THE SECOND RATIONAL HOMOLOGY OF THE TORELLI GROUP

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ABSTRACT. We calculate the second rational homology group of the Torelli group for $g \geq 6$.

1. INTRODUCTION

Let $\Sigma_{g,p}^b$ be an oriented genus g surface with p marked points and b boundary components. We often omit p or b if they vanish. The mapping class group $\mathrm{Mod}_{g,p}^b$ is the group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,p}^b$ that fix each marked point and boundary component pointwise. Deleting the marked points and gluing discs to the boundary components, we get an action of $\mathrm{Mod}_{g,p}^b$ on $\mathrm{H}_1(\Sigma_g)$ that fixes the algebraic intersection form. This gives a surjection $\mathrm{Mod}_{g,p}^b \to \mathrm{Sp}_{2g}(\mathbb{Z})$ whose kernel $\mathcal{I}_{g,p}^b$ is the Torelli group:

$$1 \longrightarrow \mathcal{I}_{g,p}^b \longrightarrow \mathrm{Mod}_{g,p}^b \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1.$$

Johnson [23, 24] proved that $\mathcal{I}_{g,p}^b$ is finitely generated for $g \geq 3$ and calculated $\mathrm{H}^1(\mathcal{I}_{g,p}^b)$. The conjugation action of $\mathrm{Mod}_{g,p}^b$ on $\mathcal{I}_{g,p}^b$ induces an action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on each $\mathrm{H}^d(\mathcal{I}_{g,p}^b)$. Let $H = \mathrm{H}^1(\Sigma_g; \mathbb{Q})$. For $g \geq 3$, it follows from Johnson's work (see [15, Theorem 3.5]) that there is an $\mathrm{Sp}_{2g}(\mathbb{Z})$-equivariant isomorphism[1]

$$\mathrm{H}^1(\mathcal{I}_{g,p}^b; \mathbb{Q}) \cong H^{\oplus(p+b)} \oplus (\wedge^3 H)/H.$$

In particular, $\mathrm{H}^1(\mathcal{I}_{g,p}^b; \mathbb{Q})$ is a finite-dimensional algebraic representation[2] of $\mathrm{Sp}_{2g}(\mathbb{Z})$.

1.1. **Main theorem.** A long-standing folk conjecture says that $\mathrm{H}^2(\mathcal{I}_{g,p}^b; \mathbb{Q})$ is also a finite-dimensional[3] algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$ for $g \gg 0$. We prove this for $g \geq 6$. When $p + b \leq 1$, we actually compute $\mathrm{H}^2(\mathcal{I}_{g,p}^b; \mathbb{Q})$. The irreducible algebraic representations of $\mathrm{Sp}_{2g}(\mathbb{Z})$ are indexed by partitions σ with at most g parts (see [11, §17]). Let $\mathbf{V}_\sigma$ be the representation corresponding to σ, so $\mathbf{V}_1 = H$ and $\mathbf{V}_{1^3} = (\wedge^3 H)/H$. We prove:

Theorem A. *For $g \geq 6$, we have*

$$\begin{aligned}
\mathrm{H}^2(\mathcal{I}_g; \mathbb{Q}) &\cong & \mathbf{V}_{1^2} && \oplus \mathbf{V}_{1^4} \oplus \mathbf{V}_{2^2,1^2} \oplus \mathbf{V}_{1^6}, \\
\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q}) &\cong & \mathbf{V}_{1^2}^{\oplus 2} \oplus \mathbf{V}_{2,1^2} \oplus \mathbf{V}_{1^4}^{\oplus 2} \oplus \mathbf{V}_{2^2,1^2} \oplus \mathbf{V}_{1^6}, \\
\mathrm{H}^2(\mathcal{I}_{g,1}; \mathbb{Q}) &\cong \mathbf{V}_0 \oplus \mathbf{V}_{1^2}^{\oplus 2} \oplus \mathbf{V}_{2,1^2} \oplus \mathbf{V}_{1^4}^{\oplus 2} \oplus \mathbf{V}_{2^2,1^2} \oplus \mathbf{V}_{1^6}.
\end{aligned}$$

In all three cases, these cohomology groups are spanned by cup products of elements of H^1.

To explain the origin of the representations in Theorem A, consider the cup product pairing $\wedge^2 \mathrm{H}^1(\mathcal{I}_{g,p}^b; \mathbb{Q}) \to \mathrm{H}^2(\mathcal{I}_{g,p}^b; \mathbb{Q})$. We have

$$\wedge^2 \mathrm{H}^1(\mathcal{I}_g; \mathbb{Q}) \cong \wedge^2((\wedge^3 H)/H) \quad \text{and} \quad \wedge^2 \mathrm{H}^1(\mathcal{I}_g^1; \mathbb{Q}) \cong \wedge^2 \mathrm{H}^1(\mathcal{I}_{g,1}; \mathbb{Q}) \cong \wedge^2(\wedge^3 H).$$

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[1]In this, the inclusion $H \hookrightarrow \wedge^3 H$ takes $h \in H$ to $h \wedge \omega$, where $\omega \in \wedge^2 H$ is the algebraic intersection form.

[2]A representation $\mathbf{V}$ of $\mathrm{Sp}_{2g}(\mathbb{Z})$ over a field $\mathbf{k}$ of characteristic 0 is algebraic if the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathbf{V}$ extends to a polynomial representation of the $\mathbf{k}$-points $\mathrm{Sp}_{2g}(\mathbf{k})$ of the algebraic group Sp_{2g}. Since $\mathrm{Sp}_{2g}(\mathbb{Z})$ is Zariski dense in $\mathrm{Sp}_{2g}(\mathbf{k})$, such an extension is unique if it exists.

[3]We emphasize that even finite-dimensionality was unknown before our work.

These decompose as[4]

$$(1.1) \qquad \wedge^2\left((\wedge^3 H)/H\right) \cong \mathbf{V}_0 \ \oplus \mathbf{V}_{1^2} \ \oplus \mathbf{V}_{2^2} \qquad\qquad \oplus \mathbf{V}_{1^4} \ \oplus \mathbf{V}_{2^2,1^2} \oplus \mathbf{V}_{1^6},$$
$$\wedge^2\left(\wedge^3 H\right) \qquad \cong \mathbf{V}_0^{\oplus 2} \oplus \mathbf{V}_{1^2}^{\oplus 3} \oplus \mathbf{V}_{2^2} \oplus \mathbf{V}_{2,1^2} \oplus \mathbf{V}_{1^4}^{\oplus 2} \oplus \mathbf{V}_{2^2,1^2} \oplus \mathbf{V}_{1^6}.$$

Hain [17] proved that the kernel of the cup product pairing on $\mathcal{I}_g$ is isomorphic to $\mathbf{V}_0 \oplus \mathbf{V}_{2^2}$. Habegger–Sorger [14] extended this, proving that for $\mathcal{I}_g^1$ the kernel is isomorphic to $\mathbf{V}_0^{\oplus 2} \oplus \mathbf{V}_{1^2} \oplus \mathbf{V}_{2^2}$. From this, it is easy to prove that the kernel for $\mathcal{I}_{g,1}$ is isomorphic to $\mathbf{V}_0 \oplus \mathbf{V}_{1^2} \oplus \mathbf{V}_{2^2}$. Deleting these kernels from (1.1) gives the representations in Theorem A.

Kupers–Randal-Williams [26] showed that in the above cases the image of the cup product pairing is the maximal algebraic subrepresentation of $\mathrm{H}^2(\mathcal{I}_{g,p}^b; \mathbb{Q})$. To prove Theorem A, we must show that this is all of $\mathrm{H}^2(\mathcal{I}_{g,p}^b; \mathbb{Q})$. We actually prove more:[5]

Theorem B. *Let* $b, p \geq 0$. *Then* $\mathrm{H}_2(\mathcal{I}_{g,p}^b; \mathbb{Q})$ *is finite dimensional for* $g \geq 5$ *and an algebraic representation of* $\mathrm{Sp}_{2g}(\mathbb{Z})$ *for* $g \geq 6$.

Remark 1.1. Our proof only works over $\mathbb{Q}$. It is not known if $\mathrm{H}_2(\mathcal{I}_{g,p}^b)$ is finitely generated.   □

Remark 1.2. This paper supersedes a paper of of Minahan [35] that uses a primitive version of our argument to prove that $\mathrm{H}_2(\mathcal{I}_g; \mathbb{Q})$ is finite-dimensional for $g \geq 51$.   □

1.2. Representation stability. Theorem A implies the following:[6]

Corollary C. *The sequence of* $\mathrm{Sp}_{2g}(\mathbb{Z})$-*representations* $\{\mathrm{H}_2(\mathcal{I}_g^1; \mathbb{Q})\}_{g=1}^\infty$ *is uniformly representation stable starting at* $g = 6$.

This was conjectured by Church–Farb [9, Conjecture 6.1]. It means that for $g \geq 6$ the maps

$$(1.2) \qquad\qquad \mathrm{H}_2(\mathcal{I}_g^1; \mathbb{Q}) \longrightarrow \mathrm{H}_2(\mathcal{I}_{g+1}^1; \mathbb{Q})$$

induced by embedding Σ_g^1 into Σ_{g+1}^1 and extending mapping classes lying in $\mathcal{I}_g^1$ to Σ_{g+1}^1 by the identity are injective and (roughly speaking) match up the decompositions of $\mathrm{H}_2(\mathcal{I}_g^1; \mathbb{Q})$ and $\mathrm{H}_2(\mathcal{I}_{g+1}^1; \mathbb{Q})$ into irreducible representations of the symplectic groups. Partial results in this direction were previously proven by Boldsen–Dollerup [4] and Miller–Patzt–Wilson [33]. However, before our work it was not even known if the maps (1.2) were injective for $g \gg 0$.

Remark 1.3. Since the representation $\mathbf{V}_{1^6}$ appears in $\mathrm{H}_2(\mathcal{I}_6^1; \mathbb{Q})$ and is not the stabilization of a representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$ for $g = 5$, the $g = 6$ in Corollary C is optimal.   □

1.3. Previous work. Questions about $\mathrm{H}_\bullet(\mathcal{I}_{g,p}^b)$ can generally be reduced to questions about $\mathrm{H}_\bullet(\mathcal{I}_g)$ (see, e.g., §6.1), so we mostly focus on this.

