1 \delta_2^{-1} \in [\pi_1(S), \pi_1(S)]$.

Recall our standing assumption that $g \geq 4$ (Assumption 1.5). This implies that the genus of S is positive. It follows (see [17, Lemma A.1]) that $[\pi_1(S), \pi_1(S)]$ is generated by based isotopy classes of simple closed separating curves that cut off one-holed tori. We can therefore write $\delta_1 \delta_2^{-1}$ as a product of such curves. These are all separated from $[\eta, \lambda]$, and we are done. \square

This completes the proof of Lemma 7.4. \square

7.5. Generators. A *symplectic splitting* of $H_{\mathbb{Z}}$ is a decomposition $H_{\mathbb{Z}} = X \oplus Y$ that is orthogonal with respect to the intersection form. For a symplectic splitting $H_{\mathbb{Z}} = X \oplus Y$, by work of Johnson [8] we can find a simple closed separating curve $\delta \in \pi_g$ such that if S (resp. T) is the subsurface to the left (resp. right) of δ , then $H_1(S) = X$ and $H_1(T) = Y$. We will say that δ *induces* the symplectic splitting $H_{\mathbb{Z}} = X \oplus Y$. For a simple closed separating curve $\delta \in [\pi_g, \pi_g]$, we will write $H_{\mathbb{Z}} = X(\delta) \oplus Y(\delta)$ for the symplectic splitting induced by δ .

Consider elements $\{x_1, \dots, x_k\}$ and $\{y_1, \dots, y_\ell\}$ of $H_{\mathbb{Z}}$. We say that the x_i and y_j are *homologically separate* if there exists a symplectic splitting $H_{\mathbb{Z}} = X \oplus Y$ with $x_i \in X$ and $y_j \in Y$ for all i and j .

Let $x \in H_{\mathbb{Z}}$ and $y \in H_{\mathbb{Z}}$ be homologically separate. By our discussion above, we can find a simple closed separating curve δ with $x \in X(\delta)$ and $y \in Y(\delta)$. Let S and T be the subsurfaces to the left and right of δ , respectively, so $X(\delta) = H_1(S)$ and $Y(\delta) = H_1(T)$. We can find $\eta, \lambda \in \pi_g$ with $\bar{\eta} = x$ and $\bar{\lambda} = y$ such that δ separates η from λ . Indeed, choose η lying in S with $\bar{\eta} = x$ and λ lying in T with $\bar{\lambda} = y$. For $h \in H_{\mathbb{Z}}$, we define

$$X(h, x, y) = \langle \delta \rangle \otimes \langle \eta, \lambda \rangle^h \in \mathcal{Q}_g.$$

By Lemma 7.4, this does not depend on any of our choices. By Lemma 6.2, this satisfies

$$q(X(h, x, y)) = \{h\} - \{h + x\} - \{h + y\} + \{h + x + y\}.$$

7.6. Summary. The following summarizes what we have accomplished in this section:

Lemma 7.5. *Let $S \subset [\pi_g, \pi_g]$ be a set of simple closed separating curves such that $[\pi_g, \pi_g]$ is π_g -normally generated by S . Then \mathcal{Q}_g is spanned by*

$$\bigcup_{\delta \in S} \{X(h, x, y) \mid x \in X(\delta) \text{ and } y \in Y(\delta) \text{ and } h \in H_{\mathbb{Z}}\}.$$

Proof. Immediate from Lemmas 7.1 and 7.2 along with the definition of $X(h, x, y)$. \square

8. RELATIONS IN THE QUOTIENT

We want to prove that the quotiented Reidemeister pairing $\mathfrak{q}: \mathcal{Q}_g \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$ is injective. For $x \in H_{\mathbb{Z}}$ and $y \in H_{\mathbb{Z}}$ homologically separate and $h \in H_{\mathbb{Z}}$, Lemma 6.2 implies that

$$\mathfrak{q}(X(h, x, y)) = \{h\} - \{h + x\} - \{h + y\} + \{h + x + y\}.$$

Our calculations will use the following five relations. It is enlightening to verify that \mathfrak{q} takes these to relations in $\mathbb{Q}[H_{\mathbb{Z}}]$.

Lemma 8.1 (Vanishing relation). *For all $h, x, y \in H_{\mathbb{Z}}$ we have $X(h, x, 0) = X(h, 0, y) = 0$.*

Proof. Immediate from Lemma 7.3. \square

Lemma 8.2 (Symmetry relation¹⁹). *Let $h \in H_{\mathbb{Z}}$, and let $x \in H_{\mathbb{Z}}$ and $y \in H_{\mathbb{Z}}$ be homologically separate. Then $X(h, x, y) = X(h, y, x)$.*

Proof. Let $H_{\mathbb{Z}} = X \oplus Y$ be a symplectic splitting with $x \in X$ and $y \in Y$. Let δ be a simple closed separating curve inducing the splitting $H_{\mathbb{Z}} = X \oplus Y$, and let $\eta, \lambda \in \pi_g$ be such that $\bar{\eta} = x$ and $\bar{\lambda} = y$ and such that δ separates η from λ . We then have

$$X(h, x, y) = \langle \delta \rangle \otimes \langle \eta, \lambda \rangle^h.$$

By definition, η lies in the subsurface to the left of δ and λ lies in the subsurface to the right of δ . Reversing the orientation of δ , we see that δ^{-1} separates λ from η . Lemma 6.1 (commutator identities) says that $\langle \lambda, \eta \rangle^h = -\langle \eta, \lambda \rangle^h$, so

$$X(h, y, x) = \langle \delta^{-1} \rangle \otimes \langle \lambda, \eta \rangle^h = \langle -\delta \rangle \otimes \langle -\eta, \lambda \rangle^h = \langle \delta \rangle \otimes \langle \eta, \lambda \rangle^h = X(h, x, y). \quad \square$$

Lemma 8.3 (Additivity relation). *Let $h \in H_{\mathbb{Z}}$, and let $x_1, x_2 \in H_{\mathbb{Z}}$ and $y \in H_{\mathbb{Z}}$ be homologically separate. Then*

$$X(h, x_1 + x_2, y) = X(h, x_1, y) + X(h + x_1, x_2, y).$$

Proof. Let $H_{\mathbb{Z}} = X \oplus Y$ be a symplectic splitting with $x_1, x_2 \in X$ and $y \in Y$. Let $\delta \in \pi_g$ be a simple closed separating curve inducing the splitting $H_{\mathbb{Z}} = X \oplus Y$. Let $\eta_1, \eta_2, \lambda \in \pi_g$ be such that $\bar{\eta}_i = x_i$ and $\bar{\lambda} = y$ and such that δ separates $\{\eta_1, \eta_2\}$ from λ . We then have

$$X(h, x_1 + x_2, y) = \langle \delta \rangle \otimes \langle \eta_1 \eta_2, \lambda \rangle^h.$$

Using Lemma 6.1 (commutator identities), this equals

$$\langle \delta \rangle \otimes (\langle \eta_1, \lambda \rangle^h + \langle \eta_2, \lambda \rangle^{h+x_1}) = X(h, x_1, y) + X(h + x_1, x_2, y). \quad \square$$

Lemma 8.4 (Inverse relation). *Let $h \in H_{\mathbb{Z}}$, and let $x \in H_{\mathbb{Z}}$ and $y \in H_{\mathbb{Z}}$ be homologically separate. Then $X(h, -x, y) = -X(h - x, x, y)$.*

¹⁹Though we will usually name the relations we are using, we will use the symmetry relation freely and without mention.

Proof. Let $H_{\mathbb{Z}} = X \oplus Y$ be a symplectic splitting with $x \in X$ and $y \in Y$. Let δ be a simple closed separating curve inducing the splitting $H_{\mathbb{Z}} = X \oplus Y$, and let $\eta, \lambda \in \pi_g$ be such that $\bar{\eta} = x$ and $\bar{\lambda} = y$ and such that δ separates η from λ . Using Lemma 6.1 (commutator identities), we then have

$$X(h, -x, y) = \langle \delta \rangle \otimes \langle \eta^{-1}, \lambda \rangle^h = \langle \delta \rangle \otimes \langle -\langle \eta, \lambda \rangle^{h-x} \rangle = -X(h-x, x, y). \quad \square$$

Lemma 8.5 (Cube relation²⁰). *Let $h \in H_{\mathbb{Z}}$, and let $x, k \in H_{\mathbb{Z}}$ and $y \in H_{\mathbb{Z}}$ be homologically separate. Then²¹ $X(h+k, x, y) = X(h, x, y) - X(h, k, y) + X(h+x, k, y)$*

Proof. We apply the additivity relation (Lemma 8.3) in two ways:

$$\begin{aligned} X(h, k+x, y) &= X(h, k, y) + X(h+k, x, y) \\ X(h, x+k, y) &= X(h, x, y) + X(h+x, k, y). \end{aligned}$$

Comparing these, we see that $X(h, k, y) + X(h+k, x, y) = X(h, x, y) + X(h+x, k, y)$. Rearranging this gives the desired relation. \square

9. A SPECIFIC GENERATING SET FOR THE QUOTIENT

Choose a symplectic basis $\mathfrak{B} = \{a_1, b_1, \dots, a_g, b_g\}$ for $H_{\mathbb{Z}}$. This basis will be fixed for the remainder of Part 2. Let $\omega(-, -)$ be the algebraic intersection pairing on $H_{\mathbb{Z}}$ and let \perp be the orthogonal complement with respect to $\omega(-, -)$. Define $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$, where \mathcal{V}_i consists of all $X(h, x, y)$ with $h \in H_{\mathbb{Z}}$ arbitrary and $x, y \in H_{\mathbb{Z}}$ nonzero satisfying:

- (\mathcal{V}_1): $x \in \langle a_d, b_d \rangle$ and $y \in \langle a_d, b_d \rangle^{\perp}$ for some $1 \leq d \leq g$.
- (\mathcal{V}_2): $x \in \langle a_d, b_d \rangle$ and $y = x + z$ with $z \in \{\pm a_e, \pm b_e\}$ for some distinct $1 \leq d, e \leq g$.
- (\mathcal{V}_3): the pair (x, y) is either $(a_d + a_e, b_d - b_e)$ or $(a_d + b_e, b_d + a_e)$ for some distinct $1 \leq d, e \leq g$. Note that in both cases the elements x and y are homologically separate and satisfy $\omega(x, y) = 0$.

Remark 9.1. In \mathcal{V}_3 , we do not include $(a_d - a_e, b_d + b_e)$ or $(a_d - b_e, b_d - a_e)$. It is enlightening to go through the proof of Lemma 9.2 below to see why they are not needed to generate \mathcal{Q}_g . \square

Our main result about \mathcal{V} is:

Lemma 9.2. *The vector space \mathcal{Q}_g is spanned by \mathcal{V} .*

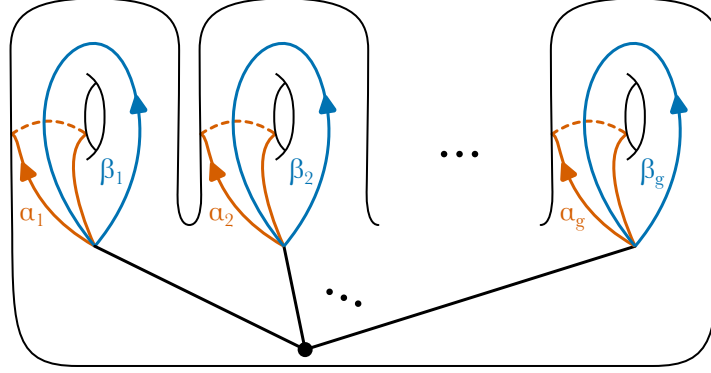
Proof. By Lemma 7.5, it is enough to find a set $S \subset [\pi_g, \pi_g]$ of simple closed separating curves that π_g -normally generate S such that for all $\delta \in S$ we have:

- (\dagger) letting $H_{\mathbb{Z}} = X(\delta) \oplus Y(\delta)$ be the induced symplectic splitting, each $X(h, x, y)$ with $x \in X(\delta)$ and $y \in Y(\delta)$ and $h \in H_{\mathbb{Z}}$ lies in the span of \mathcal{V} .

²⁰We call this the cube relation since \mathfrak{q} takes the terms it to the points in $\mathbb{Q}[H_{\mathbb{Z}}]$ forming a ‘‘cube’’ with a corner at h in the directions of x, y , and k , namely $\{h, h+x, h+y, h+k, h+x+y, h+x+k, h+y+k, h+x+y+k\}$.

²¹We think of this relation as relating $X(h, x, y)$ to $X(h+k, x, y)$, with the blue terms $X(h, k, y)$ and $X(h+x, k, y)$ being the ‘‘error’’.

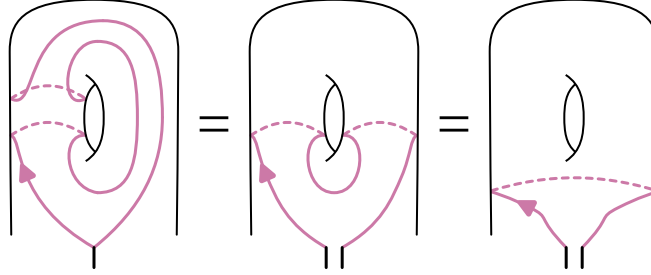
Let $B = \{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$ be the following standard generating set for π_g :



Recalling that $\mathfrak{B} = \{a_1, b_1, \dots, a_g, b_g\}$, we can choose B such that $\bar{\alpha}_i = a_i$ and $\bar{\beta}_i = b_i$ for all $1 \leq i \leq g$. Set

$$S' = \{[\alpha_d, \beta_d] \mid 1 \leq d \leq g\} \cup \{[\alpha_d, \alpha_e], [\alpha_d, \beta_e], [\beta_d, \alpha_e], [\beta_d, \beta_e] \mid 1 \leq d < e \leq g\}.$$

The group $[\pi_g, \pi_g]$ is π_g -normally generated by the elements of S' . Unfortunately, most elements of S' are not simple closed separating curves. Indeed, the $[\alpha_d, \beta_d] = \alpha_d \beta_d \alpha_d^{-1} \beta_d^{-1}$ are the only ones that are:



More generally, an *ab-pair* of curves consists of $\zeta, \eta \in \pi_g$ that are simple closed curves that only intersect at the basepoint and have algebraic intersection number ± 1 . For instance, the curves $\alpha_d, \beta_d \in \pi_g$ form an *ab-pair*. For an *ab-pair* of curves ζ and η , the commutator $[\zeta, \eta] \in \pi_g$ is a simple closed separating curve.

To fix the above issue, we will do the following: for each $\delta' \in S'$, we will write δ' as a product of simple closed separating curves δ such that a conjugate of δ satisfies (\dagger) . The desired S will consist of the (\dagger) -satisfying conjugates of all the δ that appear in these products. During this, we will freely use the relations from §8.