1.3.1. *Low genus.* As we said, for $g \geq 3$ Johnson [23, 24] proved that $\mathcal{I}_g$ is finitely generated and computed $\mathrm{H}_1(\mathcal{I}_g)$. In contrast, McCullough–Miller [30] proved that $\mathcal{I}_2$ is not finitely generated, and later Mess [32] proved that $\mathcal{I}_2$ is an infinite rank free group. Johnson–Millson (cf. [32]) and Hain [16] proved that $\mathrm{H}_3(\mathcal{I}_3; \mathbb{Q})$ and $\mathrm{H}_4(\mathcal{I}_3; \mathbb{Q})$ are infinite dimensional. Spiridonov [51] later calculated $\mathrm{H}_4(\mathcal{I}_3; \mathbb{Q})$. Some evidence that $\mathrm{H}_2(\mathcal{I}_3)$ might not be finitely generated was given by Gaifullin [13].

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[4]This calculation can easily be done using the program "LiE"; see [27].

[5]In this theorem, we switch to homology since that is more natural for our proofs.

[6]This requires not only the formula for $\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q})$ from Theorem A, but also the explicit isomorphism underlying it. A version of Corollary C without the explicit starting value $g = 6$ also follows from Theorem B along with [38, Theorem B] and [4].

1.3.2. *Second homology.* For $g \geq 3$, the only earlier finiteness result about $H_2(\mathcal{I}_g)$ was a theorem of Kassabov–Putman [25] saying that $H_2(\mathcal{I}_g)$ is spanned by the $\mathrm{Sp}_{2g}(\mathbb{Z})$-orbit of a finite set. In other words, it is finitely generated as a module over $\mathbb{Z}[\mathrm{Sp}_{2g}(\mathbb{Z})]$. Boldsen–Dollerup [4] also proved a related theorem that roughly speaking says that $H_2(\mathcal{I}_g^1)$ is spanned by classes supported on genus 6 subsurfaces for $g \geq 6$.

The torsion in $H_2(\mathcal{I}_g)$ remains mysterious. The group $H_1(\mathcal{I}_g)$ calculated by Johnson [24] contains 2-torsion coming from the Birman–Craggs–Johnson (BCJ) homomorphism. This was constructed by Johnson [21] using work of Birman–Craggs [3] on the Rochlin invariant of homology 3-spheres. Though they were unable to compute it completely, Brendle–Farb [6] constructed large parts of $H_2(\mathcal{I}_g; \mathbb{F}_2)$ that are detected by the BCJ homomorphism.

1.3.3. *High degree.* Akita [1] proved that for each $g \geq 7$ there exists some d such that $H_d(\mathcal{I}_g; \mathbb{Q})$ is infinite dimensional. Bestvina–Bux–Margalit [2] sharpened this and proved that $\mathcal{I}_g$ has cohomological dimension $3g - 5$ and that $H_{3g-5}(\mathcal{I}_g; \mathbb{Q})$ is infinite dimensional. Later Gaifullin [12] proved for $g \geq 2$ that $H_d(\mathcal{I}_g; \mathbb{Q})$ is infinite dimensional for $2g - 3 \leq d \leq 3g - 5$. For $g = 3$, this recovers the infinite dimensionality results discussed in §1.3.1 above. Other work constructing high-dimensional classes can be found in [8].

1.4. **Stability.** We prove Theorem B via a criterion with a representation stability flavor. A *coherent sequence* of representations of $\mathrm{Sp}_{2g}(\mathbb{Z})$ over a field $\mathbf{k}$ is a sequence

$$\mathbf{V}_1 \xrightarrow{f_1} \mathbf{V}_2 \xrightarrow{f_2} \mathbf{V}_3 \xrightarrow{f_3} \cdots$$

of $\mathbf{k}$-vector spaces connected by linear maps such that the following hold:

- each $\mathbf{V}_g$ is a representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$; and
- the maps $f_g \colon \mathbf{V}_g \boxtimes \mathbf{k} \to \mathbf{V}_{g+1}$ are $\mathrm{Sp}_{2g}(\mathbb{Z}) \times \mathrm{Sp}_2(\mathbb{Z})$-equivariant[7] where $\mathrm{Sp}_{2g}(Z) \times \mathrm{Sp}_2(\mathbb{Z})$ acts on $\mathbf{V}_{g+1}$ via the standard inclusion $\mathrm{Sp}_{2g}(\mathbb{Z}) \times \mathrm{Sp}_2(\mathbb{Z}) \hookrightarrow \mathrm{Sp}_{2(g+1)}(\mathbb{Z})$.

Example 1.4. Let $\iota_g \colon \Sigma_g^1 \hookrightarrow \Sigma_{g+1}^1$ be the following embedding:
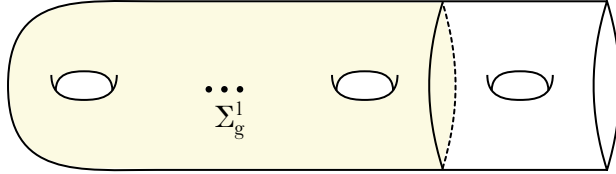
The induced map on homology $(\iota_g)_* \colon H_1(\Sigma_g^1; \mathbf{k}) \to H_1(\Sigma_{g+1}^1; \mathbf{k})$ fits into a coherent sequence

$$H_1(\Sigma_1^1; \mathbf{k}) \xrightarrow{(\iota_1)_*} H_1(\Sigma_2^1; \mathbf{k}) \xrightarrow{(\iota_2)_*} H_1(\Sigma_3^1; \mathbf{k}) \xrightarrow{(\iota_3)_*} \cdots$$

of representations of $\mathrm{Sp}_{2g}(\mathbb{Z})$. $\qquad \square$

Example 1.5. The map $\iota_g \colon \Sigma_g^1 \hookrightarrow \Sigma_{g+1}^1$ from Example 1.4 induces a map $t_g \colon \mathcal{I}_g^1 \to \mathcal{I}_{g+1}^1$ that extends mapping classes lying in $\mathcal{I}_g^1$ to Σ_{g+1}^1 by the identity. For a fixed $d \geq 0$, the map t_g induces a map $f_g \colon H_d(\mathcal{I}_g^1; \mathbf{k}) \to H_d(\mathcal{I}_{g+1}^1; \mathbf{k})$. These fit into a coherent sequence

$$H_d(\mathcal{I}_1^1; \mathbf{k}) \xrightarrow{f_1} H_d(\mathcal{I}_2^1; \mathbf{k}) \xrightarrow{f_2} H_d(\mathcal{I}_3^1; \mathbf{k}) \xrightarrow{f_3} \cdots$$

of representations of $\mathrm{Sp}_{2g}(\mathbb{Z})$. $\qquad \square$

We will use the following theorem to prove our results about the Torelli group:

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[7]Here $\mathbf{V}_g \boxtimes \mathbf{k}$ denotes the $\mathrm{Sp}_{2g}(\mathbb{Z}) \times \mathrm{Sp}_2(\mathbb{Z})$-representation that equals the tensor product of the $\mathrm{Sp}_{2g}(\mathbb{Z})$-representation $\mathbf{V}_g$ and the trivial $\mathrm{Sp}_2(\mathbb{Z})$-representation $\mathbf{k}$. As a vector space, it is isomorphic to $\mathbf{V}_g$.

Theorem D (Stability Theorem). *Consider a coherent sequence of representations of* $\mathrm{Sp}_{2g}(\mathbb{Z})$ *over a field* $\mathbf{k}$ *of characteristic* 0:

$$\mathbf{V}_1 \xrightarrow{\ f_1\ } \mathbf{V}_2 \xrightarrow{\ f_2\ } \mathbf{V}_3 \xrightarrow{\ f_3\ } \cdots$$

For some $g_0 \geq 2$, *assume that the following hold for* $g \geq g_0$:

    (i) *the cokernel of* $f_{g-1}\colon \mathbf{V}_{g-1} \to \mathbf{V}_g$ *is a finite dimensional algebraic representation of*[8] $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$; *and*

    (ii) *the coinvariants*[9] $(\mathbf{V}_g)_{\mathrm{Sp}_{2g}(\mathbb{Z})}$ *are finite dimensional.*

Then $\mathbf{V}_g$ *is finite dimensional for* $g \geq g_0$ *and an algebraic representation of* $\mathrm{Sp}_{2g}(\mathbb{Z})$ *for* $g \geq g_0 + 1$.

Remark 1.6. If in (i) the cokernel of f_{g-1} is just assumed to be finite dimensional for $g \geq g_0$, then our proof shows that $\mathbf{V}_g$ is finite dimensional for $g \geq g_0$. $\square$

Remark 1.7. We have stated Theorem D for coherent sequences of representations of $\mathrm{Sp}_{2g}(\mathbb{Z})$, but the same proof works for many other kinds of representations (possibly with worse bounds). For instance, it also works for coherent sequences of representations of $\mathrm{SL}_n(\mathbb{Z})$. $\square$

1.5. **Commentary.** Consider a coherent sequence of representations $\{\mathbf{V}_g\}$ as in Theorem D. One way to view Theorem D is that it allows finiteness results for $\mathbf{V}_g$ to be proved by induction, but without a base case. In a traditional proof by induction, to prove that $\mathbf{V}_g$ is finite-dimensional for $g \geq g_0$ you would need to prove two things:

    • as a base case, that $\mathbf{V}_{g_0}$ is finite-dimensional; and
    • for the inductive step, that the cokernel of $f_{g-1}\colon \mathbf{V}_{g-1} \to \mathbf{V}_g$ is finite-dimensional for $g \geq g_0 + 1$.

Without some further hypothesis, a finiteness assumption about the cokernels of the f_g implies nothing about the $\mathbf{V}_g$ themselves. For instance:

Example 1.8. For all $g \geq 1$, let $\mathbf{V}_g = \mathbf{k}^\infty$ with the trivial $\mathrm{Sp}_{2g}(\mathbb{Z})$-action and let $f_g\colon \mathbf{V}_g \to \mathbf{V}_{g+1}$ be the identity map. Then

$$\mathbf{V}_1 \xrightarrow{\ f_1\ } \mathbf{V}_2 \xrightarrow{\ f_2\ } \mathbf{V}_3 \xrightarrow{\ f_3\ } \cdots$$

is a coherent sequence of infinite-dimensional representations of $\mathrm{Sp}_{2g}(\mathbb{Z})$. However, the cokernel of each $f_{g-1}\colon \mathbf{V}_{g-1} \to \mathbf{V}_g$ is 0. $\square$

It is a little surprising that the weak finiteness assumption about the coinvariants in Theorem D allows such strong conclusions. Since the coinvariants are the largest trivial quotient representation, this assumption essentially just rules out things like Example 1.8.

1.6. **Torelli proof outline.** As we will show in §6, it is easy to deduce that $\mathrm{H}_2(\mathcal{I}_{g,p}^b; \mathbb{Q})$ is a finite dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$ from the special case of $\mathrm{H}_2(\mathcal{I}_g^1; \mathbb{Q})$, so we will focus our proof outline on $\mathcal{I}_g^1$. We will apply Theorem D to the coherent sequence

$$\mathrm{H}_2(\mathcal{I}_1^1; \mathbf{k}) \xrightarrow{\ f_1\ } \mathrm{H}_2(\mathcal{I}_2^1; \mathbf{k}) \xrightarrow{\ f_2\ } \mathrm{H}_2(\mathcal{I}_3^1; \mathbf{k}) \xrightarrow{\ f_3\ } \cdots$$

of representations of $\mathrm{Sp}_{2g}(\mathbb{Z})$ with $g_0 = 5$.