We start with $\delta' = [\alpha_d, \beta_d]$ for some $1 \leq d \leq g$. Since α_d and β_d form an *ab-pair*, δ' is already a simple closed separating curve, so for our product we take $\delta = \delta'$. We must show that δ satisfies (\dagger) . We have²²

$$H_{\mathbb{Z}} = X(\delta) \oplus Y(\delta) = \langle a_d, b_d \rangle^{\perp} \oplus \langle a_d, b_d \rangle.$$

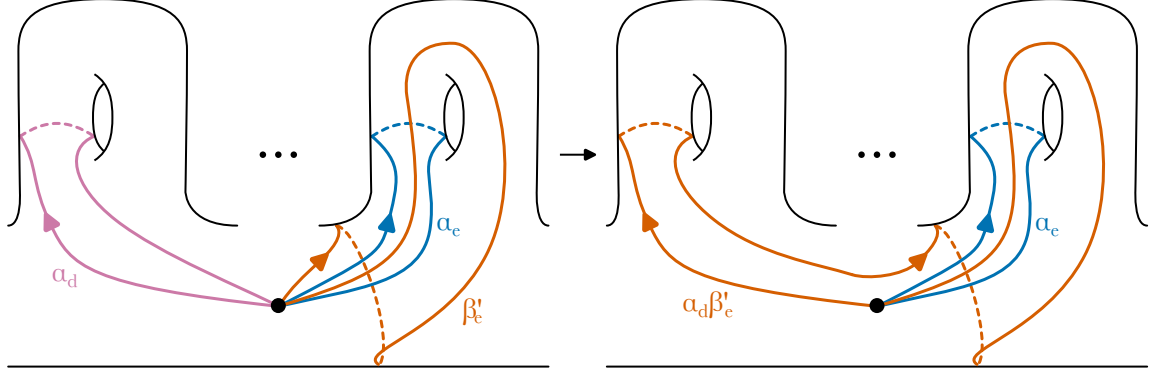
Each $X(h, x, y)$ with $x \in X(\delta)$ and $y \in Y(\delta)$ and $h \in H_{\mathbb{Z}}$ is either²³ 0 or an element of \mathcal{V}_1 , verifying (\dagger) .

There are now four remaining cases: for $1 \leq d < e \leq g$, we either have $\delta' = [\alpha_d, \alpha_e]$ or $\delta' = [\alpha_d, \beta_e]$ or $\delta' = [\beta_d, \alpha_e]$ or $\delta' = [\beta_d, \beta_e]$. All four cases are handled similarly, so we will give full details for $\delta' = [\alpha_d, \alpha_e]$ and then sketch the remaining cases.

²²It is annoying that $\langle a_d, b_d \rangle$ comes second, but this is forced by our convention that $X(\delta)$ is the homology of the subsurface to the left of δ .

²³This holds when $x = 0$ or $y = 0$; see the vanishing relation (Lemma 8.1).

We want to write $[\alpha_d, \alpha_d]$ as a product of simple closed separating curves. The intuition guiding our calculation is that the commutator bracket is very similar to an alternating bilinear pairing. Because of this, we should be able to write $[\alpha_d, \alpha_e]$ as a product of terms involving $[\alpha_d \beta'_e, \alpha_e]$ and $[\alpha_e, \beta'_e] = [\beta'_e, \alpha_e]^{-1}$ for any $\beta'_e \in \pi_g$. Choosing β'_e as in the following figure, the curves $\alpha_d \beta'_e$ and α_e (resp. α_e and β'_e) form an ab -pair, so $[\alpha_d \beta'_e, \alpha_e]$ (resp. $[\alpha_e, \beta'_e]$) is a simple closed separating curve:



The desired formula for $\delta' = [\alpha_d, \alpha_e]$ is²⁴

$$\delta' = [\alpha_d, \alpha_e] = {}^{\alpha_d}[\alpha_e, \beta'_e][\alpha_d \beta'_e, \alpha_e].$$

We must check that conjugates of $\delta = {}^{\alpha_d}[\alpha_e, \beta'_e]$ and $\delta = [\alpha_d \beta'_e, \alpha_e]$ satisfy (\dagger) . We will check that in fact $\delta = [\alpha_e, \beta'_e]$ and $\delta = [\alpha_d \beta'_e, \alpha_e]$ satisfy (\dagger) .

The curves $\alpha_e, \beta'_e \in \pi_g$ form an ab -pair, so $[\alpha_e, \beta'_e]$ is a simple closed separating curve. We also have $\overline{\beta'_e} = \overline{\beta_e} = b_e$. The exact same argument we used above for $[\alpha_e, \beta_e]$ now also verifies (\dagger) for $\delta = [\alpha_e, \beta'_e]$.

Now consider $\delta = [\alpha_d \beta'_e, \alpha_e]$. Again, since $\alpha_d \beta'_e$ and α_e form an ab -pair, this is a simple closed separating curve. The homology classes of $\{\alpha_d \beta'_e, \alpha_e\}$ are $\{a_d + b_e, a_e\}$, so

$$\begin{aligned} H_{\mathbb{Z}} &= X(\delta) \oplus Y(\delta) = \langle a_d + b_e, a_e \rangle^{\perp} \oplus \langle a_d + b_e, a_e \rangle \\ &= \langle a_d, b_d + a_e, a_r, b_r \mid 1 \leq r \leq g, r \neq d, e \rangle \oplus \langle a_d + b_e, a_e \rangle. \end{aligned}$$

Using our relations, each $X(h, x, y)$ with $x \in X(\delta)$ and $y \in Y(\delta)$ and $h \in H_{\mathbb{Z}}$ can be written as a linear combination of elements of the form $X(h', x', y')$ with

$$x' \in \{a_d, b_d + a_e, a_r, b_r \mid 1 \leq r \leq g, r \neq d, e\} \quad \text{and} \quad y' \in \{a_d + b_e, a_e\} \quad \text{and} \quad h' \in H_{\mathbb{Z}}.$$

These are either elements of \mathcal{V}_1 , or fall into one of the following cases:

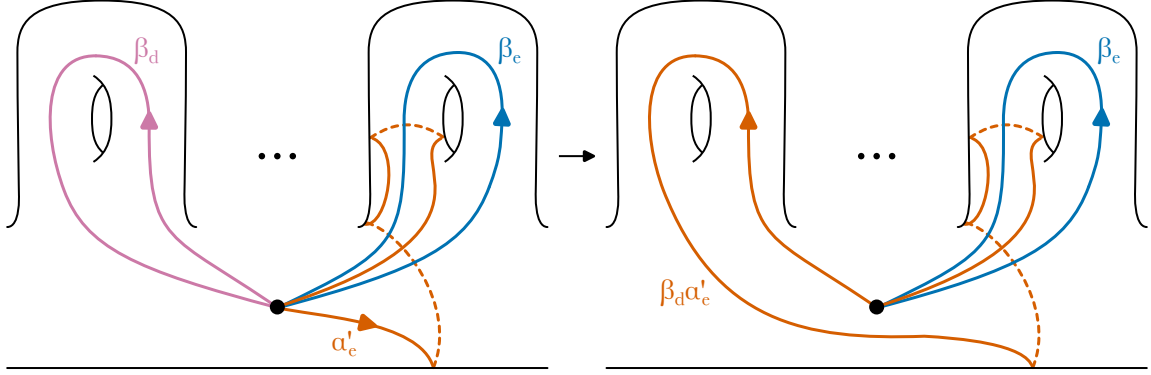
- $X(h', a_d, a_d + b_e)$, which lies in \mathcal{V}_2 .
- $X(h', b_d + a_e, a_e) = X(h', a_e, a_e + b_d)$, which lies in \mathcal{V}_2 .
- $X(h', b_d + a_e, a_d + b_e) = X(h', a_d + b_e, b_d + a_e)$, which lies in \mathcal{V}_3 .

This completes the proof of (\dagger) for $\delta = [\alpha_d \beta'_e, \alpha_e]$, and thus verifies what we must show for $\delta' = [\alpha_d, \alpha_e]$.

We must also handle $\delta' \in \{[\alpha_d, \beta_e], [\beta_d, \alpha_e], [\beta_d, \beta_d]\}$. These are all similar to $[\alpha_d, \alpha_e]$:

- For $\delta' = [\alpha_d, \beta_e]$, use $[\alpha_d, \beta_e] = {}^{\alpha_d}[\beta_e, \alpha_e][\alpha_d \alpha_e, \beta_e]$.
- For $\delta' = [\beta_d, \alpha_e]$, use $[\beta_d, \alpha_e] = [\beta_d, \alpha_e \alpha_d] {}^{\alpha_e}[\alpha_d, \beta_d]$.
- For $\delta' = [\beta_d, \beta_e]$, use $[\beta_d, \beta_e] = {}^{\beta_d}[\beta_e, \alpha'_e][\beta_d \alpha'_e, \beta_e]$ where α'_e is as shown in the following figure:

²⁴Recall from §6.1 that for G a group and $c, d \in G$, we use the notation ${}^d c = dcd^{-1}$ and $[c, d] = cdc^{-1}d^{-1}$.



This completes the proof of the lemma. \square

10. CALCULATION OF THE QUOTIENT, OUTLINE

Recall from §6 that our goal in this part of the paper is to prove the following:

Theorem 6.5. *Let $\mathfrak{q}: \mathcal{Q}_g \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$ be the quotiented Reidemeister pairing. Then \mathfrak{q} is an injection whose image has finite codimension.*

In the previous section, we constructed a generating set $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$ for \mathcal{Q}_g . In the next three sections, we use our generating set to prove Theorem 6.5. The outline is:

- In §11, we prove that the restriction of \mathfrak{q} to $\langle \mathcal{V}_1 \rangle$ is an injection.
- Now define $\mathcal{Q}_{g/1} = \mathcal{Q}_g / \langle \mathcal{V}_1 \rangle$. There is an induced map

$$\mathfrak{q}_1: \mathcal{Q}_{g/1} \rightarrow \mathbb{Q}[H_{\mathbb{Z}}] / \langle \mathfrak{q}(\mathcal{V}_1) \rangle.$$

Let $\mathcal{V}_{2/1}$ be the image of \mathcal{V}_2 in $\mathcal{Q}_{g/1}$. In §12, we prove that the restriction of \mathfrak{q}_1 to $\langle \mathcal{V}_{2/1} \rangle$ is an injection.

- Finally, define $\mathcal{Q}_{g/2} = \mathcal{Q}_g / \langle \mathcal{V}_1, \mathcal{V}_2 \rangle$. There is an induced map

$$\mathfrak{q}_2: \mathcal{Q}_{g/2} \rightarrow \mathbb{Q}[H_{\mathbb{Z}}] / \langle \mathfrak{q}(\mathcal{V}_1), \mathfrak{q}(\mathcal{V}_2) \rangle.$$

The vector space $\mathcal{Q}_{g/2}$ is spanned by the image of \mathcal{V}_3 , and in §13 we prove that \mathfrak{q}_2 is an injection.

Together the above will imply that \mathfrak{q} is an injection. To make the calculations possible, we will also have to control the quotients

$$\mathbb{Q}[H_{\mathbb{Z}}] / \langle \mathfrak{q}(\mathcal{V}_1) \rangle \quad \text{and} \quad \mathbb{Q}[H_{\mathbb{Z}}] / \langle \mathfrak{q}(\mathcal{V}_1), \mathfrak{q}(\mathcal{V}_2) \rangle.$$

What we will show is that they can be identified with subspaces $\mathbb{Q}[S]$ of $\mathbb{Q}[H_{\mathbb{Z}}]$ associated to subsets²⁵ $S \subset H_{\mathbb{Z}}$:

- $\mathbb{Q}[H_{\mathbb{Z}}] / \langle \mathfrak{q}(\mathcal{V}_1) \rangle$ will be identified with $\mathbb{Q}[\cup_{i=1}^g \langle a_i, b_i \rangle]$; and
- $\mathbb{Q}[H_{\mathbb{Z}}] / \langle \mathfrak{q}(\mathcal{V}_1), \mathfrak{q}(\mathcal{V}_2) \rangle$ will be identified with $\mathbb{Q}[\cup_{i=1}^g \{0, a_i, b_i, a_i + b_i\}]$.

The second identification implies that $\text{Im}(\mathfrak{q})$ has finite codimension, completing the proof of Theorem 6.5.

²⁵These S are just subsets of $H_{\mathbb{Z}}$. They are not closed under addition. The notation $\mathbb{Q}[S]$ simply means the set of formal \mathbb{Q} -linear combinations of elements of S .

11. CALCULATION OF THE QUOTIENT I: THE FIRST SET OF GENERATORS

We start with the first step from the outline in §10. Recall that the generating set \mathcal{V} depends on a fixed symplectic basis $\mathfrak{B} = \{a_1, b_1, \dots, a_g, b_g\}$ for $H_{\mathbb{Z}}$, and \mathcal{V}_1 is the set of all $X(h, x, y)$ with $h \in H_{\mathbb{Z}}$ arbitrary and $x, y \in H_{\mathbb{Z}}$ nonzero such that $x \in \langle a_d, b_d \rangle$ and $y \in \langle a_d, b_d \rangle^{\perp}$ for some $1 \leq d \leq g$. Our goal in this section is to prove that the restriction of the quotiented Reidemeister pairing $\mathfrak{q}: \mathcal{Q}_g \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$ to $\langle \mathcal{V}_1 \rangle$ is injective and to identify the quotient of $\mathbb{Q}[H_{\mathbb{Z}}]$ by $\langle \mathfrak{q}(\mathcal{V}_1) \rangle$.

11.1. A smaller generating set. Let \mathcal{W}_1 be the set of all $X(h, x, y) \in \mathcal{V}_1$ such that $h = 0$ and such that $x \in \langle a_d, b_d \rangle$ and $y \in \langle a_{d+1}, b_{d+1}, \dots, a_g, b_g \rangle$ for some $1 \leq d \leq g-1$. Our first order of business is to prove that \mathcal{W}_1 spans the same subspace of \mathcal{Q}_g as \mathcal{V}_1 :

Lemma 11.1. *Letting the notation be as above, we have $\langle \mathcal{W}_1 \rangle = \langle \mathcal{V}_1 \rangle$.*

Proof. Define \mathcal{V}_1^1 and \mathcal{V}_1^2 with $\mathcal{W}_1 \subset \mathcal{V}_1^2 \subset \mathcal{V}_1^1 \subset \mathcal{V}_1$ as follows. Let \mathcal{V}_1^1 be the set of all $X(h, x, y) \in \mathcal{V}_1$ such that $x \in \langle a_d, b_d \rangle$ and $y \in \langle a_{d+1}, b_{d+1}, \dots, a_g, b_g \rangle$ for some $1 \leq d \leq g-1$. Let \mathcal{V}_1^2 be the set of all $X(h, x, y) \in \mathcal{V}_1$ such that $x \in \langle a_d, b_d \rangle$ and $y \in \langle a_{d+1}, b_{d+1}, \dots, a_g, b_g \rangle$ and $h \in \langle a_d, b_d, \dots, a_g, b_g \rangle$ for some $1 \leq d \leq g-1$.

Step 1. *We prove that $\langle \mathcal{V}_1^1 \rangle = \langle \mathcal{V}_1 \rangle$.*

Consider some $X(h, x, y) \in \mathcal{V}_1$. We must show that $X(h, x, y)$ is in the span of \mathcal{V}_1^1 . Let $1 \leq d \leq g$ be such that $x \in \langle a_d, b_d \rangle$ and $y \in \langle a_d, b_d \rangle^{\perp}$. Write

$$y = y_1 + \dots + y_g \quad \text{with } y_i \in \langle a_i, b_i \rangle,$$

so $y_d = 0$. Using the additivity relation (Lemma 8.3), we see that $X(h, x, y) = X(h, x, y_1 + \dots + y_g)$ equals the following, where the colored term vanishes since $y_d = 0$:

$$X(h, x, y_1) + \dots + X(h + y_1 + \dots + y_{d-1}, x, y_d) + \dots + X(h + y_1 + \dots + y_{g-1}, x, y_g).$$

For $1 \leq i \leq g$ with $i \neq d$, the symmetry relation (Lemma 8.2) says that

$$X(h + y_1 + \dots + y_{i-1}, x, y_i) = X(h + y_1 + \dots + y_{i-1}, y_i, x).$$

From this, we see that whether or not $i < d$ or $i > d$ this term lies in \mathcal{V}_1^1 . The step follows.