Theorem D has two hypotheses. Hypothesis (ii) says that the coinvariants $\mathrm{H}_2(\mathcal{I}_g^1; \mathbb{Q})_{\mathrm{Sp}_{2g}(\mathbb{Z})}$ are finite-dimensional for $g \geq 5$, and is an immediate consequence of Kassabov–Putman's

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[8]This cokernel is a representation of $\mathrm{Sp}_{2(g-1)}(\mathbb{Z}) \times \mathrm{Sp}_2(\mathbb{Z})$, but we are ignoring the $\mathrm{Sp}_2(\mathbb{Z})$-action.

[9]If $\mathbf{W}$ is a representation of a group G, then the coinvariants $\mathbf{W}_G$ are the largest G-invariant quotient of $\mathbf{W}$. This can be expressed as $\mathbf{W}/\langle \vec{w} - x{\cdot}\vec{w} \mid \vec{w} \in W \text{ and } x \in G \rangle$.

theorem [25] that $H_2(\mathcal{I}_g^1; \mathbb{Q})$ is finitely generated as a $\mathbb{Q}[\mathrm{Sp}_{2g}(\mathbb{Z})]$-module for $g \geq 3$. In §7 we will give an alternate direct proof that these coinvariants are finite-dimensional.

Hypothesis (i) is more substantial. It asserts that for $g \geq 5$ the cokernel of

$$(1.3) \qquad f_{g-1} \colon H_2(\mathcal{I}_{g-1}^1; \mathbb{Q}) \longrightarrow H_2(\mathcal{I}_g^1; \mathbb{Q})$$

is a finite dimensional algebraic representation of $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$. Our proof of this has three steps.

1.6.1. *Step 1 (reduction to curve stabilizers).* The first step uses a variety of known results about the homology of the Torelli group to reduce what we must show to a result about curve stabilizers on a closed surface. Let α and β be the following oriented curves on Σ_g:
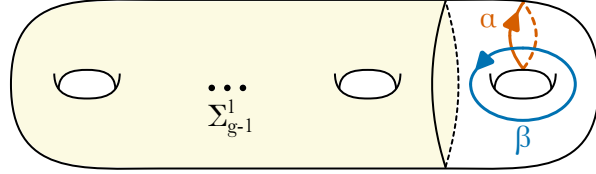
The mapping class group acts on the set of isotopy classes of simple closed curves on Σ_g. Let $(\mathcal{I}_g)_\alpha$ and $(\mathcal{I}_g)_\beta$ be the $\mathcal{I}_g$-stabilizers of α and β. From the above figure, it is clear that both of these stabilizers contain $\mathcal{I}_{g-1}^1$. Let

$$\lambda \colon H_2((\mathcal{I}_g)_\alpha; \mathbb{Q}) \oplus H_2((\mathcal{I}_g)_\beta; \mathbb{Q}) \to H_2(\mathcal{I}_g; \mathbb{Q})$$

be the sum of the maps induced by the inclusions $(\mathcal{I}_g)_\alpha \hookrightarrow \mathcal{I}_g$ and $(\mathcal{I}_g)_\beta \hookrightarrow \mathcal{I}_g$. Letting f_{g-1} be the map (1.3), we will prove:

- **Claim**: To prove that $\mathrm{coker}(f_{g-1})$ is a finite dimensional algebraic representation of $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$, it is enough to prove a similar result for $\mathrm{coker}(\lambda)$.

The idea behind the proof of this claim is as follows. Consider the following commutative diagram of homology groups:

$$H_2(\mathcal{I}_{g-1}^1; \mathbb{Q}) \xrightarrow{\ f_{g-1}\ } H_2(\mathcal{I}_g^1; \mathbb{Q}) \xrightarrow{\ \pi\ } H_2(\mathcal{I}_g; \mathbb{Q})$$
$$\xrightarrow[h_\alpha \times h_\beta]{} \qquad \xrightarrow[\lambda]{}$$
$$H_2((\mathcal{I}_g)_\alpha; \mathbb{Q}) \oplus H_2((\mathcal{I}_g)_\beta; \mathbb{Q})$$

Here π is induced by gluing a disc to $\partial \Sigma_g^1$ and extending mapping classes by id, and h_α and h_β are induced by the inclusions $\mathcal{I}_{g-1}^1 \hookrightarrow (\mathcal{I}_g)_\alpha$ and $\mathcal{I}_{g-1}^1 \hookrightarrow (\mathcal{I}_g)_\beta$. We will show:

- Both $\ker(\pi)$ and $\mathrm{coker}(\pi)$ are finite-dimensional algebraic representations of $\mathrm{Sp}_{2g}(\mathbb{Z})$. This is a standard argument using the Birman exact sequence.
- Both $\mathrm{coker}(h_\alpha)$ and $\mathrm{coker}(h_\beta)$ are finite-dimensional algebraic representations of $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$. This uses a recent theorem of the authors [36, Theorem A] about the first homology group of $\mathcal{I}_g^1$ with certain infinite-dimensional twisted coefficients.

The above claim will follow easily from these two facts and a bit of algebra. This reduces us to proving that $\mathrm{coker}(\lambda)$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$. Roughly speaking, we do this by constructing generators and relations for $\mathrm{coker}(\lambda)$.

1.6.2. *Step 2 (generators of cokernel).* We identify generators for $\mathrm{coker}(\lambda)$ by studying the action of $\mathcal{I}_g$ on a carefully chosen space. Let $a = [\alpha] \in H_1(\Sigma_g)$ and $b = [\beta] \in H_1(\Sigma_g)$. An oriented simple closed curve is an *a-curve* if its homology class is a and a *b-curve* if its homology class is b. Let $\mathcal{C}_a(\Sigma_g)$ be the simplicial complex whose p-simplices are collections $\{\gamma_0, \ldots, \gamma_k\}$ of disjoint isotopy classes of a-curves on Σ_g. Similarly, define $\mathcal{C}_b(\Sigma_g)$. Simplices

of these complexes look like:[10]

and
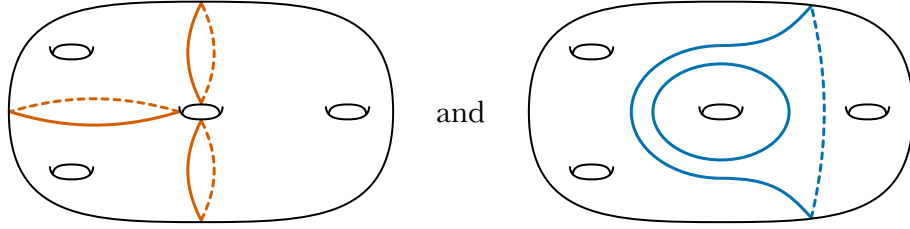
The complexes $\mathcal{C}_a(\Sigma_g)$ and $\mathcal{C}_b(\Sigma_g)$ were introduced by Putman [41], who proved that they are connected for $g \geq 3$. Hatcher–Margalit [18] found an alternate proof of this, and Minahan [34] generalized Hatcher–Margalit's proof to show that these complexes are $(g-3)$-acyclic.[11]

The space $\mathcal{C}_{ab}(\Sigma_g)$ we use is a combination of $\mathcal{C}_a(\Sigma_g)$ and $\mathcal{C}_b(\Sigma_g)$ that is inspired by the "handle graph" introduced by Putman [45] to find small generating sets for $\mathcal{I}_g$. We call it the *handle complex*. It is the union of $\mathcal{C}_a(\Sigma_g)$ and $\mathcal{C}_b(\Sigma_g)$ with certain "mixed simplices" attached that look like:

and
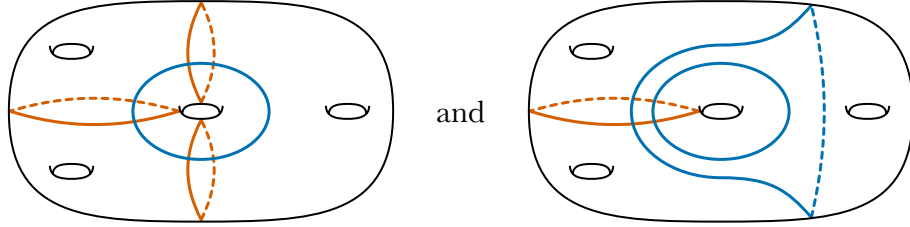
Their key property is that a mixed simplex cannot contain both two a-curves and two b-curves. We will define this carefully later. The group $\mathcal{I}_g$ acts on $\mathcal{C}_{ab}(\Sigma_g)$. We will prove:

(a) $\mathcal{C}_{ab}(\Sigma_g)$ is 1-acyclic for $g \geq 4$; and
(b) the quotient $\mathcal{C}_{ab}(\Sigma_g)/\mathcal{I}_g$ is contractible.

These two results will allow us to use the action of $\mathcal{I}_g$ on $\mathcal{C}_{ab}(\Sigma_g)$ to study $\mathrm{H}_2(\mathcal{I}_g; \mathbb{Q})$. Our main tool is (Borel) equivariant homology, which for a group G acting on a space X mixes together information about the topology of X/G and the group homology of the G-stabilizers of cells in X.

Using (a), we will show that the $\mathcal{I}_g$-equivariant homology of $C_{ab}(\Sigma_g)$ surjects onto $\mathrm{H}_2(\mathcal{I}_g; \mathbb{Q})$. The terms $\mathrm{H}_2((\mathcal{I}_g)_\alpha; \mathbb{Q})$ and $\mathrm{H}_2((\mathcal{I}_g)_\beta; \mathbb{Q})$ in the domain of λ appear in the $\mathcal{I}_g$-equivariant homology of $C_{ab}(\Sigma_g)$ since α and β are vertices of $C_{ab}(\Sigma_g)$. Using (b), we will show that rest of the equivariant homology has a tractable description, from which we will deduce generators for $\mathrm{coker}(\lambda)$.

1.6.3. *Step 3 (algebraicity of cokernel).* The third step identifies topologically some relations among our generators for $\mathrm{coker}(\lambda)$. Let $H = \mathrm{H}_1(\Sigma_{g-1}^1; \mathbb{Q})$. We use these relations to embed $\mathrm{coker}(\lambda)$ into $V = ((\wedge^2 H)/\mathbb{Q})^{\otimes 2}$. Since V is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$, this will imply that the same is true for $\mathrm{coker}(\lambda)$. This step uses recent work of the authors [37, Theorem A.6] giving tools for identifying representations given by generators and relations. This is difficult since $\mathrm{coker}(\lambda)$ has infinitely many generators and relations.