Step 2. *We prove that $\langle \mathcal{V}_1^2 \rangle = \langle \mathcal{V}_1^1 \rangle$.*

Consider some $X(h, x, y) \in \mathcal{V}_1^1$. We must show that $X(h, x, y)$ is in the span of \mathcal{V}_1^2 . Let $1 \leq d \leq g-1$ be such that $x \in \langle a_d, b_d \rangle$ and $y \in \langle a_{d+1}, b_{d+1}, \dots, a_g, b_g \rangle$. If $d = 1$ there is nothing to prove, so assume that $d > 1$. Write

$$h = h_1 + \dots + h_{d-1} + h' \quad \text{with } h_i \in \langle a_i, b_i \rangle \text{ and } h' \in \langle a_d, b_d, \dots, a_g, b_g \rangle.$$

Applying the cube relation (Lemma 8.5) repeatedly, we see that

$$\begin{aligned} X(h_1 + \dots + h_{d-1} + h', x, y) &= X(h_2 + \dots + h_{d-1} + h', x, y) - X(h_2 + \dots + h_{d-1} + h', h_1, y) \\ &\quad + X(h_2 + \dots + h_{d-1} + h' + x, h_1, y) \end{aligned}$$

$$\begin{aligned} X(h_2 + \dots + h_{d-1} + h', x, y) &= X(h_3 + \dots + h_{d-1} + h', x, y) - X(h_3 + \dots + h_{d-1} + h', h_2, y) \\ &\quad + X(h_3 + \dots + h_{d-1} + h' + x, h_2, y) \end{aligned}$$

⋮

$$X(h_{d-1} + h', x, y) = X(h', x, y) - X(h', h_{d-1}, y) + X(h' + x, h_{d-1}, y).$$

The element $X(h', x, y)$ and all the blue terms are in \mathcal{V}_1^2 , so $X(h, x, y) \in \langle \mathcal{V}_1^2 \rangle$, as desired.

Step 3. *We prove that $\langle \mathcal{W}_1 \rangle = \langle \mathcal{V}_1 \rangle$.*

Assume this is false. By Step 2, there must be some $X(h, x, y) \in \mathcal{V}_1^2$ with $X(h, x, y) \notin \mathcal{W}_1$. Choose $X(h, x, y) \in \mathcal{V}_1^2$ with $X(h, x, y) \notin \langle \mathcal{W}_1 \rangle$ in the following way:

- Elements $X(h, x, y) \in \mathcal{V}_1^2$ have $x \in \langle a_d, b_d \rangle$ and $y \in \langle a_{d+1}, b_{d+1}, \dots, a_g, b_g \rangle$ and $h \in \langle a_d, b_d, \dots, a_g, b_g \rangle$ for some $1 \leq d \leq g-1$. They do not lie in \mathcal{W}_1 precisely when $h \neq 0$. In that case, for some $d \leq d' \leq g$ we can write

$$h = h_{d'} + \dots + h_g \quad \text{with } h_i \in \langle a_i, b_i \rangle \text{ and } h_{d'} \neq 0.$$

Among all the $X(h, x, y) \in \mathcal{V}_1^2$ with $X(h, x, y) \notin \langle \mathcal{W}_1 \rangle$, choose the one with d' as large as possible.

There are now two cases. The first is $d' = d$. By the additivity relation (Lemma 8.3),

$$X(h_{d+1} + \dots + h_g, h_d + x, y) = X(h_{d+1} + \dots + h_g, h_d, y) + X(h_d + \dots + h_g, x, y).$$

Since $d' = d$ is as large as possible, both blue terms are in the span of \mathcal{W}_1 . This implies that $X(h_d + \dots + h_g, x, y)$ is also in the span of \mathcal{W}_1 , a contradiction.

The second case is $d+1 \leq d' \leq g$. The additivity relation (Lemma 8.3) implies that

$$X(h_{d'+1} + \dots + h_g, x, h_{d'} + y) = X(h_{d'+1} + \dots + h_g, x, h_{d'}) + X(h_{d'} + \dots + h_g, x, y).$$

Since d' is as large as possible, both blue terms are in the span of \mathcal{W}_1 . This implies that $X(h_d + \dots + h_g, x, y)$ is also in the span of \mathcal{W}_1 , a contradiction. \square

11.2. Main result. We now prove the main result of this section:

Proposition 11.2. *The restriction of \mathfrak{q} to $\langle \mathcal{V}_1 \rangle$ is injective, and²⁶*

$$\mathbb{Q}[H_{\mathbb{Z}}] = \langle \mathfrak{q}(\mathcal{V}_1) \rangle \oplus \mathbb{Q}\left[\bigcup_{i=1}^g \langle a_i, b_i \rangle\right].$$

Proof. Let $\widehat{\mathfrak{q}}$ be the composition

$$\mathcal{Q}_g \xrightarrow{\mathfrak{q}} \mathbb{Q}[H_{\mathbb{Z}}] \twoheadrightarrow \mathbb{Q}[H_{\mathbb{Z}}]/\mathbb{Q}[\bigcup_{i=1}^g \langle a_i, b_i \rangle]$$

and let $\mathcal{Q}_g(1) = \langle \mathcal{V}_1 \rangle$. We will prove that the restriction of $\widehat{\mathfrak{q}}$ to $\mathcal{Q}_g(1)$ is an isomorphism.

Let Q be the codomain of $\widehat{\mathfrak{q}}$. We construct an inverse $\mathfrak{p}: Q \rightarrow \mathcal{Q}_g(1)$ to $\widehat{\mathfrak{q}}|_{\mathcal{Q}_g(1)}$ as follows. We can identify Q with the set of formal \mathbb{Q} -linear combinations of terms of the form $\{z_{d_1} + \dots + z_{d_r}\}$ where:

- $1 \leq d_1 < d_2 < \dots < d_r \leq g$ with $r \geq 2$; and
- for $1 \leq i \leq r$, the term z_{d_i} is a nonzero element of $\langle a_{d_i}, b_{d_i} \rangle$.

Define

$$\mathfrak{p}(\{z_{d_1} + \dots + z_{d_r}\}) = \sum_{j=1}^{r-1} X(0, z_{d_j}, z_{d_{j+1}} + \dots + z_{d_r}).$$

To see that this is an inverse to $\widehat{\mathfrak{q}}|_{\mathcal{Q}_g(1)}$, we must check two things:

Claim. *For $\{z_{d_1} + \dots + z_{d_r}\}$ as above, we have*

$$\widehat{\mathfrak{q}}(\mathfrak{p}(\{z_{d_1} + \dots + z_{d_r}\})) = \{z_{d_1} + \dots + z_{d_r}\}.$$

²⁶Here $\mathbb{Q}[\bigcup_{i=1}^g \langle a_i, b_i \rangle]$ is the set of formal \mathbb{Q} -linear combinations of $\{h\}$ for $h \in H_{\mathbb{Z}}$ an element such that $h \in \langle a_i, b_i \rangle$ for some $1 \leq i \leq g$. This is not a disjoint union since all these terms contain $0 \in H_{\mathbb{Z}}$.

For this, we calculate as follows. Terms of $\mathbb{Q}[\bigcup_{i=1}^g \langle a_i, b_i \rangle]$ are in blue, and vanish in Q :

$$\begin{aligned} \mathfrak{q}(\mathfrak{p}(\{z_{d_1} + \cdots + z_{d_r}\})) &= \mathfrak{q}\left(\sum_{i=1}^{r-1} X(0, z_{d_i}, z_{d_{i+1}} + \cdots + z_{d_r})\right) \\ &= \sum_{i=1}^{r-1} \{0\} - \{z_{d_i}\} - \{z_{d_{i+1}} + \cdots + z_{d_r}\} + \{z_{d_i} + \cdots + z_{d_r}\}. \end{aligned}$$

Deleting the indicated blue terms gives a telescoping sum adding up to $\{z_{d_1} + \cdots + z_{d_r}\} - \{z_{d_r}\}$. Deleting this final blue term gives $\{z_{d_1} + \cdots + z_{d_r}\}$, as desired.

Claim. *The composition $\mathfrak{p} \circ \widehat{\mathfrak{q}}$ is the identity on $\mathcal{Q}_g(1)$.*

We check this on the generating set \mathcal{W}_1 given by Lemma 11.1. An element of \mathcal{W}_1 can be written as $X(0, x_{e_1}, x_{e_2} + \cdots + x_{e_s})$ where:

- $1 \leq e_1 < \cdots < e_s \leq g$ with $s \geq 2$; and
- for $1 \leq i \leq s$, the term x_{e_i} is a nonzero element of $\langle a_{e_i}, b_{e_i} \rangle$.

If $s = 2$, then when we apply $\widehat{\mathfrak{q}}$ to this the only term that survives in the quotient Q is $\{x_{e_1} + x_{e_2}\}$, which is taken by \mathfrak{p} back to $X(0, x_{e_1}, x_{e_2})$. If $s \geq 3$, then when we apply $\widehat{\mathfrak{q}}$ to this two terms survive in the quotient Q :

$$\widehat{\mathfrak{q}}(X(0, x_{e_1}, x_{e_2} + \cdots + x_{e_s})) = -\{x_{e_2} + \cdots + x_{e_s}\} + \{x_{e_1} + \cdots + x_{e_s}\}.$$

Applying \mathfrak{p} to this gives

$$-\sum_{i=2}^{s-1} X(0, x_{e_i}, x_{e_{i+1}} + \cdots + x_{e_s}) + \sum_{i=1}^{s-1} X(0, x_{e_i}, x_{e_{i+1}} + \cdots + x_{e_s}).$$

All terms cancel except $X(0, x_{e_1}, x_{e_2} + \cdots + x_{e_s})$, as desired. \square

12. CALCULATION OF THE QUOTIENT II: THE SECOND SET OF GENERATORS

We now move on the set \mathcal{V}_2 of generators, which we recall consists of all $X(h, x, x + y)$ with $h \in H_{\mathbb{Z}}$ arbitrary such that for some distinct $1 \leq d, e \leq g$ we have $x \in \{a_d, b_d\}$ and $y \in \{\pm a_e, \pm b_e\}$. As notation, define $\mathcal{Q}_{g/1} = \mathcal{Q}_g / \langle \mathcal{V}_1 \rangle$. For any generator $X(h, x, y)$ of \mathcal{Q}_g , let $X_1(h, x, y)$ be its image in $\mathcal{Q}_{g/1}$.

12.1. Y-elements. We start by proving that for $X(h, x, x + y) \in \mathcal{V}_2$, its image $X_1(h, x, x + y) \in \mathcal{Q}_{g/1}$ does not depend on y :

Lemma 12.1. *Let $h \in H_{\mathbb{Z}}$ and $x \in \{a_d, b_d\}$ for some $1 \leq d \leq g$. For some $1 \leq e, e' \leq g$ with $e, e' \neq d$, let $y \in \{\pm a_e, \pm b_e\}$ and $y' \in \{\pm a_{e'}, \pm b_{e'}\}$. Then $X_1(h, x, x + y) = X_1(h, x, x + y')$.*

Proof. Recall our standing assumption that $g \geq 4$ (Assumption 1.5). Because of this, it is enough to prove the claim when $e \neq e'$. Using the additivity relation (Lemma 8.3), we have

$$\begin{aligned} X(h, x, x + y + y') &= X(h, x, x + y) + X(h + x + y, x, y'), \\ X(h, x, x + y' + y) &= X(h, x, x + y') + X(h + x + y', x, y). \end{aligned}$$

The blue terms here lie in \mathcal{V}_1 and thus die in $\mathcal{Q}_{g/1}$. The lemma follows. \square

Using this lemma, if $h \in H_{\mathbb{Z}}$ and $x \in \{a_d, b_d\}$ for some $1 \leq d \leq g$, then we can define $Y(h, x) \in \mathcal{Q}_{g/1}$ to be the image of any corresponding element $X(h, x, x + y) \in \mathcal{V}_2$.

12.2. h-values of Y-elements. We now prove that if $h \in H_{\mathbb{Z}}$ and $x \in \{a_d, b_d\}$ for some $1 \leq d \leq g$, then $Y(h, x)$ only depends on the projection of h to $\langle a_d, b_d \rangle$:

Lemma 12.2. *Let $h, h' \in H_{\mathbb{Z}}$ and let $x \in \{a_d, b_d\}$ for some $1 \leq d \leq g$. Assume that $h' - h \in \langle a_d, b_d \rangle^{\perp}$. Then $Y(h, x) = Y(h', x)$.*

Proof. Set $k = h' - h$, so $h' = h + k$. We can write k as a sum of elements lying in subspaces of the form $\langle a_e, b_e \rangle$ with $e \neq d$, and it is enough to prove the lemma for k an element of such an $\langle a_e, b_e \rangle$. Using our standing assumption $g \geq 4$ (see Assumption 1.5), we can find $1 \leq e' \leq g$ with $e' \neq d, e$. Using the cube relation (Lemma 8.5) and the additivity relation (Lemma 8.3), we see that

$$\begin{aligned} X(h+k, x, x+a_{e'}) &= X(h, x, x+a_{e'}) - X(h, k, x+a_{e'}) + X(h+x, k, x+a_{e'}) \\ &= X(h, x, x+a_{e'}) - X(h, k, x) - X(h+x, k, a_{e'}) \\ &\quad + X(h+x, k, x) + X(h+2x, k, a_{e'}). \end{aligned}$$

The blue terms lie in the span of \mathcal{V}_1 , and thus vanish in $\mathcal{Q}_{g/1}$. We conclude that

$$Y(h, x) = X_1(h, x, x+a_{e'}) = X_1(h+k, x, x+a_{e'}) = Y(h+k, x). \quad \square$$

Letting $\mathcal{V}_{2/1}$ be the image of \mathcal{V}_2 in $\mathcal{Q}_{g/1}$, we see from the above claim that $\mathcal{V}_{2/1}$ consists of all $Y(h, x)$ with $h \in \langle a_d, b_d \rangle$ and $x \in \{a_d, b_d\}$ for some $1 \leq d \leq g$.