1.7. **Outline.** Theorem D is proved in Part 1. Theorem B is proved in Parts $2 - 4$, which perform the three steps outlined above.

1.8. **Acknowledgments.** We would like to thank Nate Harman for helpful discussions about the material in §4 and Xiyan Zhong for corrections.

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[10]Our convention is that a-curves are orange and b-curves are blue, and we often omit the orientations.
[11]This means that $\widetilde{\mathrm{H}}_k(\mathcal{C}_a) = \widetilde{\mathrm{H}}_k(\mathcal{C}_b) = 0$ for $k \leq g - 3$. It is not clear if they are simply connected.

Part 1. The stability theorem

The proof of Theorem D (the stability theorem) is divided into two parts: §2 addresses finite dimensionality, and §3 – §5 addresses algebraicity.

2. A criterion for finite dimensionality

The following theorem strengthens part of Theorem D (the stability theorem). For a commutative ring $\mathbf{k}$ and a group G, we call a $\mathbf{k}[G]$-module $\mathbf{V}$ a *representation* of G over $\mathbf{k}$. We say that $\mathbf{V}$ is *finite dimensional* if $\mathbf{V}$ is finitely generated as a $\mathbf{k}$-module.

Theorem 2.1 (Finite dim criterion). *Let $\mathbf{k}$ be a commutative Noetherian ring and let $g \geq 2$. Let $\mathbf{V}_{g-1}$ be an $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$-representation over $\mathbf{k}$ and $\mathbf{V}_g$ be an $\mathrm{Sp}_{2g}(\mathbb{Z})$-representation over $\mathbf{k}$. Let $f \colon \mathbf{V}_{g-1} \boxtimes \mathbf{k} \to \mathbf{V}_g$ be an $\mathrm{Sp}_{2(g-1)}(\mathbb{Z}) \times \mathrm{Sp}_2(\mathbb{Z})$-equivariant map. Assume:*

(i) the cokernel of f is finite dimensional; and
(ii) the coinvariants $(\mathbf{V}_g)_{\mathrm{Sp}_{2g}(\mathbb{Z})}$ are finite dimensional.

Then $\mathbf{V}_g$ is finite dimensional.

Proof. Let $G = 1 \times \mathrm{Sp}_2(\mathbb{Z}) \subset \mathrm{Sp}_{2g}(\mathbb{Z})$ and let $\mathbf{W}$ be the image of f. The group G acts trivially on $\mathbf{W}$, and assumption (i) says that the $\mathbf{k}$-submodule $\mathbf{W}$ of $\mathbf{V}_g$ has finite codimension.[12] For $m \in \mathrm{Sp}_{2g}(\mathbb{Z})$, the $\mathbf{k}$-submodule $m \cdot \mathbf{W}$ of $\mathbf{V}_g$ also has finite codimension and the subgroup mGm^{-1} of $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts trivially on it. The group $\mathrm{Sp}_{2g}(\mathbb{Z})$ is generated by finitely many conjugates of[13] G. Pick $m_1, \ldots, m_n \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that $\mathrm{Sp}_{2g}(\mathbb{Z})$ is generated by the subgroups $\{m_i G m_i^{-1}\}_{i=1}^n$. Set

$$\mathbf{U} = \bigcap_{i=1}^n m_i \cdot \mathbf{W}.$$

Each $m_i G m_i^{-1}$ acts trivially on $\mathbf{U}$, so $\mathrm{Sp}_{2g}(\mathbb{Z})$ itself acts trivially on $\mathbf{U}$. Moreover, since the intersection of finitely many finite codimension submodules is a finite codimension submodule, the submodule $\mathbf{U}$ has finite codimension. To prove that $\mathbf{V}_g$ is finite dimensional, it is therefore enough to prove that $\mathbf{U}$ is finite dimensional.

Consider the short exact sequence of $\mathrm{Sp}_{2g}(\mathbb{Z})$-representations

(2.1) $$0 \longrightarrow \mathbf{U} \longrightarrow \mathbf{V}_g \longrightarrow \mathbf{V}_g/\mathbf{U} \longrightarrow 0.$$

Since $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts trivially on $\mathbf{U}$, we have

$$\mathrm{H}_0(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathbf{U}) = \mathbf{U}_{\mathrm{Sp}_{2g}(\mathbb{Z})} = \mathbf{U},$$

where the subscript indicates that we are taking coinvariants. The long exact sequence in $\mathrm{Sp}_{2g}(\mathbb{Z})$-homology associated to (2.1) thus contains the segment

$$\mathrm{H}_1(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathbf{V}_g/\mathbf{U}) \longrightarrow \mathbf{U} \longrightarrow \mathrm{H}_0(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathbf{V}_g).$$

Since $\mathrm{Sp}_{2g}(\mathbb{Z})$ is finitely generated and the quotient $\mathbf{V}_g/\mathbf{U}$ is finite dimensional, it follows[14] that $\mathrm{H}_1(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathbf{V}_g/\mathbf{U})$ is finite dimensional. Also, by assumption (ii) the coinvariants

$$\mathrm{H}_0(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathbf{V}_g) = (\mathbf{V}_g)_{\mathrm{Sp}_{2g}(\mathbb{Z})}$$

are finite dimensional. It follows that $\mathbf{U}$ is finite dimensional, as desired. $\square$

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[12]This simply means that $\mathbf{V}_g/\mathbf{W}$ is a finitely generated $\mathbf{k}$-module.

[13]One way to see this is to observe that G contains the transvection along a primitive element of $0 \times \mathbb{Z}^2 \subset \mathbb{Z}^{2g}$, all transvections along primitive elements of $\mathbb{Z}^{2g}$ are conjugate in $\mathrm{Sp}_{2g}(\mathbb{Z})$, and $\mathrm{Sp}_{2g}(\mathbb{Z})$ is generated by finitely many transvections along primitive elements of $\mathbb{Z}^{2g}$.

[14]This uses the fact that our base ring $\mathbf{k}$ is Noetherian.

3. A criterion for algebraicity I: statement and motivation for proof

Theorem 2.1 from §2 is a strengthening of one part of Theorem D (the stability theorem). We now turn to the following, which is a strengthening of the other part:

Theorem 3.1 (Algebraicity criterion). *Let $\mathbf{k}$ be a field of characteristic 0 and let $g \geq 3$. Let $\mathbf{V}_{g-2}$ be an $\mathrm{Sp}_{2(g-2)}(\mathbb{Z})$-representation over $\mathbf{k}$ and $\mathbf{V}_g$ be an $\mathrm{Sp}_{2g}(\mathbb{Z})$-representation over $\mathbf{k}$. Let $f\colon \mathbf{V}_{g-2} \boxtimes \mathbf{k} \to \mathbf{V}_g$ be an $\mathrm{Sp}_{2(g-2)}(\mathbb{Z}) \times \mathrm{Sp}_4(\mathbb{Z})$-equivariant map. Assume:*

(a) the representation $\mathbf{V}_g$ is finite dimensional; and
(b) the cokernel of f is an algebraic representation of $\mathrm{Sp}_{2(g-2)}(\mathbb{Z})$.

Then $\mathbf{V}_g$ is an algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$.

Theorem D is an immediate consequence[15] of Theorems 2.1 and 3.1.

3.1. Motivation for proof.
To motivate what we do, let us consider one way a counterexample to Theorem 3.1 might arise. Fix some $\ell \geq 2$. The prototypical non-algebraic representations of $\mathrm{Sp}_{2g}(\mathbb{Z})$ are those that factor through the finite group $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$. The group $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ contains the subgroup $\mathrm{Sp}_{2(g-2)}(\mathbb{Z}/\ell) \times \mathrm{Sp}_4(\mathbb{Z}/\ell)$. Via the projections

$$\mathrm{Sp}_{2(g-2)}(\mathbb{Z}/\ell) \times \mathrm{Sp}_4(\mathbb{Z}/\ell) \to \mathrm{Sp}_{2(g-2)}(\mathbb{Z}/\ell) \quad \text{and} \quad \mathrm{Sp}_{2(g-2)}(\mathbb{Z}/\ell) \times \mathrm{Sp}_4(\mathbb{Z}/\ell) \to \mathrm{Sp}_4(\mathbb{Z}/\ell),$$

we can regard representations of $\mathrm{Sp}_{2(g-2)}(\mathbb{Z}/\ell)$ and $\mathrm{Sp}_4(\mathbb{Z}/\ell)$ as representations of the product $\mathrm{Sp}_{2(g-2)}(\mathbb{Z}/\ell) \times \mathrm{Sp}_4(\mathbb{Z}/\ell)$. Assume there exists a nontrivial finite-dimensional representation $\mathbf{V}$ of the finite group $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ over $\mathbf{k}$ with the following property:

- The restriction of $\mathbf{V}$ to $\mathrm{Sp}_{2(g-2)}(\mathbb{Z}/\ell) \times \mathrm{Sp}_4(\mathbb{Z}/\ell)$ decomposes as $\mathbf{W} \oplus \mathbf{W}'$ with $\mathrm{Sp}_4(\mathbb{Z}/\ell)$ acting trivially on $\mathbf{W}$ and $\mathrm{Sp}_{2(g-2)}(\mathbb{Z}/\ell)$ acting trivially on $\mathbf{W}'$.

Let $\mathbf{V}_g = \mathbf{V}$ and $\mathbf{V}_{g-2} = \mathbf{W}$, regarded as representations of $\mathrm{Sp}_{2g}(\mathbb{Z})$ and $\mathrm{Sp}_{2(g-2)}(\mathbb{Z})$ via the surjections

$$\mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}/\ell) \quad \text{and} \quad \mathrm{Sp}_{2(g-2)}(\mathbb{Z}) \longrightarrow \mathrm{Sp}_{2(g-2)}(\mathbb{Z}/\ell).$$

Consider the map $f\colon \mathbf{V}_{g-2} \boxtimes \mathbf{k} \to \mathbf{V}_g$ coming from the inclusion $\mathbf{W} \hookrightarrow \mathbf{V}$. We claim that this is a counterexample to Theorem 3.1. This requires checking two things:

- The representation $\mathrm{coker}(f) = \mathbf{V}/\mathbf{W} \cong \mathbf{W}'$ is an algebraic representation of $\mathrm{Sp}_{2(g-2)}(\mathbb{Z})$. This holds since the $\mathrm{Sp}_{2(g-2)}(\mathbb{Z})$-action on $\mathbf{W}'$ is trivial.[16]
- The representation $\mathbf{V}_g = \mathbf{V}$ is a non-algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$. This holds since the (nontrivial) $\mathrm{Sp}_{2g}(\mathbb{Z})$-action on it factors through a finite group.