12.3. Mapping Y-elements. Consider the composition

$$\mathcal{Q}_g \xrightarrow{\mathfrak{q}} \mathbb{Q}[H_{\mathbb{Z}}] = \langle \mathfrak{q}(\mathcal{V}_1) \rangle \oplus \mathbb{Q}[\bigcup_{i=1}^g \langle a_i, b_i \rangle] \longrightarrow \mathbb{Q}[\bigcup_{i=1}^g \langle a_i, b_i \rangle],$$

where the equality comes from Proposition 11.2. This induces a map

$$\mathfrak{q}_{/1}: \mathcal{Q}_{g/1} \rightarrow \mathbb{Q}[\bigcup_{i=1}^g \langle a_i, b_i \rangle].$$

For $h \in \langle a_i, b_i \rangle$, we will still denote the corresponding element of $\mathbb{Q}[\bigcup_{i=1}^g \langle a_i, b_i \rangle]$ by $\{h\}$. The following lemma calculates the image of $Y(h, x)$ under the map $\mathfrak{q}_{/1}$:

Lemma 12.3. *For $Y(h, x) \in \mathcal{V}_{2/1}$, we have $\mathfrak{q}_{/1}(Y(h, x)) = \{h\} - 2\{h+x\} + \{h+2x\}$.*

Proof. Let $1 \leq d \leq g$ be such that $h \in \langle a_d, b_d \rangle$ and $x \in \{a_d, b_d\}$. Pick $1 \leq e \leq g$ with $e \neq d$, so $Y(h, x) = X_1(h, x, x+a_e)$. We then have

$$\mathfrak{q}(X(h, x, x+a_e)) = \{h\} - \{h+x\} - \{h+x+a_e\} + \{h+2x+a_e\}.$$

To project this into $\mathbb{Q}[\bigcup_{i=1}^g \langle a_i, b_i \rangle]$, we can add the images under \mathfrak{q} of any elements of \mathcal{V}_1 . Adding

$$\begin{aligned} \mathfrak{q}(X(h, x, a_e)) - \mathfrak{q}(X(h, 2x, a_e)) &= (\{h\} - \{h+x\} - \{h+a_e\} + \{h+x+a_e\}) \\ &\quad - (\{h\} - \{h+2x\} - \{h+a_e\} + \{h+2x+a_e\}), \end{aligned}$$

we get $\{h\} - 2\{h+x\} + \{h+2x\} \in \mathbb{Q}[\langle a_d, b_d \rangle]$. Since this lies in $\mathbb{Q}[\bigcup_{i=1}^g \langle a_i, b_i \rangle]$, it is the projection of $\mathfrak{q}(X(h, x, x+a_e))$. The lemma follows. \square

12.4. Y-relation. We now give a basic relation between the $Y(h, x)$:

Lemma 12.4. *Let $1 \leq d \leq g$ and let $h \in \langle a_d, b_d \rangle$. Then*

$$Y(h, a_d) - 2Y(h+b_d, a_d) + Y(h+2b_d, a_d) = Y(h, b_d) - 2Y(h+a_d, b_d) + Y(h+2a_d, b_d).$$

Proof. For an arbitrary $k \in H_{\mathbb{Z}}$, we first claim that there is a relation

$$\begin{aligned} & X(k, a_1, a_2) - X(k + b_1, a_1, a_2) - X(k + b_2, a_1, a_2) + X(k + b_1 + b_2, a_1, a_2) \\ &= X(k, b_1, b_2) - X(k + a_1, b_1, b_2) - X(k + a_2, b_1, b_2) + X(k + a_1 + a_2, b_1, b_2) \end{aligned}$$

in \mathcal{Q}_g . To see this, observe that \mathfrak{q} takes this to a true relation in $\mathbb{Q}[H_{\mathbb{Z}}]$, and each term of it lies in \mathcal{V}_1 . Proposition 11.2 says that the restriction of \mathfrak{q} to $\langle \mathcal{V}_1 \rangle$ is an injection, so we conclude that the above is a relation in \mathcal{Q}_g , as claimed.

Recall our standing assumption that $g \geq 4$ (Assumption 1.5). Using this, pick distinct $1 \leq e, e' \leq g$ with $e, e' \neq d$. Let $\phi \in \mathrm{Sp}_{2g}(\mathbb{Z})$ be a symplectic automorphism taking (a_1, b_1, a_2, b_2) to $(a_d + a_e, b_d - b_{e'}, a_d + a_{e'}, b_d - b_e)$, which exists since both are partial symplectic bases. Choose $k \in H_{\mathbb{Z}}$ such that $\phi(k) = h$. The group $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts on \mathcal{Q}_g , so we can apply ϕ to the above relation and get a new relation

$$\begin{aligned} & X(h, a_d + a_e, a_d + a_{e'}) - X(h + b_d - b_{e'}, a_d + a_e, a_d + a_{e'}) \\ & \quad - X(h + b_d - b_e, a_d + a_e, a_d + a_{e'}) + X(h + 2b_d - b_{e'} - b_e, a_d + a_e, a_d + a_{e'}) \\ = & X(h, b_d - b_{e'}, b_d - b_e) - X(h + a_d + a_e, b_d - b_{e'}, b_d - b_e) \\ & \quad - X(h + a_d + a_{e'}, b_d - b_{e'}, b_d - b_e) + X(h + 2a_d + a_e + a_{e'}, b_d - b_{e'}, b_d - b_e). \end{aligned}$$

Each term of this maps to a corresponding term in the desired relation between the $Y(h, x)$. For instance, using the additivity relation (Lemma 8.3) we have

$$\begin{aligned} X_1(h, a_d + a_e, a_d + a_{e'}) &= X_1(h, a_d, a_d + a_{e'}) + X_1(h + a_d, a_e, a_d + a_{e'}) = Y(h, a_d), \\ X_1(h + b_d - b_{e'}, a_d + a_e, a_d + a_{e'}) &= X_1(h + b_d - b_{e'}, a_d, a_d + a_{e'}) \\ & \quad + X_1(h + b_d - b_{e'} + a_d, a_e, a_d + a_{e'}) = Y(h + b_d, a_d). \end{aligned}$$

Here the blue terms are images of elements of $\langle \mathcal{V}_1 \rangle$ and thus die in $\overline{\mathcal{Q}}_g$, and for the final equality we are using Lemma 12.2 to discard the $-b_{e'}$. The other terms are similar. \square

12.5. A smaller generating set. We now show that only some of the $Y(h, x)$ are needed to generate $\langle \mathcal{V}_{2/1} \rangle$. Define \mathcal{W}_2 to be the union of the following two sets:

- $\{Y(h, a_d) \mid 1 \leq d \leq g, h \in \langle a_d, b_d \rangle\}$; and
- $\{Y(h, b_d) \mid 1 \leq d \leq g, h \in \langle a_d, b_d \rangle \text{ with } a_d\text{-coordinate } 0 \text{ or } 1\}$.

We then have:

Lemma 12.5. *Letting the notation be as above, we have $\langle \mathcal{W}_2 \rangle = \langle \mathcal{V}_{2/1} \rangle$.*

Proof. Let $1 \leq d \leq g$. For all $h \in \langle a_d, b_d \rangle$ we must show that $\langle \mathcal{W}_2 \rangle$ contains $Y(h, b_d)$. Assume for the sake of contradiction that it does not, and let $Y(h, b_d)$ be an element not lying in $\langle \mathcal{W}_2 \rangle$ such that the a_d -coordinate λ of h is as small as possible. We thus either have $\lambda \geq 2$ or $\lambda \leq -1$. If $\lambda \geq 2$, then the relation

$$\begin{aligned} & Y(h - 2a_d, a_d) - 2Y(h - 2a_d + b_d, a_d) + Y(h - 2a_d + 2b_d, a_d) \\ &= Y(h - 2a_d, b_d) - 2Y(h - a_d, b_d) + Y(h, b_d) \end{aligned}$$

from Lemma 12.4 allows us to write $Y(h, b_d)$ in terms of elements that lie in $\langle \mathcal{W}_2 \rangle$, a contradiction. The case where $\lambda \leq -1$ is similar. \square

12.6. Main result. We now prove the main result of this section:

Proposition 12.6. *The restriction of $\mathfrak{q}_{/1}$ to $\langle \mathcal{V}_{2/1} \rangle$ is injective, and*

$$\mathbb{Q}\left[\bigcup_{i=1}^g \langle a_i, b_i \rangle\right] = \langle \mathfrak{q}_{/1}(\mathcal{V}_{2/1}) \rangle \oplus \mathbb{Q}\left[\bigcup_{i=1}^g \{0, a_i, b_i, a_i + b_i\}\right].$$

Proof. By Lemma 12.5, we have $\langle \mathcal{V}_{2/1} \rangle = \langle \mathcal{W}_2 \rangle$. We will prove that $\mathfrak{q}_{/1}$ takes the elements of \mathcal{W}_2 to linearly independent elements of $\mathbb{Q}[\bigcup_{i=1}^g \langle a_i, b_i \rangle]$ spanning a subspace that is a complement to $\mathbb{Q}[\bigcup_{i=1}^g \{0, a_i, b_i, a_i + b_i\}]$. For this, fix some $1 \leq d \leq g$. Let $\mathcal{W}_2(d)$ be the set of all $Y(h, x) \in \mathcal{W}_2$ such that $h, x \in \langle a_d, b_d \rangle$. By Lemma 12.3, for $Y(h, x) \in \mathcal{W}_2(d)$ we have

$$\mathfrak{q}_{/1}(Y(h, x)) = \{h\} - 2\{h + x\} + \{h + 2x\} \in \mathbb{Q}[\langle a_d, b_d \rangle].$$

It is enough to prove that $\mathfrak{q}_{/1}$ takes the elements of $\mathcal{W}_2(d)$ to linearly independent elements of $\mathbb{Q}[\langle a_d, b_d \rangle]$ such that

$$\mathbb{Q}[\langle a_d, b_d \rangle] = \langle \mathfrak{q}_{/1}(\mathcal{W}_2(d)) \oplus \mathbb{Q}[\{0, a_d, b_d, a_d + b_d\}] \rangle.$$

The key to this is the following easy piece of linear algebra:

Claim. *Let $\{\vec{e}_n\}_{n \in \mathbb{Z}}$ be a basis for a \mathbb{Q} -vector space V of countable dimension. For each $n \in \mathbb{Z}$, let $\vec{f}_n = \vec{e}_n - 2\vec{e}_{n+1} + \vec{e}_{n+2} \in V$. Then the $\{\vec{f}_n\}_{n \in \mathbb{Z}}$ are linearly independent elements of V spanning a complement to $\langle \vec{e}_0, \vec{e}_1 \rangle$.*

Proof of claim. Immediate from the fact that

- $\{\vec{e}_0, \vec{e}_1, \vec{f}_0, \vec{f}_1, \vec{f}_2, \dots\}$ is a basis for $\langle \vec{e}_0, \vec{e}_1, \vec{e}_2, \dots \rangle$; and
- $\{\vec{e}_1, \vec{e}_0, \vec{f}_{-1}, \vec{f}_{-2}, \vec{f}_{-3}, \dots\}$ is a basis for $\langle \vec{e}_1, \vec{e}_0, \vec{e}_{-1}, \dots \rangle$. □

For $n, m \in \mathbb{Z}$, let $\vec{e}_{n,m} = na_d + mb_d \in \langle a_d, b_d \rangle$. The $\vec{e}_{n,m}$ form a basis for $\mathbb{Q}[\langle a_d, b_d \rangle]$. Define the following subspaces of $\mathbb{Q}[\langle a_d, b_d \rangle]$:

- for $n_0 \in \mathbb{Z}$, the subspace $L_{n_0} = \langle \vec{e}_{n_0, m} \mid m \in \mathbb{Z} \rangle$; and
- for $m_0 \in \mathbb{Z}$, the subspace $M_{m_0} = \langle \vec{e}_{n, m_0} \mid n \in \mathbb{Z} \rangle$.

For $m_0 \in \mathbb{Z}$, we have

$$\mathfrak{q}_{/1}(Y(\vec{e}_{n, m_0}, a_0)) = \vec{e}_{n, m_0} - 2\vec{e}_{n+1, m_0} + \vec{e}_{n+2, m_0} \in M_{m_0} \quad \text{for all } n \in \mathbb{Z}.$$

By the above claim, these are linearly independent elements of M_{m_0} spanning a complement to $\langle \vec{e}_{0, m_0}, \vec{e}_{1, m_0} \rangle$. We have

$$\mathbb{Q}[\langle a_d, b_d \rangle] = \bigoplus_{m_0 \in \mathbb{Z}} M_{m_0},$$

so this implies that $\mathfrak{q}_{/1}$ takes the elements of $\{Y(\vec{e}_{n, m}, a_d) \mid n, m \in \mathbb{Z}\}$ to linearly independent elements of $\mathbb{Q}[\langle a_d, b_d \rangle]$ spanning a complement to $L_0 \oplus L_1$. For $n_0 \in \{0, 1\}$, we have

$$\mathfrak{q}_{/1}(Y(\vec{e}_{n_0, m}, b_0)) = \vec{e}_{n_0, m} - 2\vec{e}_{n_0, m+1} + \vec{e}_{n_0, m+2} \in L_{n_0} \quad \text{for all } m \in \mathbb{Z}.$$

By the above claim, these are linearly independent elements of L_{n_0} spanning a complement to $\langle \vec{e}_{n_0, 0}, \vec{e}_{n_0, 1} \rangle$. Putting this all together, we conclude that $\mathfrak{q}_{/1}$ takes the elements of

$$\mathcal{W}_2(d) = \{Y(\vec{e}_{n, m}, a_d) \mid n, m \in \mathbb{Z}\} \cup \{Y(\vec{e}_{n, m}, b_d) \mid n \in \{0, 1\}, m \in \mathbb{Z}\}$$

to linearly independent elements of $\mathbb{Q}[\langle a_d, b_d \rangle]$ spanning a complement to $\langle \vec{e}_{0, 0}, \vec{e}_{0, 1}, \vec{e}_{1, 0}, \vec{e}_{1, 1} \rangle$, as desired. □

13. CALCULATION OF THE QUOTIENT III: THE THIRD SET OF GENERATORS

We conclude with the final set \mathcal{V}_3 of generators, which we recall consists of all $X(h, x, y)$ with $h \in H_{\mathbb{Z}}$ arbitrary and the pair (x, y) equal to either $(a_d + a_e, b_d - b_e)$ or $(a_d + b_e, b_d + a_e)$ for some distinct $1 \leq d, e \leq g$. As notation, define $\mathcal{Q}_{g/2} = \mathcal{Q}_g / \langle \mathcal{V}_1, \mathcal{V}_2 \rangle$. For any generator $X(h, x, y)$ of \mathcal{Q}_g , let $X_2(h, x, y)$ be its image in $\mathcal{Q}_{g/2}$.

13.1. ZW-elements. We start by proving that for $X(h, x, y) \in \mathcal{V}_3$, its image $X_2(h, x, y) \in \mathcal{Q}_{g/2}$ does not depend on h .

Lemma 13.1. *For distinct $1 \leq d, e \leq g$, let (x, y) be either $(a_d + a_e, b_d - b_e)$ or $(a_d + b_e, b_d + a_e)$. Then for $h, h' \in H\mathbb{Z}$ we have $X_2(h, x, y) = X_2(h', x, y)$.*

Proof. We will give the details for $(x, y) = (a_d + a_e, b_d - b_e)$; the other case is similar. It is enough to prove that for²⁷ $k \in \{a_1, b_1, \dots, a_g, b_g\}$, we have $X_2(h, x, y) = X_2(h + k, x, y)$. All these values of k are handled using the cube relation (Lemma 8.5). We will give the details for $k = a_d$ and leave the other cases to the reader.

The elements $a_d + a_e \in H\mathbb{Z}$ and $a_d, b_d - b_e \in H\mathbb{Z}$ are homologically separate. By the cube relation (Lemma 8.5), we thus have

$$\begin{aligned} X(h + a_d, a_d + a_e, b_d - b_e) &= X(x, a_d + a_e, b_d - b_e) \\ &\quad - X(x, a_d + a_e, a_d) + X(x + b_d - b_e, a_d + a_e, a_d). \end{aligned}$$

Both blue terms lie in \mathcal{V}_2 and thus vanish in $\mathcal{Q}_{g/2}$. The lemma follows. \square

By this lemma, for distinct $1 \leq d, e \leq g$ we can define $Z(a_d + a_e, b_d - b_e) = X_2(h, a_d + a_e, b_d - b_e)$ and $W(a_d + b_e, b_d + a_e) = X_2(h, a_d + b_e, b_d + a_e)$ for any $h \in H\mathbb{Z}$. Let $\mathcal{V}_{3/2}$ be the set of these $Z(x, y)$ and $W(x, y)$. Lemma 9.2 says that $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$ spans \mathcal{Q}_g , so $\mathcal{V}_{3/2}$ spans $\mathcal{Q}_{g/2}$.

13.2. ZW-relations. These elements satisfy several relations:

Lemma 13.2. *For distinct $1 \leq d, e \leq g$, we have $Z(a_d + a_e, b_d - b_e) = -Z(a_e + a_d, b_e - b_d)$ and $W(a_d + b_e, b_d + a_e) = W(a_e + b_d, b_e + a_d)$.*

Proof. The first follows from the inverse relation (Lemma 8.4) and the second follows from the symmetry relation (Lemma 8.2). \square

Lemma 13.3. *For distinct $1 \leq d, e, f \leq g$ we have*

$$Z(a_d + a_f, b_d - b_f) = Z(a_d + a_e, b_d - b_e) + Z(a_e + a_f, b_e - b_f).$$

Proof. Since $(b_d - b_e) + (b_e - b_f) = b_d - b_f$, the additivity relation (Lemma 8.3) says that

$$(13.1) \quad X(0, a_d + a_e + a_f, b_d - b_f) = X(0, a_d + a_e + a_f, b_d - b_e) + X(b_d - b_e, a_d + a_e + a_f, b_e - b_f).$$

The additivity relation also implies that

$$X(0, a_d + a_e + a_f, b_d - b_f) = X(0, a_d + a_f, b_d - b_f) + X(a_d + a_e, a_e, b_d - b_f).$$

The blue term lies in the span of \mathcal{V}_1 , and thus vanishes in $\mathcal{Q}_{g/2}$. We therefore have

$$X_2(0, a_d + a_e + a_f, b_d - b_f) = X_2(0, a_d + a_f, b_d - b_f) = Z(a_d + a_f, b_d - b_f).$$

Similarly, we have

$$\begin{aligned} X_2(0, a_d + a_e + a_f, b_d - b_e) &= Z(a_d + a_e, b_d - b_e), \\ X_2(b_d - b_e, a_d + a_e + a_f, b_e - b_f) &= Z(a_e + a_f, b_e - b_f). \end{aligned}$$

Plugging all of this into (13.1) gives the desired relation. \square

Lemma 13.4. *For distinct $1 \leq d, e, f \leq g$ we have*

$$W(a_d + b_f, b_d + a_f) = Z(a_d + a_e, b_d - b_e) + W(a_e + b_f, b_e + a_f).$$

Proof. Since $(b_d - b_e) + (b_e + a_f) = b_d + a_f$, the additivity relation (Lemma 8.3) says that

$$X(0, a_d + a_e + b_f, b_d + a_f) = X(0, a_d + a_e + b_f, b_d - b_e) + X(b_d - b_e, a_d + a_e + b_f, b_e + a_f).$$

Just like in the proof of Lemma 13.3, this projects to the desired relation. \square

²⁷You might think we also need to handle things like $k = -a_1$, but since h is arbitrary this is not necessary.

13.3. **A smaller generating set.** Let

$$\mathcal{W}_3 = \{Z(a_d + a_{d+1}, b_d - b_{d+1}) \mid 1 \leq d \leq g-1\} \cup \{W(a_1 + b_2, b_1 + a_2)\}.$$

We prove that this spans $\mathcal{Q}_{g/2}$:

Lemma 13.5. *Letting the notation be as above, we have $\langle \mathcal{W}_3 \rangle = \mathcal{Q}_{g/2}$.*

Proof. It is enough to prove that $\langle \mathcal{W}_3 \rangle$ contains $\mathcal{V}_{3/2}$. We do this in two steps:

Claim. *For all distinct $1 \leq d, e \leq g$, we have $Z(a_d + a_e, b_d - b_e) \in \langle \mathcal{W}_3 \rangle$.*

Using Lemma 13.2, we can assume that $d < e$. Lemma 13.3 then implies that

$$Z(a_d + a_e, b_d - b_e) = \sum_{i=d}^{e-1} Z(a_i + a_{i+1}, b_i - b_{i+1}) \in \langle \mathcal{W}_3 \rangle.$$

Claim. *For all distinct $1 \leq d, e \leq g$, we have $W(a_d + b_e, b_d + a_e) \in \langle \mathcal{W}_3 \rangle$.*

We will prove this in the case where $d, e \notin \{1, 2\}$. The cases where one or both are in $\{1, 2\}$ are similar (but easier). Start by using Lemmas 13.2 and 13.4 to see that

$$\begin{aligned} W(a_2 + b_e, b_2 + a_e) &= W(a_e + b_2, b_e + a_2) \\ &= Z(a_e + a_1, b_e - b_1) + W(a_1 + b_2, b_1 + a_2) \in \langle \mathcal{W}_3 \rangle. \end{aligned}$$

Using this, another application of Lemma 13.4 says that

$$W(a_d + b_e, b_d + a_e) = Z(a_d + a_2, b_d - b_2) + W(a_2 + b_e, b_2 + a_e) \in \langle \mathcal{W}_3 \rangle. \quad \square$$

13.4. **Mapping ZW-elements.** Consider the composition

$$\mathcal{Q}_g \xrightarrow{\mathfrak{q}} \mathbb{Q}[H_{\mathbb{Z}}] = \langle \mathfrak{q}(\mathcal{V}_1), \mathfrak{q}(\mathcal{V}_2) \rangle \oplus \mathbb{Q}[\bigcup_{i=1}^g \langle 0, a_i, b_i, a_i + b_i \rangle] \twoheadrightarrow \mathbb{Q}[\bigcup_{i=1}^g \langle 0, a_i, b_i, a_i + b_i \rangle],$$

where the equality comes from Proposition 12.6. This induces a map

$$\mathfrak{q}_{/2}: \mathcal{Q}_{g/2} \rightarrow \mathbb{Q}[\bigcup_{i=1}^g \langle 0, a_i, b_i, a_i + b_i \rangle].$$

For $h \in \langle 0, a_i, b_i, a_i + b_i \rangle$, we will still denote the corresponding element of $\mathbb{Q}[\bigcup_{i=1}^g \langle 0, a_i, b_i, a_i + b_i \rangle]$ by $\{h\}$. For $1 \leq d \leq g$, let $\theta_d = \{0\} - \{a_d\} - \{b_d\} + \{a_d + b_d\}$. The following lemma calculates the images of $Z(x, y)$ and $W(x, y)$ under the map $\mathfrak{q}_{/2}$:

Lemma 13.6. *For distinct $1 \leq d, e \leq g$, we have*

$$\mathfrak{q}_{/2}(Z(a_d + a_e, b_d - b_e)) = \theta_d - \theta_e \quad \text{and} \quad \mathfrak{q}_{/2}(W(a_d + b_e, b_d + a_e)) = \theta_d + \theta_e.$$

Consequently, the image of $\mathfrak{q}_{/2}$ is $\langle \theta_d \mid 1 \leq d \leq g \rangle$.

Proof. The two calculations are similar, so we will give the details for the first. Note that

$$\mathfrak{q}(X(0, a_d + a_e, b_d - b_e)) = \{0\} - \{a_d + a_e\} - \{b_d - b_e\} + \{a_d + a_e + b_d - b_e\}.$$

To project this into $\mathbb{Q}[\bigcup_{i=1}^g \langle 0, a_i, b_i, a_i + b_i \rangle]$, we can add the images under \mathfrak{q} of any elements of \mathcal{V}_1 or \mathcal{V}_2 . Adding

$$\mathfrak{q}(X(0, a_d, a_e)) + \mathfrak{q}(X(0, b_d, -b_e)) - \mathfrak{q}(X(0, a_d + b_d, a_e - b_e)),$$

we get

$$(\{0\} - \{a_d\} - \{b_d\} + \{a_d + b_d\}) + (\{0\} - \{a_e\} - \{-b_e\} + \{a_e - b_e\}).$$

Project this to $\mathbb{Q}[\bigcup_{i=0}^g \langle a_i, b_i \rangle]$ and add

$$\begin{aligned} \mathfrak{q}_{/1}(Y(-b_e, b_e)) - \mathfrak{q}_{/1}(Y(a_e - b_e, b_e)) &= (\{-b_e\} - 2\{0\} + \{b_e\}) \\ &\quad - (\{a_e - b_e\} - 2\{a_e\} + \{a_e + b_e\}). \end{aligned}$$

We get

$$(\{0\} - \{a_d\} - \{b_d\} + \{a_d + b_d\}) + (-\{0\} + \{a_e\} + \{-b_e\} - \{a_e - b_e\}) = \theta_d - \theta_e. \quad \square$$

13.5. Main result. We now prove the main result of this section. This will complete the proof of Theorem 6.5 outlined in §10.

Proposition 13.7. *The map $q_{/2}: \mathcal{Q}_{g/2} \rightarrow \mathbb{Q}[\bigcup_{i=1}^g \langle 0, a_i, b_i, a_i + b_i \rangle]$ is injective.*

Proof. Lemma 13.6 says that $q_{/2}$ takes $\mathcal{Q}_{g/2}$ onto $\langle \theta_d \mid 1 \leq d \leq g \rangle$. Since the θ_d are linearly independent, the image of $q_{/2}$ is g -dimensional. Lemma 13.5 says that \mathcal{W}_3 spans $\mathcal{Q}_{g/2}$. Since \mathcal{W}_3 contains g elements, we deduce that $q_{/2}$ takes the elements of \mathcal{W}_3 to linearly independent elements. We conclude that $q_{/2}$ is injective. \square

Part 3. Relations in the kernel of the coinvariant Reidemeister pairing

Let $\bar{\tau}$ be the coinvariant Reidemeister pairing. Having proved Theorem 6.3 in Part 2, the remaining conclusion of Theorem 5.1 that must be proved is that $\ker(\bar{\tau})$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$. In this part of the paper, we will construct enough relations in $\ker(\bar{\tau})$ to force it to be a subrepresentation of $((\wedge^2 H)/\mathbb{Q})^{\otimes 2}$. See the introductory §14 for an outline.

14. INTRODUCTION TO PART 3

This section fixes some notation and does some preliminary calculations, and then outlines what we will do in the rest of Part 3.

14.1. Intersection pairing. Let ω be the algebraic intersection pairing on H . Like in §2.5, we will identify ω with an $\mathrm{Sp}_{2g}(\mathbb{Q})$ -invariant element $\omega \in \wedge^2 H$. If $\{a_1, b_1, \dots, a_g, b_g\}$ is a symplectic basis for H , then

$$\omega = a_1 \wedge b_1 + \dots + a_g \wedge b_g.$$

The line spanned by ω is an $\mathrm{Sp}_{2g}(\mathbb{Q})$ -invariant copy of \mathbb{Q} in $\wedge^2 H$, and whenever we talk about $(\wedge^2 H)/\mathbb{Q}$ we mean the quotient by this line.

14.2. Coinvariants. As we discussed in §2.4, the group π_g is the point-pushing subgroup of $\mathcal{I}_{g,1}$. The coinvariants $(\mathcal{C}_g)_{\mathcal{I}_{g,1}}$ are thus a quotient of $(\mathcal{C}_g)_{\pi_g}$. However:

Lemma 14.1. *We have $(\mathcal{C}_g)_{\mathcal{I}_{g,1}} \cong (\mathcal{C}_g)_{\pi_g} \cong (\wedge^2 H)/\mathbb{Q}$.*

Proof. Lemma 2.8 says that that $(\mathcal{C}_g)_{\pi_g} \cong (\wedge^2 H)/\mathbb{Q}$. The induced action of $\mathcal{I}_{g,1}/\pi_g \cong \mathcal{I}_g$ on $(\wedge^2 H)/\mathbb{Q}$ is trivial since \mathcal{I}_g acts trivially on H . It follows that nothing has to be killed when passing from $(\mathcal{C}_g)_{\pi_g}$ to $(\mathcal{C}_g)_{\mathcal{I}_{g,1}}$, as desired. \square

The product $\mathcal{I}_{g,1} \times \mathcal{I}_{g,1}$ acts on $\mathcal{C}_g^{\otimes 2}$, and Lemma 14.1 implies that:

Corollary 14.2. *We have $(\mathcal{C}_g^{\otimes 2})_{\mathcal{I}_{g,1} \times \mathcal{I}_{g,1}} \cong (\mathcal{C}_g^{\otimes 2})_{\pi_g \times \pi_g} \cong ((\wedge^2 H)/\mathbb{Q})^{\otimes 2}$.*

14.3. Reidemeister kernel. Recall from §5 that the coinvariant Reidemeister pairing is a linear map

$$\bar{\tau}: (\mathcal{C}_g^{\otimes 2})_{\mathcal{I}_{g,1}} \rightarrow \mathbb{Q}[H_{\mathbb{Z}}].$$

In this, $\mathcal{I}_{g,1}$ acts on $\mathcal{C}_g^{\otimes 2}$ via the diagonal map $\Delta: \mathcal{I}_{g,1} \rightarrow \mathcal{I}_{g,1} \times \mathcal{I}_{g,1}$. To help distinguish this from the $\mathcal{I}_{g,1} \times \mathcal{I}_{g,1}$ -action, we will write this using $\Delta(\mathcal{I}_{g,1})$. With this notation, the coinvariant Reidemeister pairing is a linear map

$$\bar{\tau}: (\mathcal{C}_g^{\otimes 2})_{\Delta(\mathcal{I}_{g,1})} \rightarrow \mathbb{Q}[H_{\mathbb{Z}}].$$

Define $\mathcal{K}_g = \ker(\bar{\tau})$, so we have an exact sequence

$$0 \longrightarrow \mathcal{K}_g \longrightarrow (\mathcal{C}_g^{\otimes 2})_{\Delta(\mathcal{I}_{g,1})} \xrightarrow{\bar{\tau}} \mathbb{Q}[H_{\mathbb{Z}}].$$

Theorem 6.3 says that \mathcal{K}_g is generated by the set of elements of the form $\underline{(\gamma_1)^{h_1} \otimes (\gamma_2)^{h_2}}$ with $\gamma_1 \in [\pi_g, \pi_g]$ and $\gamma_2 \in [\pi_g, \pi_g]$ separated and $h_1, h_2 \in H_{\mathbb{Z}}$ arbitrary.