3.2. Outline.
It turns out that the potential existence of such $\mathbf{V}$ is the main obstacle to proving Theorem 3.1. Ruling this out requires a long detour into the representation theory of finite groups, which we do in §4. We then prove Theorem 3.1 in §5.

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[15]The only potentially non-obvious point here is that in Theorem D, we are given maps

$$\mathbf{V}_{g-2} \xrightarrow{f_{g-2}} \mathbf{V}_{g-1} \xrightarrow{f_{g-1}} \mathbf{V}_g$$

between representations of the appropriate symplectic groups such that $\mathrm{coker}(f_{g-2})$ is a finite dimensional algebraic representation of $\mathrm{Sp}_{2(g-2)}(\mathbb{Z})$ and $\mathrm{coker}(f_{g-1})$ is a finite dimensional algebraic representation of $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$. To apply Theorem 3.1, we need for $\mathrm{coker}(f_{g-1} \circ f_{g-2})$ to be a finite dimensional algebraic representation of $\mathrm{Sp}_{2(g-2)}(\mathbb{Z})$. This follows from the fact that algebraic representations of $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$ restrict to algebraic representations of $\mathrm{Sp}_{2(g-2)}(\mathbb{Z})$ along with the fact that the collection of algebraic representations of $\mathrm{Sp}_{2(g-2)}(\mathbb{Z})$ is closed under extensions and subquotients.

[16]The group $\mathrm{Sp}_{2(g-2)}(\mathbb{Z}) \times \mathrm{Sp}_4(\mathbb{Z})$ acts on $\mathrm{coker}(f) = \mathbf{W}'$ and this action is nontrivial since $\mathrm{Sp}_4(\mathbb{Z})$ acts nontrivially, but we only care about the action of $\mathrm{Sp}_{2(g-2)}(\mathbb{Z})$.

4. A CRITERION FOR ALGEBRAICITY II: UNMIXED REPRESENTATIONS OF FINITE GROUPS

This section studies the restriction to product subgroups of representations of finite groups.

4.1. Mixed representations.

Let G_1 and G_2 be finite groups. Via the projections

$$G_1 \times G_2 \to G_1 \quad \text{and} \quad G_1 \times G_2 \to G_2,$$

the irreducible representations of G_1 and G_2 over a field are irreducible representations of $G_1 \times G_2$. We will call these the *unmixed* irreducible representations of $G_1 \times G_2$. The other irreducible representations of $G_1 \times G_2$ are the *mixed* irreducible representations; they have the property that their restrictions to G_1 and G_2 are both nontrivial.[17] A general representation of $G_1 \times G_2$ over a field of characteristic 0 is *unmixed* if all of its irreducible factors are unmixed. If at least one of its irreducible factors is mixed, then it is *mixed*.

4.2. Universally mixed subgroups.

Let G be a finite group and let $G_1 \times G_2$ be a subgroup of G. We say that $G_1 \times G_2$ is *universally mixed* in G if for all all representations $\mathbf{V}$ of G over algebraically closed fields of characteristic 0, the restriction $\mathrm{Res}^G_{G_1 \times G_2} \mathbf{V}$ is either mixed or trivial.[18] We remark that it is enough to check this for irreducible representations $\mathbf{V}$.

4.3. $\mathbf{GL_2}$ over finite fields.

The following gives an important example of this property:

Lemma 4.1. *Let $\mathbb{F}_q$ be a finite field with*[19] *$q > 3$. Then $\mathrm{GL}_1(\mathbb{F}_q) \times \mathrm{GL}_1(\mathbb{F}_q)$ is universally mixed in $\mathrm{GL}_2(\mathbb{F}_q)$.*

Proof. Let $G = \mathrm{GL}_2(\mathbb{F}_q)$, let $A = \mathrm{GL}_1(\mathbb{F}_q) \times 1$, and let $B = 1 \times \mathrm{GL}_1(\mathbb{F}_q)$. Let $\mathbf{k}$ be an algebraically closed field of characteristic 0 and let $\mathbf{V}$ be a nontrivial irreducible representation of G over $\mathbf{k}$. It is enough to prove that $\mathrm{Res}^G_{A \times B} \mathbf{V}$ is a mixed representation of $A \times B$.

If $\mathbf{V}$ is 1-dimensional then the only way that the restriction of $\mathbf{V}$ to $A \times B$ can be unmixed is if the restriction of $\mathbf{V}$ to either A or B is trivial. Since the normal closures in G of both A and B are[20] all of G, this would imply that $\mathbf{V}$ is the trivial representation of G, contrary to our assumptions. We can thus assume without loss of generality that $\mathbf{V}$ has dimension greater than 1.

If Γ is a group and $\chi \colon \Gamma \to \mathbf{k}^\times$ is a character, then denote by $\mathbf{k}_\chi$ the associated 1-dimensional representation of Γ. Since $A \times B$ is abelian and $\mathbf{k}$ is algebraically closed, proving that the restriction of $\mathbf{V}$ to $A \times B$ is mixed is equivalent to finding nontrivial characters $\chi_1 \colon A \to \mathbf{k}^\times$ and $\chi_2 \colon B \to \mathbf{k}^\times$ such that letting $\chi_1\chi_2 \colon A \times B \to \mathbf{k}^\times$ be their product, the restriction of $\mathbf{V}$ to $A \times B$ contains $\mathbf{k}_{\chi_1\chi_2}$ as a subrepresentation.

For $a, b \in \mathbf{k}^\times$ let $\mathrm{diag}(a,b) \in A \times B$ be the associated diagonal matrix. Let $Z < G$ be the central subgroup of diagonal matrices $\mathrm{diag}(z,z)$. By Schur's Lemma, there is a character $\lambda \colon Z \to \mathbf{k}^\times$ called the *central character* such that for all $z \in Z$, the element z acts on $\mathbf{V}$ as multiplication by $\lambda(z)$.

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[17]If $\mathbf{k}$ is an algebraically closed field of characteristic 0, the mixed irreducible representations of $G_1 \times G_2$ over $\mathbf{k}$ are of the form $\mathbf{V} \otimes \mathbf{W}$ with $\mathbf{V}$ a nontrivial irreducible representation of G_1 and $\mathbf{W}$ a nontrivial irreducible representation of G_2.

[18]This implies that the same holds for representations $\mathbf{V}$ of G over arbitrary fields of characteristic 0.

[19]While the lemma is true for $q = 2$ for the (uninteresting) reason that $\mathrm{GL}_1(\mathbb{F}_2) = 1$, it is false for $q = 3$. Indeed, let $G = \mathrm{GL}_2(\mathbb{F}_3)$ and let $P \subset G$ be the subgroup of upper triangular matrices. Define a character $\rho \colon P \to \mathbb{C}^\times$ via the formula

$$\rho \begin{pmatrix} a & u \\ 0 & b \end{pmatrix} = \begin{cases} 1 & \text{if } a = 1, \\ -1 & \text{if } a = -1. \end{cases}$$

Note that we write ρ like this since $1, -1 \in \mathbb{F}_3$ are different from $1, -1 \in \mathbb{C}^\times$ despite the fact that they look the same. Let $\mathbb{C}_\rho$ be the associated 1-dimensional representation of P and let $\mathbf{V} = \mathrm{Ind}^G_P \mathbb{C}_\rho$. It is an easy exercise to show that $\mathbf{V}$ restricts to an unmixed representation of $\mathrm{GL}_1(\mathbb{F}_3) \times \mathrm{GL}_1(\mathbb{F}_3)$.

[20]This uses the fact that $q \neq 2$.

If $\chi_1 \colon A \to \mathbf{k}^\times$ is a character and $\mathbf{L}$ is a subrepresentation of $\mathrm{Res}_A^G \mathbf{V}$ with $\mathbf{L} \cong \mathbf{k}_{\chi_1}$, then for $b \in \mathbb{F}_q^\times$ and $\vec{x} \in \mathbf{L}$ we have

$$\mathrm{diag}(1,b) \cdot \vec{x} = \mathrm{diag}(b^{-1},1)\,\mathrm{diag}(b,b) \cdot \vec{x} = \chi_1(\mathrm{diag}(b,1))^{-1}\lambda(\mathrm{diag}(b,b))\vec{x}.$$

In other words, $\mathbf{L}$ is also preserved by B. Moreover, letting $\chi_2 \colon B \to \mathbf{k}^\times$ be defined via the formula

(4.1) $$\chi_2(\mathrm{diag}(1,b)) = \chi_1(\mathrm{diag}(b,1))^{-1}\lambda(\mathrm{diag}(b,b)) \quad \text{for all } b \in \mathbb{F}_q^\times,$$

as an $A \times B$-representation we have $\mathbf{L} \cong \mathbf{k}_{\chi_1\chi_2}$. It follows that to prove the lemma, it is enough to find such an $\mathbf{L}$ with both χ_1 and χ_2 nontrivial.

Let $a_0 \in \mathbb{F}_q^\times$ be a generator of this cyclic group. Since $q > 3$ and $A \cong \mathbb{F}_q^\times$ and $\mathbf{k}$ is algebraically closed, we can find a nontrivial character $\chi_1 \colon A \to \mathbf{k}^\times$ with

$$\chi_1(\mathrm{diag}(a_0,1)) \neq \lambda(\mathrm{diag}(a_0,a_0)).$$

It follows that the character $\chi_2 \colon B \to \mathbf{k}^\times$ defined in (4.1) satisfies $\chi_2(\mathrm{diag}(0,a_0)) \neq 1$, and in particular is also nontrivial. It is therefore enough to prove that $\mathrm{Res}_A^G \mathbf{V}$ contains a subrepresentation $\mathbf{L}$ with $\mathbf{L} \cong \mathbf{k}_{\chi_1}$. In fact, we will prove the following:

Claim. *For all irreducible representations $\mathbf{V}$ of G of dimension greater than 1, the representation $\mathrm{Res}_A^G \mathbf{V}$ contains a copy of the left regular representation $\mathbf{k}[A]$ of A. In particular, it contains a copy of every irreducible representation of A.*

Let $U < G$ be the unipotent subgroup consisting of matrices

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \text{ with } u \in \mathbb{F}_q$$

and let $M < G$ be the mirabolic ("miraculous parabolic") subgroup consisting of matrices

$$\begin{pmatrix} a & u \\ 0 & 1 \end{pmatrix} \text{ with } a \in \mathbb{F}_q^\times \text{ and } u \in \mathbb{F}_q.$$

The group U is isomorphic to the additive group of $\mathbb{F}_q$. Let $\rho \colon U \to \mathbf{k}^\times$ be an arbitrary nontrivial character and let $\mathbf{W} = \mathrm{Ind}_U^M \mathbf{k}_\rho$. Then:

- $\mathbf{W}$ is an irreducible representation of M (see [39, Theorem 6.1]); and
- $\mathrm{Ind}_M^G \mathbf{W}$ is the direct sum of one copy of every representation of G of dimension greater than 1 (see [39, Theorem 16.1]).

In particular, since our representation $\mathbf{V}$ is irreducible and has dimension greater than 1 we can apply Frobenius reciprocity and see that

$$\mathbf{k} = \mathrm{Hom}_G(\mathrm{Ind}_M^G \mathbf{W}, \mathbf{V}) = \mathrm{Hom}_M(\mathbf{W}, \mathrm{Res}_M^G \mathbf{V}).$$

Since $\mathbf{W}$ is an irreducible representation of M, it follows that $\mathbf{W}$ injects into $\mathrm{Res}_M^G \mathbf{V}$. Recalling that we are trying to prove that $\mathrm{Res}_A^G \mathbf{V}$ contains a copy of $\mathbf{k}[A]$, it is thus enough to prove that $\mathrm{Res}_A^M \mathbf{W} \cong \mathbf{k}[A]$. For this, we have $M = U \rtimes A$, so

$$\mathbf{W} = \mathrm{Ind}_U^M \mathbf{k}_\rho = \bigoplus_{a \in \mathbb{F}_q^\times} \mathrm{diag}(a,1) \cdot \mathbf{k}_\rho.$$

The action of A on $\mathbf{W}$ permutes these one-dimensional factors simply transitively. It follows that the restriction of $\mathbf{W}$ to A is isomorphic to $\mathbf{k}[A]$, as desired. $\qquad\square$

4.4. **$\mathbf{GL_n}$ over finite fields.** The following generalizes Lemma 4.1:

Proposition 4.2. *Let $\mathbb{F}_q$ be a finite field and $n_1, n_2 \geq 1$. Set $n = n_1 + n_2$. Assume that $q > 3$ if $n_1 = 1$ or $n_2 = 1$. Then $\mathrm{GL}_{n_1}(\mathbb{F}_q) \times \mathrm{GL}_{n_2}(\mathbb{F}_q)$ is universally mixed in $\mathrm{GL}_n(\mathbb{F}_q)$.*

The proof of Proposition 4.2 requires some preliminary lemmas.

Lemma 4.3. *For $i = 1, 2$ let Γ_i be a finite group and let $G_i < \Gamma_i$ be a subgroup. Let $\mathbf{V}$ be a representation of $\Gamma_1 \times \Gamma_2$ over a field of characteristic 0 whose restriction to $G_1 \times G_2$ is mixed. Then $\mathbf{V}$ is mixed.*

Proof. Let $\mathbf{W}$ be a mixed irreducible subrepresentation of $\mathrm{Res}^{\Gamma_1 \times \Gamma_2}_{G_1 \times G_2} \mathbf{V}$. By definition, both G_1 and G_2 act nontrivially on $\mathbf{W}$. Let $\mathbf{W}'$ be the irreducible subrepresentation of $\mathbf{V}$ such that $\mathrm{Res}^{\Gamma_1 \times \Gamma_2}_{G_1 \times G_2} \mathbf{W}'$ contains $\mathbf{W}$. Since G_1 and G_2 act nontrivially on $\mathbf{W}$, both Γ_1 and Γ_2 act nontrivially on $\mathbf{W}'$. It follows that $\mathbf{W}'$ is a mixed irreducible representation of $\Gamma_1 \times \Gamma_2$, so $\mathbf{V}$ is a mixed representation of $\Gamma_1 \times \Gamma_2$. $\square$

Lemma 4.4 (Poison subgroup). *Let Γ be a finite group and let $\Gamma_1 \times \Gamma_2$ be a subgroup of Γ. Let $G < \Gamma$ be a subgroup, and for $i = 1, 2$ let $G_i < \Gamma_i$ be a subgroup such that $G_1 \times G_2 < G$. Assume that $G_1 \times G_2$ is universally mixed in G and that the Γ-normal closure of $G_1 \times G_2$ contains $\Gamma_1 \times \Gamma_2$. Then $\Gamma_1 \times \Gamma_2$ is a universally mixed subgroup of Γ.*

Proof. Let $\mathbf{V}$ be a representation of Γ over an algebraically closed field of characteristic 0. We must prove that $\mathrm{Res}^{\Gamma}_{\Gamma_1 \times \Gamma_2} \mathbf{V}$ is either trivial or mixed. Since $G_1 \times G_2$ is universally mixed in G, the restriction

$$\mathrm{Res}^{G}_{G_1 \times G_2} \mathrm{Res}^{\Gamma}_{G} \mathbf{V} = \mathrm{Res}^{\Gamma}_{G_1 \times G_2} \mathbf{V}$$

is either trivial or mixed.

If $\mathrm{Res}^{\Gamma}_{G_1 \times G_2} \mathbf{V}$ is trivial, then the kernel of the action of Γ on $\mathbf{V}$ contains $G_1 \times G_2$. Since the Γ-normal closure of $G_1 \times G_2$ contains $\Gamma_1 \times \Gamma_2$, this implies that the kernel of the action of Γ on $\mathbf{V}$ contains $\Gamma_1 \times \Gamma_2$, i.e., that $\mathrm{Res}^{\Gamma}_{\Gamma_1 \times \Gamma_2} \mathbf{V}$ is trivial. If instead $\mathrm{Res}^{\Gamma}_{G_1 \times G_2} \mathbf{V}$ is mixed, then Lemma 4.3 implies that $\mathrm{Res}^{\Gamma}_{\Gamma_1 \times \Gamma_2} \mathbf{V}$ is also mixed. $\square$

Proof of Proposition 4.2. We divide the proof into two cases.

Case 1. $q > 3$.

Lemma 4.1 says that $\mathrm{GL}_1(\mathbb{F}_q) \times \mathrm{GL}_1(\mathbb{F}_q)$ is universally mixed in $\mathrm{GL}_2(\mathbb{F}_q)$. Embed $\mathrm{GL}_2(\mathbb{F}_q)$ into $\mathrm{GL}_n(\mathbb{F}_q)$ such that $\mathrm{GL}_1(\mathbb{F}_q) \times 1 \subset \mathrm{GL}_2(\mathbb{F}_q)$ maps to $\mathrm{GL}_{n_1}(\mathbb{F}_q) \times 1$ and $1 \times \mathrm{GL}_1(\mathbb{F}_q) \subset \mathrm{GL}_2(\mathbb{F}_q)$ maps to $1 \times \mathrm{GL}_{n_2}(\mathbb{F}_q)$. Since $q \neq 2$, the normal closure of $\mathrm{GL}_1(\mathbb{F}_q) \times \mathrm{GL}_1(\mathbb{F}_q)$ in $\mathrm{GL}_n(\mathbb{F}_q)$ is $\mathrm{GL}_n(\mathbb{F}_q)$, so this embedding satisfies the conditions of the poison subgroup lemma (Lemma 4.4). This lemma implies that $\mathrm{GL}_{n_1}(\mathbb{F}_q) \times \mathrm{GL}_{n_2}(\mathbb{F}_q)$ is universally mixed in $\mathrm{GL}_n(\mathbb{F}_q)$, as desired.

Case 2. $q \in \{2, 3\}$. *Our hypotheses thus say that $n_1, n_2 \geq 2$.*

Since $q^2 > 3$, Lemma 4.1 says that $\mathrm{GL}_1(\mathbb{F}_{q^2}) \times \mathrm{GL}_1(\mathbb{F}_{q^2})$ is universally mixed in $\mathrm{GL}_2(\mathbb{F}_{q^2})$. Embed the group $\mathrm{GL}_2(\mathbb{F}_{q^2})$ of $\mathbb{F}_{q^2}$-linear automorphisms of $\mathbb{F}_{q^2}^2$ in the group $\mathrm{GL}_4(\mathbb{F}_q)$ of $\mathbb{F}_q$-linear automorphisms of $\mathbb{F}_q^4$ by regarding $\mathbb{F}_{q^2}$ as a 2-dimensional vector space over $\mathbb{F}_q$. This maps $\mathrm{GL}_1(\mathbb{F}_{q^2}) \times 1$ to $\mathrm{GL}_2(\mathbb{F}_q) \times 1$ and $1 \times \mathrm{GL}_1(\mathbb{F}_{q^2})$ to $1 \times \mathrm{GL}_2(\mathbb{F}_q)$.

We can now embed $\mathrm{GL}_4(\mathbb{F}_q)$ into $\mathrm{GL}_n(\mathbb{F}_q)$ such that $\mathrm{GL}_2(\mathbb{F}_q) \times 1$ maps to $\mathrm{GL}_{n_1}(\mathbb{F}_q) \times 1$ and $1 \times \mathrm{GL}_2(\mathbb{F}_q)$ maps to $1 \times \mathrm{GL}_{n_2}(\mathbb{F}_q)$. The normal closure of $\mathrm{GL}_1(\mathbb{F}_{q^2}) \times \mathrm{GL}_1(\mathbb{F}_{q^2})$ in $\mathrm{GL}_n(\mathbb{F}_q)$ is $\mathrm{GL}_n(\mathbb{F}_q)$, so this embedding satisfies the conditions of the poison subgroup lemma (Lemma 4.4). We conclude that $\mathrm{GL}_{n_1}(\mathbb{F}_q) \times \mathrm{GL}_{n_2}(\mathbb{F}_q)$ is universally mixed in $\mathrm{GL}_n(\mathbb{F}_q)$. $\square$

4.5. **$\mathbf{GL_n}$ over integers mod ℓ.** Here is another example of a universally mixed subgroup:

Proposition 4.5. *Let $\ell \geq 2$ and $n_1, n_2 \geq 2$. Set $n = n_1 + n_2$. Then $\mathrm{GL}_{n_1}(\mathbb{Z}/\ell) \times \mathrm{GL}_{n_2}(\mathbb{Z}/\ell)$ is universally mixed in $\mathrm{GL}_n(\mathbb{Z}/\ell)$.*

Proof. Write ℓ as a product of powers of distinct primes: $\ell = p_1^{d_1} \cdots p_m^{d_m}$. By the Chinese remainder theorem, we have
$$\mathrm{GL}_n(\mathbb{Z}/\ell) \cong \mathrm{GL}_n(\mathbb{Z}/p_1^{d_1}) \times \cdots \times \mathrm{GL}_n(\mathbb{Z}/p_m^{d_m}).$$
Lemma 4.6 below says that $\mathrm{GL}_{n_1}(\mathbb{Z}/p_i^{d_i}) \times \mathrm{GL}_{n_2}(\mathbb{Z}/p_i^{d_i})$ is universally mixed in $\mathrm{GL}_n(\mathbb{Z}/p_i^{d_i})$. Assuming this, let $\mathbf{V}$ be a representation of $\mathrm{GL}_n(\mathbb{Z}/\ell)$ over an algebraically closed field of characteristic 0.