14.4. Main theorem. Let us recall the theorem we are trying to prove:

Theorem 5.1. *Let $\bar{\tau}: (\mathcal{C}_g \otimes \mathcal{C}_g)_{\Delta(\mathcal{I}_{g,1})} \rightarrow \mathbb{Q}[H_{\mathbb{Z}}]$ be the coinvariant Reidemeister pairing. Then both $\ker(\bar{\tau})$ and $\text{coker}(\bar{\tau}) = \text{coker}(\tau)$ are finite-dimensional. Moreover, $\ker(\bar{\tau})$ is an algebraic representation of $\text{Sp}_{2g}(\mathbb{Z})$.*

In Part 2, we proved Theorem 6.3, which among other things said that $\text{coker}(\bar{\tau})$ is finite-dimensional. To complete the proof of Theorem 5.1, we must therefore prove that $\mathcal{K}_g = \ker(\bar{\tau})$ is a finite-dimensional algebraic representation of $\text{Sp}_{2g}(\mathbb{Z})$. Using Corollary 14.2, we obtain a surjective map

$$\mathbf{a}: (\mathcal{C}_g^{\otimes 2})_{\Delta(\mathcal{I}_{g,1})} \twoheadrightarrow (\mathcal{C}_g^{\otimes 2})_{\mathcal{I}_{g,1} \times \mathcal{I}_{g,1}} \cong ((\wedge^2 H)/\mathbb{Q})^{\otimes 2}$$

whose codomain is a finite-dimensional algebraic representation of²⁸ $\text{Sp}_{2g}(\mathbb{Z})$. We will call this the *algebraization map*. It is far from an isomorphism; indeed, the coinvariant Reidemeister pairing takes $(\mathcal{C}_g^{\otimes 2})_{\Delta(\mathcal{I}_{g,1})}$ to an infinite-dimensional representation of $\text{Sp}_{2g}(\mathbb{Z})$. However, we will prove that this is the only obstruction to \mathbf{a} being an isomorphism:

Theorem 14.3. *The restriction of the algebraization map \mathbf{a} to \mathcal{K}_g is an injection.*

This will imply that \mathcal{K}_g is a subrepresentation of the finite-dimensional algebraic representation $((\wedge^2 H)/\mathbb{Q})^{\otimes 2}$ of $\text{Sp}_{2g}(\mathbb{Z})$. By (♠), this will imply that \mathcal{K}_g is a finite-dimensional algebraic representation of $\text{Sp}_{2g}(\mathbb{Z})$, as was claimed by Theorem 5.1. The rest of this paper is devoted to the proof of Theorem 14.3. We divide the proof into four steps:

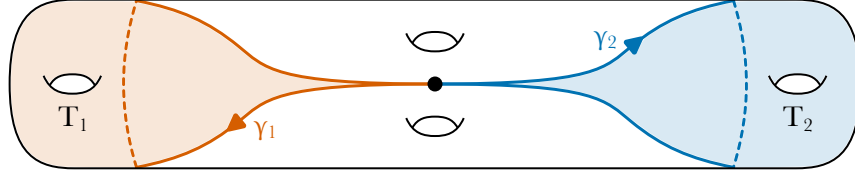
- §15 does some preliminary calculations in \mathcal{K}_g .
- §16 constructs a refined generating set for \mathcal{K}_g .
- §17 identifies some redundancies among these generators.
- §18 uses these generators and relations to prove Theorem 14.3.

15. PRELIMINARY CALCULATIONS IN \mathcal{K}_g

In this section, we make a preliminary study of \mathcal{K}_g .

²⁸Actually, it is an algebraic representation of $\text{Sp}_{2g}(\mathbb{Z}) \times \text{Sp}_{2g}(\mathbb{Z})$, but we only care about the diagonal subgroup $\text{Sp}_{2g}(\mathbb{Z})$ since the domain is only a representation of this diagonal subgroup.

15.1. Pairs of genus-1 curves. Recall from Theorem 6.3 that \mathcal{K}_g is spanned by elements $\langle \gamma_1 \rangle^{h_1} \otimes \langle \gamma_2 \rangle^{h_2}$ with $\gamma_1 \in [\pi_g, \pi_g]$ and $\gamma_2 \in [\pi_g, \pi_g]$ separated and $h_1, h_2 \in H_{\mathbb{Z}}$ arbitrary. Our first result is that we can take the γ_i to be simple closed curves that bound on their right sides genus-1 subsurfaces T_i such that $T_1 \cap T_2$ is the basepoint:

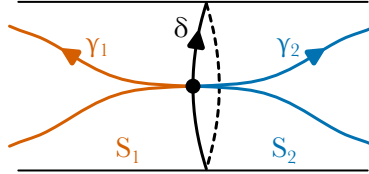


Lemma 15.1. *The vector space \mathcal{K}_g is spanned by $\langle \gamma_1 \rangle^{h_1} \otimes \langle \gamma_2 \rangle^{h_2}$ where:*

- for $i = 1, 2$, the curve $\gamma_i \in [\pi_g, \pi_g]$ is a simple closed separating curve that bounds on its right side a genus-1 subsurface T_i and $h_i \in H_{\mathbb{Z}}$ is arbitrary; and
- the intersection of T_1 and T_2 is the basepoint.

Proof. Theorem 6.3 says that \mathcal{K}_g is spanned by $\langle \gamma_1 \rangle^{h_1} \otimes \langle \gamma_2 \rangle^{h_2}$ with $\gamma_1 \in [\pi_g, \pi_g]$ and $\gamma_2 \in [\pi_g, \pi_g]$ separated and $h_1, h_2 \in H_{\mathbb{Z}}$ arbitrary. Fixing such $\gamma_1, \gamma_2 \in [\pi_g, \pi_g]$ and $h_1, h_2 \in H_{\mathbb{Z}}$, we must show that $\langle \gamma_1 \rangle^{h_1} \otimes \langle \gamma_2 \rangle^{h_2}$ can be written as a linear combination of the indicated generators.

Let $\delta \in \pi_g$ separate γ_1 from γ_2 and let S_1 and S_2 be the subsurfaces to the left and right of δ , respectively:



For $i = 1, 2$, the curve γ_i is in the image of the map $[\pi_1(S_i), \pi_1(S_i)] \rightarrow [\pi_g, \pi_g]$. Since $[\pi_1(S_i), \pi_1(S_i)]$ is $\pi_1(S_i)$ -normally generated by simple closed curves bounding genus-1 subsurfaces on their right sides (see, e.g., [17, Lemma A.1]²⁹), we can write³⁰

$$\gamma_i = (c_{i,1} \delta_{i,1})^{\epsilon_{i,1}} \dots (c_{i,n_i} \delta_{i,n_i})^{\epsilon_{i,n_i}}$$

where each $\delta_{i,j} \in [\pi_1(S_i), \pi_1(S_i)]$ is a simple closed curve bounding a genus-1 subsurface of S_i on its right side, each $c_{i,j}$ is an element of $\pi_1(S_i)$, and each $\epsilon_{i,j}$ is ± 1 .

Regard these expressions as occurring in π_g . By construction, for $1 \leq j \leq n_1$ and $1 \leq k \leq n_2$ the curves $\delta_{1,j}$ and $\delta_{2,k}$ only intersect at the basepoint and bound genus-1 subsurfaces on their right sides that only intersect at the basepoint. The desired expression is then

$$\begin{aligned} \langle \gamma_1 \rangle^{h_1} \otimes \langle \gamma_2 \rangle^{h_2} &= \left(\sum_{j=1}^{n_1} \epsilon_{1,j} \langle \delta_{1,j} \rangle^{h_1 + \bar{c}_{1,j}} \right) \otimes \left(\sum_{k=1}^{n_2} \epsilon_{2,k} \langle \delta_{2,k} \rangle^{h_2 + \bar{c}_{2,k}} \right) \\ &= \sum_{j=1}^{n_1} \sum_{k=1}^{n_2} \epsilon_{1,j} \epsilon_{2,k} \langle \delta_{1,j} \rangle^{h_1 + \bar{c}_{1,j}} \otimes \langle \delta_{2,k} \rangle^{h_2 + \bar{c}_{2,k}}. \quad \square \end{aligned}$$

²⁹This reference gives generation rather than $\pi_1(S_i)$ -normal generation. However, it requires the basepoint to lie in the interior, while ours lies on the boundary. The proof of [17, Lemma A.1] shows that in this case we only get $\pi_1(S_i)$ -normal generation.

³⁰Here we are using our convention that ${}^a b = a b a^{-1}$.

15.2. Exponents do not matter. We next prove that the exponents h_i are unnecessary:

Lemma 15.2. *Let $\gamma_1 \in [\pi_g, \pi_g]$ and $\gamma_2 \in [\pi_g, \pi_g]$ be separated and let $h_1, h_2 \in H_{\mathbb{Z}}$ be arbitrary. Then $\langle \gamma_1 \rangle^{h_1} \otimes \langle \gamma_2 \rangle^{h_2} = \langle \gamma_1 \rangle \otimes \langle \gamma_2 \rangle$.*

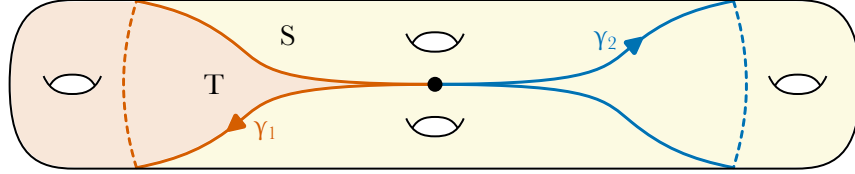
Proof. The proof has two steps. The first step handles the generators given by Lemma 15.1 above, and the second step reduces the lemma to those generators.

Step 1. *The lemma is true if γ_1 and γ_2 are simple closed separating curves that bound on their right sides genus-1 subsurfaces that only intersect at the basepoint.*

Identify the group π_g of inner automorphisms of π_g with the point-pushing subgroup of $\mathcal{I}_{g,1}$. Since the group $\Delta(\mathcal{I}_{g,1})$ acts trivially on $\mathcal{K}_g \subset (\mathcal{C}_g^{\otimes 2})_{\Delta(\mathcal{I}_{g,1})}$, the subgroup $\Delta(\pi_g)$ of $\Delta(\mathcal{I}_{g,1})$ acts trivially. Letting $h = h_2 - h_1$, we therefore have the following (c.f. §6.7):

$$\langle \gamma_1 \rangle^{h_1} \otimes \langle \gamma_2 \rangle^{h_2} = \langle \gamma_1 \rangle \otimes \langle \gamma_2 \rangle^{h_2 - h_1} = \langle \gamma_1 \rangle \otimes \langle \gamma_2 \rangle^h.$$

Let $T \cong \Sigma_1^1$ be the genus-1 subsurface bounded by γ_1 on its right side and let $S \cong \Sigma_{g-1}^1$ be the subsurface bounded by γ_2 on its left side:

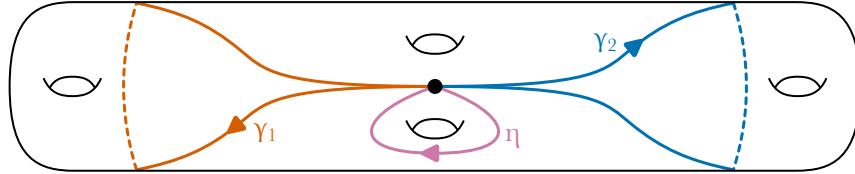


We have $H_{\mathbb{Z}} = H_1(T) \oplus H_1(S)$. Write $h = t + s$ with $t \in H_1(T)$ and $s \in H_1(S)$, so our goal is to prove that $\langle \gamma_1 \rangle \otimes \langle \gamma_2 \rangle^{t+s} = \langle \gamma_1 \rangle \otimes \langle \gamma_2 \rangle^s$.

We first prove that

$$(15.1) \quad \langle \gamma_1 \rangle \otimes \langle \gamma_2 \rangle^{t+s} = \langle \gamma_1 \rangle \otimes \langle \gamma_2 \rangle^s.$$

For this, pick $\lambda \in \pi_1(T) \subset \pi_g$ with $\bar{\lambda} = -t$. Choose a simple closed nonseparating curve $\eta \in \pi_1(S) \subset \pi_g$ that intersects γ_2 as depicted in the following figure:



The curve $\gamma_2 \eta$ is also a simple closed nonseparating curve, and since $\gamma_2 \in [\pi_g, \pi_g]$ the curves $\gamma_2 \eta$ and η are homologous. Using work of Johnson [8], we can find $f \in \mathcal{I}_{g,1}$ that is supported on S such that $f(\eta) = \gamma_2 \eta$. Since $\Delta(\mathcal{I}_{g,1})$ acts trivially on $\mathcal{K}_g \subset (\mathcal{C}_g^{\otimes 2})_{\Delta(\mathcal{I}_{g,1})}$ and also fixes $t, s \in H_{\mathbb{Z}}$, we therefore have

$$(15.2) \quad \langle \gamma_1 \rangle \otimes \langle \eta, \lambda \rangle^{t+s} = \langle f(\gamma_1) \rangle \otimes \langle f(\eta), f(\lambda) \rangle^{f(t)+f(s)} = \langle \gamma_1 \rangle \otimes \langle \gamma_2 \eta, \lambda \rangle^{t+s}.$$

Using our commutator identities (Lemma 6.1) along with the fact that $\bar{\gamma}_2 = 0$, we have

$$\langle \gamma_2 \eta, \lambda \rangle^{t+s} = \langle \gamma_2, \lambda \rangle^{t+s} + \langle \eta, \lambda \rangle^{t+s+\bar{\gamma}_2} = \langle \gamma_2, \lambda \rangle^{t+s} + \langle \eta, \lambda \rangle^{t+s}.$$

Plugging this into (15.2) and canceling the term $\langle \gamma_1 \rangle \otimes \langle \eta, \lambda \rangle^{t+s}$, we get $\langle \gamma_1 \rangle \otimes \langle \gamma_2, \lambda \rangle^{t+s} = 0$.

Using the fact that $\bar{\lambda} = -t$, we conclude that

$$\begin{aligned} 0 &= \langle \gamma_1 \rangle \otimes \langle \gamma_2, \lambda \rangle^{t+s} = \langle \gamma_1 \rangle \otimes \langle \gamma_2 \lambda \gamma_2^{-1} \lambda^{-1} \rangle^{t+s} = \langle \gamma_1 \rangle \otimes \langle \gamma_2 \rangle^{t+s} - \langle \gamma_1 \rangle \otimes \langle \gamma_2 \rangle^{t+s+\bar{\lambda}} \\ &= \langle \gamma_1 \rangle \otimes \langle \gamma_2 \rangle^{t+s} - \langle \gamma_1 \rangle \otimes \langle \gamma_2 \rangle^{t+s+(-t)} = \langle \gamma_1 \rangle \otimes \langle \gamma_2 \rangle^{t+s} - \langle \gamma_1 \rangle \otimes \langle \gamma_2 \rangle^s, \end{aligned}$$

as was claimed in (15.1).