Since $\mathrm{GL}_{n_1}(\mathbb{Z}/p_i^{d_i}) \times \mathrm{GL}_{n_2}(\mathbb{Z}/p_i^{d_i})$ is universally mixed in $\mathrm{GL}_n(\mathbb{Z}/p_i^{d_i})$, it is also universally mixed in $\mathrm{GL}_n(\mathbb{Z}/\ell)$. The restriction of $\mathbf{V}$ to each $\mathrm{GL}_{n_1}(\mathbb{Z}/p_i^{d_i}) \times \mathrm{GL}_{n_2}(\mathbb{Z}/p_i^{d_i})$ is thus either trivial or mixed. If all these restrictions are trivial, then the restriction of $\mathbf{V}$ to $\mathrm{GL}_{n_1}(\mathbb{Z}/\ell) \times \mathrm{GL}_{n_2}(\mathbb{Z}/\ell)$ is trivial. If one of these restrictions is mixed, then Lemma 4.3 implies that the restriction of $\mathbf{V}$ to $\mathrm{GL}_{n_1}(\mathbb{Z}/\ell) \times \mathrm{GL}_{n_2}(\mathbb{Z}/\ell)$ is mixed. $\qquad\square$

The above proof required:[21]

Lemma 4.6. *Let p^d be a prime power and $n_1, n_2 \geq 1$. Set $n = n_1 + n_2$. If $n_1 = 1$ or $n_2 = 1$, then assume that $p > 3$. Then $\mathrm{GL}_{n_1}(\mathbb{Z}/p^d) \times \mathrm{GL}_{n_2}(\mathbb{Z}/p^d)$ is universally mixed in $\mathrm{GL}_n(\mathbb{Z}/p^d)$.*

Proof. The proof is by induction on d. The base case $d = 1$ is Proposition 4.2. For the inductive step, assume that $d \geq 2$ and that the lemma holds for $\mathrm{GL}_n(\mathbb{Z}/p^{d-1})$. For $m \geq 1$, define

$$(4.2) \qquad \mathrm{KL}_m(\mathbb{Z}/p^d) = \ker(\mathrm{GL}_m(\mathbb{Z}/p^d) \to \mathrm{GL}_m(\mathbb{Z}/p^{d-1})).$$

We will prove in Lemma 4.7 below that $\mathrm{KL}_{n_1}(\mathbb{Z}/p^d) \times \mathrm{KL}_{n_2}(\mathbb{Z}/p^d)$ is universally mixed in $\mathrm{GL}_n(\mathbb{Z}/p^d)$. To see that this implies the lemma, consider a representation $\mathbf{V}$ of $\mathrm{GL}_n(\mathbb{Z}/p^d)$ over an algebraically closed field of characteristic 0. Since $\mathrm{KL}_{n_1}(\mathbb{Z}/p^d) \times \mathrm{KL}_{n_2}(\mathbb{Z}/p^d)$ is universally mixed in $\mathrm{GL}_n(\mathbb{Z}/p^d)$, the restriction of $\mathbf{V}$ to $\mathrm{KL}_{n_1}(\mathbb{Z}/p^d) \times \mathrm{KL}_{n_2}(\mathbb{Z}/p^d)$ is either trivial or mixed.

If it is mixed, then Lemma 4.3 implies that the restriction of $\mathbf{V}$ to $\mathrm{GL}_{n_1}(\mathbb{Z}/p^d) \times \mathrm{GL}_{n_2}(\mathbb{Z}/p^d)$ is mixed. If it is trivial, then the kernel of the action of $\mathrm{GL}_n(\mathbb{Z}/p^d)$ on $\mathbf{V}$ contains $\mathrm{KL}_{n_1}(\mathbb{Z}/p^d) \times \mathrm{KL}_{n_2}(\mathbb{Z}/p^d)$. The $\mathrm{GL}_n(\mathbb{Z}/p^d)$-normal closure of $\mathrm{KL}_{n_1}(\mathbb{Z}/p^d) \times \mathrm{KL}_{n_2}(\mathbb{Z}/p^d)$ is $\mathrm{KL}_n(\mathbb{Z}/p^d)$, so we deduce that the kernel of the action of $\mathrm{GL}_n(\mathbb{Z}/p^d)$ on $\mathbf{V}$ contains $\mathrm{KL}_n(\mathbb{Z}/p^d)$. This implies that the action of $\mathrm{GL}_n(\mathbb{Z}/p^d)$ on $\mathbf{V}$ factors through $\mathrm{GL}_n(\mathbb{Z}/p^{d-1})$.

Regarding $\mathbf{V}$ as a representation of $\mathrm{GL}_n(\mathbb{Z}/p^{d-1})$, our inductive hypothesis implies that the restriction of $\mathbf{V}$ to $\mathrm{GL}_{n_1}(\mathbb{Z}/p^{d-1}) \times \mathrm{GL}_{n_2}(\mathbb{Z}/p^{d-1})$ is either trivial or mixed. This implies the analogous result for $\mathrm{GL}_{n_1}(\mathbb{Z}/p^d) \times \mathrm{GL}_{n_2}(\mathbb{Z}/p^d)$, finishing the proof. $\qquad\square$

The following lemma was invoked during the proof of Lemma 4.6. It uses the notation $\mathrm{KL}_m(\mathbb{Z}/p^d)$ from (4.2).

Lemma 4.7. *Let p^d be a prime power with $d \geq 2$ and let $n_1, n_2 \geq 1$. Set $n = n_1 + n_2$. If $n_1 = 1$ or $n_2 = 1$, then assume that[22] $p \geq 3$. Then $\mathrm{KL}_{n_1}(\mathbb{Z}/p^d) \times \mathrm{KL}_{n_2}(\mathbb{Z}/p^d)$ is universally mixed in $\mathrm{GL}_n(\mathbb{Z}/p^d)$.*

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[21]The hypotheses of Lemma 4.6 are more general than is needed for Proposition 4.5, but are exactly what is needed for the proof of Lemma 4.6.

[22]The condition $p \geq 3$ is not a typo. Lemma 4.6 required $p > 3$ here, but Lemma 4.7 is true when $p \geq 3$ and the proof naturally gives the more general statement.

Proof. Let $m \geq 1$. We start by clarifying the nature of $\mathrm{KL}_m(\mathbb{Z}/p^d)$. Elements of $\mathrm{KL}_m(\mathbb{Z}/p^d)$ can be written as $\mathbb{I}_m + p^{d-1}A$ with A an $m \times m$ matrix over $\mathbb{Z}/p^d$. The value of $\mathbb{I}_m + p^{d-1}A$ only depends on the image of A in the set of matrices over $\mathbb{Z}/p \cong \mathbb{F}_p$. Since

$$(\mathbb{I}_n + p^{d-1}A)(\mathbb{I}_n + p^{d-1}B) = \mathbb{I}_n + p^{d-1}(A+B) + p^{2d-2}AB = \mathbb{I}_n + p^{d-1}(A+B),$$

we deduce that $\mathrm{KL}_m(\mathbb{Z}/p^d)$ is isomorphic to the additive group $\mathrm{Mat}_m(\mathbb{F}_p)$ of $m \times m$ matrices over $\mathbb{F}_p$. Under this isomorphism, the conjugation action of $\mathrm{GL}_m(\mathbb{Z}/p^d)$ on its normal subgroup $\mathrm{KL}_m(\mathbb{Z}/p^d)$ is identified with the conjugation action of $\mathrm{GL}_m(\mathbb{Z}/p)$ on $\mathrm{Mat}_m(\mathbb{F}_p)$ via the surjection $\mathrm{GL}_m(\mathbb{Z}/p^d) \twoheadrightarrow \mathrm{GL}_m(\mathbb{Z}/p)$.

We return to proving that $\mathrm{KL}_{n_1}(\mathbb{Z}/p^d) \times \mathrm{KL}_{n_2}(\mathbb{Z}/p^d)$ is universally mixed in $\mathrm{GL}_n(\mathbb{Z}/p^d)$. Arguing via the poison subgroup lemma (Lemma 4.4) as in the proof of Case 1 of the proof of Proposition 4.2, to prove that

$$\mathrm{KL}_{n_1}(\mathbb{Z}/p^d) \times \mathrm{KL}_{n_2}(\mathbb{Z}/p^d) \cong \mathrm{Mat}_{n_1}(\mathbb{F}_p) \times \mathrm{Mat}_{n_2}(\mathbb{F}_p)$$

is universally mixed in $\mathrm{GL}_n(\mathbb{Z}/p^d)$, it is enough handle the following small n_i cases:

Case 1. $p \geq 3$ *and* $n_1 = n_2 = 1$, *so* $n = 2$.

Consider a representation $\mathbf{W}$ of $\mathrm{GL}_2(\mathbb{Z}/p^d)$ over an algebraically closed field $\mathbf{k}$ of characteristic 0. If the restriction of $\mathbf{W}$ to $\mathrm{KL}_1(\mathbb{Z}/p^d) \times \mathrm{KL}_1(\mathbb{Z}/p^d)$ is trivial, we are done. Assume, therefore, that this restriction is nontrivial. We must prove that it is mixed. For this, we will study it as a representation of the larger group $\mathrm{KL}_2(\mathbb{Z}/p^d)$. Let $\mathbf{U}$ be the restriction of $\mathbf{W}$ to $\mathrm{KL}_2(\mathbb{Z}/p^d)$. We know that the restriction of $\mathbf{U}$ to $\mathrm{KL}_1(\mathbb{Z}/p^d) \times \mathrm{KL}_1(\mathbb{Z}/p^d)$ is nontrivial, and our goal is to prove that this restriction is mixed.

Since $\mathrm{KL}_2(\mathbb{Z}/p^d) \cong \mathrm{Mat}_2(\mathbb{F}_p)$ is abelian and $\mathbf{k}$ is algebraically closed, the irreducible representations of $\mathrm{KL}_2(\mathbb{Z}/p^d)$ over $\mathbf{k}$ are the one-dimensional representations associated to characters $\chi \in \mathrm{Hom}(\mathrm{KL}_2(\mathbb{Z}/p^d), \mathbf{k}^\times)$. For $\chi \in \mathrm{Hom}(\mathrm{KL}_2(\mathbb{Z}/p^d), \mathbf{k}^\times)$, let $\mathbf{U}_\chi$ denote the χ-eigenspace:

$$\mathbf{U}_\chi = \left\{ x \in \mathbf{U} \mid m \cdot x = \chi(m)x \text{ for all } m \in \mathrm{KL}_2(\mathbb{Z}/p^d) \right\}.$$

We thus have

$$\mathbf{U} = \bigoplus_{\chi \in \mathrm{Hom}(\mathrm{KL}_2(\mathbb{Z}/p^d), \mathbf{k}^\times)} \mathbf{U}_\chi.$$

To prove that the restriction of $\mathbf{U}$ to $\mathrm{KL}_1(\mathbb{Z}/p^d) \times \mathrm{KL}_1(\mathbb{Z}/p^d)$ is mixed, we must find some $\chi \in \mathrm{Hom}(\mathrm{KL}_2(\mathbb{Z}/p^d), \mathbf{k}^\times)$ such that:

- $\mathbf{U}_\chi \neq 0$; and
- χ restricts to a nontrivial character on both $\mathrm{KL}_1(\mathbb{Z}/p^d) \times 1$ and $1 \times \mathrm{KL}_1(\mathbb{Z}/p^d)$.