To complete the proof, it is now enough to prove that

$$(15.3) \quad \underline{(\gamma_1) \otimes (\gamma_2)^s} = \underline{(\gamma_1) \otimes (\gamma_2)}.$$

For this, note that since $s \in H_1(S)$ we have that $\underline{(\gamma_1) \otimes (\gamma_2)^s}$ is in the image of the map

$$(15.4) \quad H_1([\pi_1(S), \pi_1(S)]; \mathbb{Q}) \longrightarrow \mathcal{K}_g \subset (\mathcal{C}_g^{\otimes 2})_{\Delta(\mathcal{I}_{g,1})}$$

taking the homology class of $\zeta \in [\pi_1(S), \pi_1(S)]$ to $\underline{(\gamma_1) \otimes (\zeta)}$. Let $\mathcal{I}(S)$ denote the Torelli group of S . By extending mapping classes on S to $\Sigma_{g,1}$ by the identity, we get an inclusion $\mathcal{I}(S) \hookrightarrow \mathcal{I}_{g,1}$. Since $\mathcal{I}(S)$ fixes γ_1 , the map (15.4) factors through the $\mathcal{I}(S)$ -coinvariants as

$$(15.5) \quad H_1([\pi_1(S), \pi_1(S)]; \mathbb{Q})_{\mathcal{I}(S)} \longrightarrow \mathcal{K}_g \subset (\mathcal{C}_g^{\otimes 2})_{\Delta(\mathcal{I}_{g,1})}.$$

Lemma 3.5 implies that³¹

$$(15.6) \quad H_1([\pi_1(S), \pi_1(S)]; \mathbb{Q})_{\mathcal{I}(S)} \cong \wedge^2 H_1(S; \mathbb{Q}).$$

The free group $\pi_1(S)$ also acts on $[\pi_1(S), \pi_1(S)]$ by conjugation, and it is classical that

$$(15.7) \quad H_1([\pi_1(S), \pi_1(S)]; \mathbb{Q})_{\pi_1(S)} \cong \wedge^2 H_1(S; \mathbb{Q}).$$

See, e.g., [20, Theorem C]. The isomorphisms (15.6) and (15.7) give two quotients of $H_1([\pi_1(S), \pi_1(S)]; \mathbb{Q})$ that happen to be isomorphic, and thus two different maps

$$[\pi_1(S), \pi_1(S)] \longrightarrow H_1([\pi_1(S), \pi_1(S)]; \mathbb{Q}) \twoheadrightarrow \wedge^2 H_1(S; \mathbb{Q}).$$

Examining the proofs of Lemma 3.5 and [20, Theorem C], we see that these are actually the same map. This implies that $\pi_1(S)$ -conjugate elements of $[\pi_1(S), \pi_1(S)]$ map to the same element of $H_1([\pi_1(S), \pi_1(S)]; \mathbb{Q})_{\mathcal{I}(S)}$. Choosing $\sigma \in \pi_1(S)$ with $\bar{\sigma} = s$, the images of $\sigma\gamma_2$ and γ_2 in $H_1([\pi_1(S), \pi_1(S)]; \mathbb{Q})_{\mathcal{I}(S)}$ are therefore the same. Mapping this to $(\mathcal{C}_g^{\otimes 2})_{\Delta(\mathcal{I}_{g,1})}$ via (15.5), we conclude that $\underline{(\gamma_1) \otimes (\gamma_2)^s}$ and $\underline{(\gamma_1) \otimes (\gamma_2)}$ are the same, as was claimed in (15.3).

Step 2. *The lemma is true for general γ_i .*

Use Lemma 15.1 to write

$$\underline{(\gamma_1) \otimes (\gamma_2)} = \sum_{j=1}^n c_j \underline{(\delta_{1,j})^{k_{1,j}} \otimes (\delta_{2,j})^{k_{2,j}}}$$

where for $1 \leq j \leq n$ we have $c_j \in \mathbb{Z}$ and the following holds:

- for $i = 1, 2$, the curve $\delta_{i,j} \in [\pi_g, \pi_g]$ is a simple closed separating curve that bounds on its right side a genus-1 subsurface $T_{i,j}$ and $k_{i,j} \in H_{\mathbb{Z}}$ is arbitrary; and
- the intersection of $T_{1,j}$ and $T_{2,j}$ is the basepoint.

We then have

$$(15.8) \quad \underline{(\gamma_1)^{h_1} \otimes (\gamma_2)^{h_2}} = \sum_{j=1}^n c_j \underline{(\delta_{1,j})^{h_1+k_{1,j}} \otimes (\delta_{2,j})^{h_2+k_{2,j}}}.$$

Applying Step 1 to each term in this sum, we get that

$$(15.9) \quad \sum_{j=1}^n c_j \underline{(\delta_{1,j})^{h_1+k_{1,j}} \otimes (\delta_{2,j})^{h_2+k_{2,j}}} = \sum_{j=1}^n c_j \underline{(\delta_{1,j})^{k_{1,j}} \otimes (\delta_{2,j})^{k_{2,j}}} = \underline{(\gamma_1) \otimes (\gamma_2)}.$$

Combining (15.8) and (15.9), we conclude that $\underline{(\gamma_1)^{h_1} \otimes (\gamma_2)^{h_2}} = \underline{(\gamma_1) \otimes (\gamma_2)}$, as desired. \square

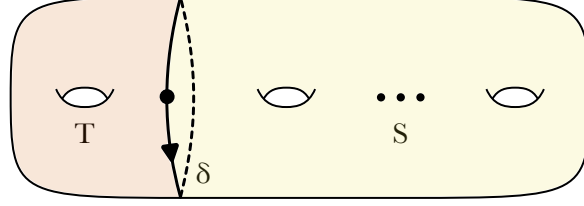
³¹Though when we proved Lemma 3.5 we were working under our standing assumption that $g \geq 4$ (Assumption 1.5), the proof only requires $g \geq 3$ and thus applies to S .

16. A REFINED GENERATING SET FOR \mathcal{K}_g

Our goal in this section is to construct a refined generating set for \mathcal{K}_g .

16.1. Symplectic terminology. A *symplectic summand* of $H_{\mathbb{Z}}$ is a subgroup V of $H_{\mathbb{Z}}$ such that $H_{\mathbb{Z}} = V \oplus V^{\perp}$, where \perp is taken with respect to the algebraic intersection pairing. A symplectic summand V of $H_{\mathbb{Z}}$ is isomorphic to \mathbb{Z}^{2h} for an integer h called its *genus*. For a subgroup W of $H_{\mathbb{Z}}$, let $W_{\mathbb{Q}}$ denote the subspace $W \otimes \mathbb{Q}$ of $H = H_{\mathbb{Z}} \otimes \mathbb{Q}$.

16.2. Generators. Fix a genus-1 symplectic summand V of $H_{\mathbb{Z}}$ and some $\kappa \in \wedge^2 V_{\mathbb{Q}}^{\perp}$. We construct elements $[[V, \kappa]]$ and $[[\kappa, V]]$ of \mathcal{K}_g in the following way. By work of Johnson [8], we can find a simple closed separating curve $\delta \in [\pi_g, \pi_g]$ bounding on its right side a subsurface $T \cong \Sigma_1^1$ with $H_1(T) = V$. Let $S \cong \Sigma_{g-1}^1$ be the subsurface to the left of δ :



Regard $\pi_1(S)$ and $H_1(S)$ as subgroups of π_g and $H_1(\Sigma_g)$, so $H_1(S) = V^{\perp}$. Let

$$\begin{aligned} \phi_L: [\pi_1(S), \pi_1(S)] &\longrightarrow \mathcal{K}_g \subset (\mathcal{C}_g^{\otimes 2})_{\Delta(\mathcal{I}_{g,1})}, \\ \phi_R: [\pi_1(S), \pi_1(S)] &\longrightarrow \mathcal{K}_g \subset (\mathcal{C}_g^{\otimes 2})_{\Delta(\mathcal{I}_{g,1})} \end{aligned}$$

be the maps defined by

$$\phi_L(\eta) = \langle \delta \rangle \otimes \langle \eta \rangle \quad \text{and} \quad \phi_R(\eta) = \langle \eta \rangle \otimes \langle \delta \rangle \quad \text{for } \eta \in [\pi_1(S), \pi_1(S)].$$

Since their targets are \mathbb{Q} -vector spaces, these maps factor through $H_1([\pi_1(S), \pi_1(S)]; \mathbb{Q})$. Letting $\pi_1(S)$ act on $H_1([\pi_1(S), \pi_1(S)]; \mathbb{Q})$ via the conjugation action of $\pi_1(S)$ on $[\pi_1(S), \pi_1(S)]$, it follows from Lemma 15.2 that both induced maps $H_1([\pi_1(S), \pi_1(S)]; \mathbb{Q}) \rightarrow \mathcal{K}_g$ are $\pi_1(S)$ -invariant. They therefore both factor through the coinvariants

$$H_1([\pi_1(S), \pi_1(S)]; \mathbb{Q})_{\pi_1(S)} \cong \wedge^2 H_1(S; \mathbb{Q}) = \wedge^2 V_{\mathbb{Q}}^{\perp},$$

where the first isomorphism is classical (see, e.g., [20, Theorem C]). Let

$$\begin{aligned} \bar{\phi}_L: \wedge^2 V_{\mathbb{Q}}^{\perp} &\longrightarrow \mathcal{K}_g \subset (\mathcal{C}_g^{\otimes 2})_{\Delta(\mathcal{I}_{g,1})}, \\ \bar{\phi}_R: \wedge^2 V_{\mathbb{Q}}^{\perp} &\longrightarrow \mathcal{K}_g \subset (\mathcal{C}_g^{\otimes 2})_{\Delta(\mathcal{I}_{g,1})} \end{aligned}$$

be these two induced maps. Recalling that $\kappa \in \wedge^2 V_{\mathbb{Q}}^{\perp}$, we define

$$[[\delta, \kappa]] = \bar{\phi}_L(\kappa) \quad \text{and} \quad [[\kappa, \delta]] = \bar{\phi}_R(\kappa).$$

We claim this only depends on V :

Lemma 16.1. *Let V be a genus-1 symplectic summand of $H_{\mathbb{Z}}$ and let $\kappa \in \wedge^2 V_{\mathbb{Q}}^{\perp}$. Let $\delta_1, \delta_2 \in [\pi_g, \pi_g]$ be simple closed separating curves such that δ_i bounds on its right side a subsurface $T_i \cong \Sigma_1^1$ with $H_1(T_i) = V$. Then $[[\delta_1, \kappa]] = [[\delta_2, \kappa]]$ and $[[\kappa, \delta_1]] = [[\kappa, \delta_2]]$.*

Proof. By work of Johnson [8], we can find $f \in \mathcal{I}_{g,1}$ such that $f(\delta_1) = \delta_2$. Recall that $\Delta: \mathcal{I}_{g,1} \rightarrow \mathcal{I}_{g,1} \times \mathcal{I}_{g,1}$ is the diagonal map and $\mathcal{K}_g \subset (\mathcal{C}_g^{\otimes 2})_{\Delta(\mathcal{I}_{g,1})}$. By construction,

$$[[\delta_2, \kappa]] = \Delta(f)([[\delta_1, \kappa]]) \quad \text{and} \quad \Delta(f)([[\kappa, \delta_1]]) = [[\kappa, \delta_2]].$$

Since $\Delta(\mathcal{I}_{g,1})$ acts trivially on $\mathcal{K}_g \subset (\mathcal{C}_g^{\otimes 2})_{\Delta(\mathcal{I}_{g,1})}$, the lemma follows. \square

Because of this lemma, we can define $[[V, \kappa]] = [[\delta, \kappa]]$ and $[[\kappa, V]] = [[\kappa, \delta]]$.

16.3. Refined generating set. The following says that the $\llbracket V, \kappa \rrbracket$ and $\llbracket \kappa, V \rrbracket$ generate \mathcal{K}_g and identifies some relations between them:

Lemma 16.2. *The vector space \mathcal{K}_g is generated by the $\llbracket V, \kappa \rrbracket$ and $\llbracket \kappa, V \rrbracket$ as V ranges over genus-1 symplectic summand of $H_{\mathbb{Z}}$ and κ ranges over elements of $\wedge^2 V_{\mathbb{Q}}^{\perp}$. Moreover, for a genus-1 symplectic summand V and $\kappa_1, \kappa_2 \in \wedge^2 V_{\mathbb{Q}}^{\perp}$ and $\lambda_1, \lambda_2 \in \mathbb{Q}$ we have relations*

$$\begin{aligned}\llbracket V, \lambda_1 \kappa_1 + \lambda_2 \kappa_2 \rrbracket &= \lambda_1 \llbracket V, \kappa_1 \rrbracket + \lambda_2 \llbracket V, \kappa_2 \rrbracket, \\ \llbracket \lambda_1 \kappa_1 + \lambda_2 \kappa_2, V \rrbracket &= \lambda_1 \llbracket \kappa_1, V \rrbracket + \lambda_2 \llbracket \kappa_2, V \rrbracket.\end{aligned}$$

Proof. The elements $\llbracket V, \kappa \rrbracket$ and $\llbracket \kappa, V \rrbracket$ generate \mathcal{K}_g since by Lemma 15.2 they contain all the generators for \mathcal{K}_g identified by Lemma 15.1. The indicated relations are all immediate from the construction of $\llbracket V, \kappa \rrbracket$ and $\llbracket \kappa, V \rrbracket$. \square

17. REDUNDANCIES AMONG GENERATORS FOR \mathcal{K}_g

The generating set for \mathcal{K}_g given by Lemma 16.2 has some redundancies.

17.1. Commutator projection. Describing these redundancies requires some preliminaries. Let F be a free group. The group F acts on conjugation on $[F, F]$, and it is classical that the coinvariants of the induced action on $H_1([F, F])$ satisfy

$$H_1([F, F])_F \cong \wedge^2 H_1(F).$$

See, e.g., [20, Theorem C]. We have used this isomorphism several times already. Let $\rho: [F, F] \rightarrow \wedge^2 H_1(F)$ be the composition

$$[F, F] \hookrightarrow H_1([F, F]) \twoheadrightarrow H_1([F, F])_F \cong \wedge^2 H_1(F).$$

We will call this the *commutator projection* map. For $z \in F$, let \bar{z} be the image of z in $H_1(F)$. The commutator projection map satisfies

$$\rho([x, y]) = \bar{x} \wedge \bar{y} \quad \text{for all } x, y \in F.$$

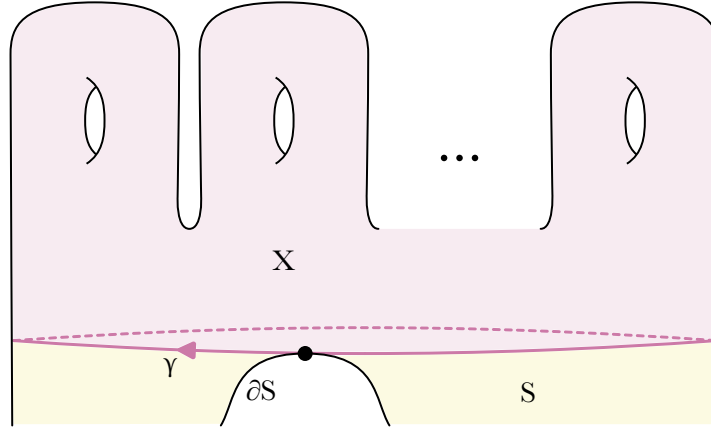
17.2. Subsurface intersection form. Let W be a genus- h symplectic summand of $H_{\mathbb{Z}}$. Alternating bilinear forms on W can be identified with elements of $\wedge^2 W$. In particular, the restriction to W of the algebraic intersection form can be identified with an element ω_W of $\wedge^2 W \subset \wedge^2 H_{\mathbb{Z}}$. If $\{a_1, b_1, \dots, a_h, b_h\}$ is a symplectic basis for W , then

$$\omega_W = a_1 \wedge b_1 + \dots + a_h \wedge b_h.$$

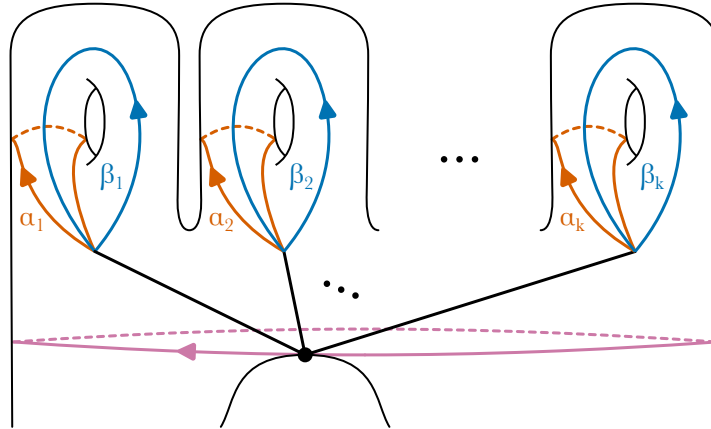
The importance for us of these elements comes from:

Lemma 17.1. *Let $S \cong \Sigma_h^1$ be a subsurface of Σ_g such that the basepoint $*$ of Σ_g lies on ∂S and let $\rho: [\pi_1(S), \pi_1(S)] \rightarrow \wedge^2 H_1(S)$ be the commutator projection map. Let $\gamma \in [\pi_1(S), \pi_1(S)]$ be a simple closed separating curve bounding on its right side a subsurface $X \cong \Sigma_k^1$ of S with $\partial X \cap \partial S = \{*\}$. Then $\rho(\gamma) = \omega_{H_1(X)}$.*

Proof. We can draw S and γ and X as follows:



As in the following figure, we can then find a generating set $\{\alpha_1, \beta_1, \dots, \alpha_k, \beta_k\}$ for $\pi_1(X)$ such that $\gamma = [\alpha_1, \beta_1] \cdots [\alpha_k, \beta_k]$:



We then have $\rho(\gamma) = \rho([\alpha_1, \beta_1]) + \cdots + \rho([\alpha_k, \beta_k]) = \bar{\alpha}_1 \wedge \bar{\beta}_1 + \cdots + \bar{\alpha}_k \wedge \bar{\beta}_k = \omega_{H_1(X)}$. \square

17.3. Redundancy. With the above preliminaries, the following identifies the redundancies between our generators:

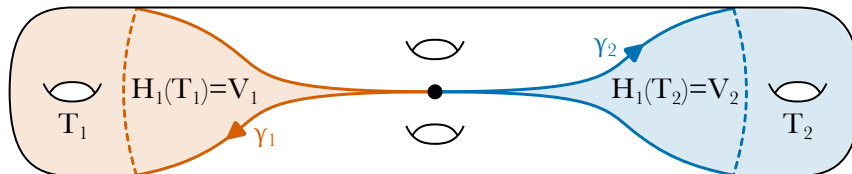
Lemma 17.2. *In \mathcal{K}_g , we have the following relations:*

- (a) *For all orthogonal genus-1 symplectic summands V_1 and V_2 of $H_{\mathbb{Z}}$, the relation $[[V_1, \omega_{V_2}]] = [[\omega_{V_1}, V_2]]$.*
- (b) *For all genus-1 symplectic summands V of $H_{\mathbb{Z}}$, the relation $[[V, \omega_{V^\perp}]] = [[\omega_{V^\perp}, V]]$.*

Proof. We start by verifying (a). Let V_1 and V_2 be orthogonal genus-1 symplectic summands of $H_{\mathbb{Z}}$. Using work of Johnson [8], we can find $\gamma_1, \gamma_2 \in \pi_g$ such that:

- for $i = 1, 2$, the curve γ_i is a simple closed separating curve bounding a genus-1 surface $T_i \cong \Sigma_1^1$ on its right side with $H_1(T_i) = V_i$; and
- the intersection of T_1 and T_2 is the basepoint.

See here:

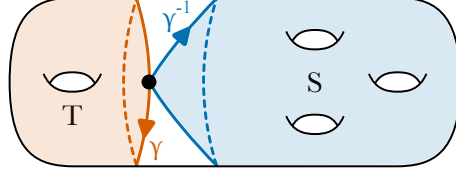


Using Lemma 17.1, we have

$$[[V_1, \omega_{V_2}]] = \langle \gamma_1 \rangle \otimes \langle \gamma_2 \rangle \quad \text{and} \quad [[\omega_{V_1}, V_2]] = \langle \gamma_1 \rangle \otimes \langle \gamma_2 \rangle,$$

so $[[V_1, \omega_{V_2}]] = [[\omega_{V_1}, V_2]]$, as claimed in (a).

We next verify (b). Let V be a genus-1 symplectic summand of $H_{\mathbb{Z}}$. Again using work of Johnson [8], we can find a simple close separating curve $\gamma \in [\pi_g, \pi_g]$ that bounds a genus-1 subsurface $T \cong \Sigma_1^1$ on its right side with $H_1(T) = V$. Note that γ^{-1} can be homotoped to be disjoint from γ and bound a subsurface $S \cong \Sigma_{g-1}^1$ on its right side such that $H_1(S) = V^{\perp}$ and such that S and T only intersect at the basepoint:



Using Lemma 17.1, we have

$$[[V, \omega_{V^{\perp}}]] = \langle \gamma \rangle \otimes \langle \gamma^{-1} \rangle \quad \text{and} \quad [[\omega_{V^{\perp}}, V]] = \langle \gamma^{-1} \rangle \otimes \langle \gamma \rangle.$$

It follows that

$$[[V, \omega_{V^{\perp}}]] = \langle \gamma \rangle \otimes \langle \gamma^{-1} \rangle = -\langle \gamma \rangle \otimes \langle \gamma \rangle = \langle \gamma^{-1} \rangle \otimes \langle \gamma \rangle = [[\omega_{V^{\perp}}, V]],$$

as claimed by (b). \square

18. IDENTIFYING \mathcal{K}_g

Recall that the algebraization map is the map

$$\mathbf{a}: (\mathcal{C}_g^{\otimes 2})_{\Delta(\mathcal{I}_{g,1})} \rightarrow (\mathcal{C}_g^{\otimes 2})_{\mathcal{I}_{g,1} \times \mathcal{I}_{g,1}} \cong ((\wedge^2 H)/\mathbb{Q})^{\otimes 2}.$$

See Corollary 14.2 for this isomorphism. We close the paper by proving Theorem 14.3, whose statement we recall:

Theorem 14.3. *The restriction of the algebraization map \mathbf{a} to \mathcal{K}_g is an injection.*

Proof. For $\kappa \in \wedge^2 H$, let $\bar{\kappa}$ be the image of κ in $(\wedge^2 H)/\mathbb{Q}$. For a genus-1 symplectic summand V of $H_{\mathbb{Z}}$ and $\kappa \in \wedge^2 V_{\mathbb{Q}}^{\perp}$, it is immediate from Lemma 17.1 that

$$\mathbf{a}([[V, \kappa]]) = \bar{\omega}_V \otimes \bar{\kappa} \quad \text{and} \quad \mathbf{a}([\kappa, V]) = \bar{\kappa} \otimes \bar{\omega}_V.$$

Here we are identifying $\wedge^2 V_{\mathbb{Q}}^{\perp}$ with the corresponding subspace of $\wedge^2 H$ to allow us to talk about $\bar{\kappa} \in (\wedge^2 H)/\mathbb{Q}$.

Now define \mathfrak{K}_g to be the vector space with the following presentation:

- **Generators.** For all genus-1 symplectic summands V of $H_{\mathbb{Z}}$ and all $\kappa \in \wedge^2 V_{\mathbb{Q}}^{\perp}$, generators $[[V, \kappa]]'$ and $[\kappa, V]'$.
- **Relations.** The following families of relations:
 - For all genus-1 symplectic summands V of $H_{\mathbb{Z}}$ and all $\kappa_1, \kappa_2 \in \wedge^2 V_{\mathbb{Q}}^{\perp}$ and all $\lambda_1, \lambda_2 \in \mathbb{Q}$, the linearity relations

$$\begin{aligned} [[V, \lambda_1 \kappa_1 + \lambda_2 \kappa_2]]' &= \lambda_1 [[V, \kappa_1]]' + \lambda_2 [[V, \kappa_2]]' \quad \text{and} \\ [\lambda_1 \kappa_1 + \lambda_2 \kappa_2, V]' &= \lambda_1 [\kappa_1, V]' + \lambda_2 [\kappa_2, V]'. \end{aligned}$$

- For all orthogonal genus-1 symplectic summands V and W of $H_{\mathbb{Z}}$, the relation

$$[[V, \omega_W]]' = [[\omega_V, W]]'.$$

– For all genus-1 symplectic summands V of $H_{\mathbb{Z}}$, the relation

$$\llbracket V, \omega_{V^\perp} \rrbracket' = \llbracket \omega_{V^\perp}, V \rrbracket'.$$

Define a map $\pi: \mathfrak{K}_g \rightarrow \mathcal{K}_g$ on generators $\llbracket V, \kappa \rrbracket'$ and $\llbracket \kappa, V \rrbracket'$ by letting

$$\pi(\llbracket V, \kappa \rrbracket') = \llbracket V, \kappa \rrbracket \quad \text{and} \quad \pi(\llbracket \kappa, V \rrbracket') = \llbracket V, \kappa \rrbracket.$$

This makes sense since by Lemmas 16.2 and 17.2 it takes relations to relations. Moreover, since the image of π contains all the generators of \mathfrak{K}_g identified by Lemma 16.2 it follows that π is surjective.

The composition $\mathfrak{a} \circ \pi: \mathfrak{K}_g \rightarrow ((\wedge^2 H)/\mathbb{Q})^{\otimes 2}$ satisfies

$$\mathfrak{a} \circ \pi(\llbracket V, \kappa \rrbracket') = \bar{\omega}_V \otimes \bar{\kappa} \quad \text{and} \quad \mathfrak{a} \circ \pi(\llbracket \kappa, V \rrbracket') = \bar{\kappa} \otimes \bar{\omega}_V.$$

In [15, Theorem A.6], the authors proved that this map $\mathfrak{a} \circ \pi$ is injective. The paper [15] calls the image of $\mathfrak{a} \circ \pi$ the *symmetric kernel*. It is the kernel of a contraction

$$((\wedge^2 H)/\mathbb{Q})^{\otimes 2} \rightarrow \text{Sym}^2(H).$$

Since π is surjective and $\mathfrak{a} \circ \pi$ is injective, it follows that \mathfrak{a} is injective,³² as desired. \square

REFERENCES

- [1] T. Akita, Homological infiniteness of Torelli groups, *Topology* 40 (2001), no. 2, 213–221. [arXiv:alg-geom/9712006](#) (Cited on page 1.)
- [2] M. Bestvina, K. Bux, & D. Margalit, The dimension of the Torelli group, *J. Amer. Math. Soc.* 23 (2010), no. 1, 61–105. [arXiv:0709.0287](#) (Cited on page 1.)
- [3] J. S. Birman, Mapping class groups and their relationship to braid groups, *Comm. Pure Appl. Math.* 22 (1969), 213–238. (Cited on page 6.)
- [4] B. Farb & D. Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, 49, Princeton Univ. Press, Princeton, NJ, 2012. (Cited on pages 4 and 22.)
- [5] A. Gaifullin, On infinitely generated homology of Torelli groups, preprint 2018. [arXiv:1803.09311](#) (Cited on page 1.)
- [6] J. L. Harer, The virtual cohomological dimension of the mapping class group of an orientable surface, *Invent. Math.* 84 (1986), no. 1, 157–176. (Cited on page 1.)
- [7] D. Johnson, Homeomorphisms of a surface which act trivially on homology *Proc. Amer. Math. Soc.* 75 (1979), no. 1, 119–125. (Cited on page 11.)
- [8] D. Johnson, Conjugacy relations in subgroups of the mapping class group and a group-theoretic description of the Rochlin invariant, *Math. Ann.* 249 (1980), no. 3, 243–263. (Cited on pages 23, 40, 42, 44, and 45.)
- [9] D. Johnson, The structure of the Torelli group. I. A finite set of generators for \mathcal{I} , *Ann. of Math.* (2) 118 (1983), no. 3, 423–442. (Cited on page 1.)
- [10] D. Johnson, The structure of the Torelli group. II. A characterization of the group generated by twists on bounding curves, *Topology* 24 (1985), no. 2, 113–126. (Cited on page 1.)
- [11] D. Johnson, The structure of the Torelli group. III. The abelianization of \mathcal{T} , *Topology* 24 (1985), no. 2, 127–144. (Cited on pages 1, 14, and 15.)
- [12] N. Kawazumi, On the stable cohomology algebra of extended mapping class groups for surfaces, in *Groups of diffeomorphisms*, 383–400, *Adv. Stud. Pure Math.*, 52, Math. Soc. Japan, Tokyo. (Cited on page 1.)
- [13] E. Looijenga, Stable cohomology of the mapping class group with symplectic coefficients and of the universal Abel-Jacobi map, *J. Algebraic Geom.* 5 (1996), no. 1, 135–150. [arXiv:alg-geom/9401005](#) (Cited on page 1.)
- [14] I. Madsen & M. Weiss, The stable moduli space of Riemann surfaces: Mumford’s conjecture, *Ann. of Math.* (2) 165 (2007), no. 3, 843–941. [arXiv:math/0212321](#) (Cited on page 1.)
- [15] D. Minahan & A. Putman, Presentations of representations, preprint 2025. (Cited on page 46.)
- [16] D. Minahan & A. Putman, The second rational homology group of the Torelli group, preprint 2025. (Cited on page 2.)

³²And also that π is an isomorphism, so \mathcal{K}_g is also isomorphic to the symmetric kernel.

- [17] A. Putman, Cutting and pasting in the Torelli group, *Geom. Topol.* 11 (2007), no. 2, 829–865. [arXiv:math/0608373](https://arxiv.org/abs/math/0608373) (Cited on pages 1, 7, 8, 22, 23, and 39.)
- [18] A. Putman, Abelian covers of surfaces and the homology of the level L mapping class group, *J. Topol. Anal.* 3 (2011), no. 3, 265–306. [arXiv:0907.1718](https://arxiv.org/abs/0907.1718) (Cited on pages 3, 4, and 12.)
- [19] A. Putman, The Johnson homomorphism and its kernel, *J. Reine Angew. Math.* 735 (2018), 109–141. [arXiv:0904.0467](https://arxiv.org/abs/0904.0467) (Cited on pages 8 and 11.)
- [20] A. Putman, The commutator subgroups of free groups and surface groups, *Enseign. Math.* 68 (2022), no. 3-4, 389–408. [arXiv:2101.05905](https://arxiv.org/abs/2101.05905) (Cited on pages 6, 13, 41, 42, and 43.)
- [21] A. Putman, The stable cohomology of the moduli space of curves with level structures, preprint 2022. [arXiv:2209.06183](https://arxiv.org/abs/2209.06183) (Cited on pages 3 and 12.)
- [22] A. Putman, Realizing homology classes by simple closed curves, informal note, <https://www.nd.edu/~andyp/notes/SimpleClosedCurves.pdf>. (Cited on page 21.)
- [23] K. Reidemeister, Homotopiegruppen von komplexen, *Abh. Math. Sem. Univ. Hamburg* 10 (1934), no. 1, 211–215. (Cited on page 11.)
- [24] K. Reidemeister, Complexes and homotopy chains, *Bull. Amer. Math. Soc.* 56 (1950), 297–307. (Cited on page 11.)
- [25] T. Sakasai, A survey of Magnus representations for mapping class groups and homology cobordisms of surfaces, in *Handbook of Teichmüller theory. Volume III*, IRMA Lect. Math. Theor. Phys., 17, 531–594. [arXiv:1005.5501](https://arxiv.org/abs/1005.5501) (Cited on page 2.)
- [26] X. Zhong, Prym representations and twisted cohomology of the mapping class group with level structures, preprint 2024. [arXiv:2401.13869](https://arxiv.org/abs/2401.13869) (Cited on pages 1 and 3.)

DEPT OF MATHEMATICS; UNIVERSITY OF CHICAGO; CHICAGO, IL 60637
Email address: dminahan@uchicago.edu

DEPT OF MATHEMATICS; UNIVERSITY OF NOTRE DAME; 255 HURLEY HALL; NOTRE DAME, IN 46556
Email address: andyp@nd.edu