Since the restriction of $\mathbf{U}$ to $\mathrm{KL}_1(\mathbb{Z}/p^d) \times \mathrm{KL}_1(\mathbb{Z}/p^d)$ is nontrivial, we can find some $\chi_0 \in \mathrm{Hom}(\mathrm{KL}_2(\mathbb{Z}/p^d), \mathbf{k}^\times)$ such that $\mathbf{U}_{\chi_0} \neq 0$ and χ_0 restricts to a nontrivial character on either $\mathrm{KL}_1(\mathbb{Z}/p^d) \times 1$ or $1 \times \mathrm{KL}_1(\mathbb{Z}/p^d)$. We will assume that χ_0 restricts to a nontrivial character on $\mathrm{KL}_1(\mathbb{Z}/p^d) \times 1$; the other case is identical up to changes in notation.

The conjugation action of $\mathrm{GL}_2(\mathbb{Z}/p^d)$ on its normal subgroup $\mathrm{KL}_2(\mathbb{Z}/p^d)$ induces an action of $\mathrm{GL}_2(\mathbb{Z}/p^d)$ on $\mathrm{Hom}(\mathrm{KL}_2(\mathbb{Z}/p^d), \mathbf{k}^\times)$. What is more, the action of $\mathrm{GL}_2(\mathbb{Z}/p^d)$ on

$$\mathbf{U} = \mathrm{Res}^{\mathrm{GL}_2(\mathbb{Z}/p^d)}_{\mathrm{KL}_2(\mathbb{Z}/p^d)} \mathbf{W}$$

permutes the $\mathbf{U}_\chi$ with $g \in \mathrm{GL}_2(\mathbb{Z}/p^d)$ taking $\mathbf{U}_\chi$ to $\mathbf{U}_{g \cdot \chi}$. Since $\mathbf{U}_{\chi_0} \neq 0$, we also have $\mathbf{U}_{g \cdot \chi_0} \neq 0$. It follows that it is enough to find some $g \in \mathrm{GL}_2(\mathbb{Z}/p^d)$ such that $g \cdot \chi_0$ restricts to a nontrivial character on both $\mathrm{KL}_1(\mathbb{Z}/p^d) \times 1$ and $1 \times \mathrm{KL}_1(\mathbb{Z}/p^d)$.

As we said when describing $\mathrm{KL}_m(\mathbb{Z}/p^d)$ above, the action of $\mathrm{GL}_2(\mathbb{Z}/p^d)$ on $\mathrm{KL}_2(\mathbb{Z}/p^d) \cong \mathrm{Mat}_2(\mathbb{F}_p)$ comes from the conjugation action of $\mathrm{GL}_2(\mathbb{F}_p)$ on $\mathrm{Mat}_2(\mathbb{F}_p)$ via the surjection

$$\mathrm{GL}_2(\mathbb{Z}/p^d) \longrightarrow \mathrm{GL}_2(\mathbb{Z}/p) = \mathrm{GL}_2(\mathbb{F}_p).$$

Since

$$\mathrm{Mat}_2(\mathbb{F}_p) \cong \mathrm{Hom}(\mathbb{F}_p^2, \mathbb{F}_p^2) = (\mathbb{F}_p^2)^* \otimes \mathbb{F}_p^2,$$

we see that as a representation of $\mathrm{GL}_2(\mathbb{F}_p)$ the vector space $\mathrm{Mat}_2(\mathbb{F}_p)$ is self-dual. Since $\mathrm{char}(\mathbf{k}) = 0$, this implies that there is an $\mathrm{GL}_2(\mathbb{F}_p)$-equivariant isomorphism

$$\mathrm{Hom}(\mathrm{KL}_2(\mathbb{Z}/p^d), \mathbf{k}^\times) = \mathrm{Hom}(\mathrm{Mat}_2(\mathbb{F}_p), \mathbf{k}^\times) \cong \mathrm{Mat}_2(\mathbb{F}_p).$$

Let $X_0 \in \mathrm{Mat}_2(\mathbb{F}_p)$ be the image of $\chi_0 \in \mathrm{Hom}(\mathrm{KL}_2(\mathbb{Z}/p^d), \mathbf{k}^\times)$ under this isomorphism. Since χ_0 restricts to a nontrivial character on $\mathrm{KL}_1(\mathbb{Z}/p^d) \times 1$, the $(1,1)$-entry of the 2×2 matrix X_0 is nonzero. Our goal is to find some $g \in \mathrm{GL}_2(\mathbb{F}_p)$ such that the both the $(1,1)$- and the $(2,2)$-entries of gX_0g^{-1} are nonzero.

If the $(2,2)$-entry of X_0 is already nonzero, there is nothing to prove, so we can assume it is zero. Write

$$X_0 = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \text{ with } a, b, c \in \mathbb{F}_p \text{ and } a \neq 0.$$

It is enough to deal with the following three cases:

- $b = c = 0$. Since $p \geq 3$, what we want follows from

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2a & 0 \\ a & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2a & -2a \\ a & -a \end{pmatrix}.$$

- $b \neq 0$. Since $p \geq 3$, we can find some $x \in \mathbb{F}_p$ with $x \neq 0$ and $a - bx \neq 0$ and what we want follows from

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ ax + c & bx \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} = \begin{pmatrix} a - bx & b \\ ax + c - bx^2 & bx \end{pmatrix}.$$

- $c \neq 0$. Since $p \geq 3$, we can find some $x \in \mathbb{F}_p$ with $x \neq 0$ and $a + cx \neq 0$ and what we want follows from

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a + cx & b \\ c & 0 \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a + cx & -ax - cx^2 + b \\ c & -cx \end{pmatrix}.$$

Case 2. $p = 2$ *and either* $(n_1, n_2) = (1, 2)$ *or* $(n_1, n_2) = (2, 1)$, *so* $n = 3$.

For $\mathbb{F}_2$, the argument in Case 1 fails only at the last step, and in fact keeping in mind that 1 is the only nonzero element of $\mathbb{F}_2$ one can check for instance that there does not exist

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_2) \text{ such that } \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix}^{-1} = \begin{pmatrix} 1 & r \\ s & 1 \end{pmatrix} \text{ with } r, s \in \mathbb{F}_2.$$

This is why we must go up to 3×3 matrices.

The cases $(n_1, n_2) = (1, 2)$ and $(n_1, n_2) = (2, 1)$ are identical up to changes in notation, so we will explain what to do for $(n_1, n_2) = (1, 2)$. An argument identical to the one in in Case 1 shows that it is enough to prove the following:

- Consider

$$X_0 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathrm{Mat}_3(\mathbb{F}_2) \text{ with either } a_{11} \neq 0 \text{ or } \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \neq 0.$$

Then there exists some $g \in \mathrm{GL}_3(\mathbb{F}_2)$ such that

$$gX_0g^{-1} = \begin{pmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{pmatrix} \in \mathrm{Mat}_3(\mathbb{F}_2) \text{ with both } a'_{11} \neq 0 \text{ and } \begin{pmatrix} a'_{22} & a'_{23} \\ a'_{32} & a'_{33} \end{pmatrix} \neq 0.$$

Since $\mathrm{Mat}_3(\mathbb{F}_2)$ has $2^9 = 512$ elements there are finitely many cases to check, and since $\mathrm{GL}_3(\mathbb{F}_2)$ has 168 elements this is easily done with a computer. We omit the details. $\quad\square$

4.6. **$\mathrm{Sp}_{2\mathbf{g}}$ over integers mod ℓ.** Our final example of a universally mixed subgroup is:

Proposition 4.8. *Let $\ell \geq 2$ and $g_1, g_2 \geq 2$. Set $g = g_1 + g_2$. Then $\mathrm{Sp}_{2g_1}(\mathbb{Z}/\ell) \times \mathrm{Sp}_{2g_2}(\mathbb{Z}/\ell)$ is universally mixed in $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$.*

Proof. Proposition 4.5 says that $\mathrm{GL}_{g_1}(\mathbb{Z}/\ell) \times \mathrm{GL}_{g_2}(\mathbb{Z}/\ell)$ is universally mixed in $\mathrm{GL}_g(\mathbb{Z}/\ell)$. We will prove the proposition by embedding $\mathrm{GL}_g(\mathbb{Z}/\ell)$ into $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ and applying the poison subgroup lemma (Lemma 4.4).

Let $\{a_1, b_1, \ldots, a_g, b_g\}$ be a symplectic basis for $(\mathbb{Z}/\ell)^{2g}$. Let $I \cong (\mathbb{Z}/\ell)^g$ be the span of $\{a_1, \ldots, a_g\}$ and let $J \cong (\mathbb{Z}/\ell)^g$ be the span of $\{b_1, \ldots, b_g\}$, so $(\mathbb{Z}/\ell)^{2g} = I \oplus J$. Embed $\mathrm{GL}_g(\mathbb{Z}/\ell)$ into $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ by letting a matrix $M \in \mathrm{GL}_g(\mathbb{Z}/\ell)$ act on $I \cong (\mathbb{Z}/\ell)^g$ via the action of $M \in \mathrm{GL}_g(\mathbb{Z}/\ell)$ on $(\mathbb{Z}/\ell)^g$ and on J via the action of $(M^t)^{-1} \in \mathrm{GL}_g(\mathbb{Z}/\ell)$ on $(\mathbb{Z}/\ell)^g$.

Under this embedding, $\mathrm{GL}_{g_1}(\mathbb{Z}/\ell) \times 1$ maps to $\mathrm{Sp}_{2g_1}(\mathbb{Z}/\ell) \times 1$ and $1 \times \mathrm{GL}_{g_2}(\mathbb{Z}/\ell)$ maps to $1 \times \mathrm{Sp}_{2g_2}(\mathbb{Z}/\ell)$. Moreover, the normal closure of $\mathrm{GL}_{g_1}(\mathbb{Z}/\ell) \times \mathrm{GL}_{g_2}(\mathbb{Z}/\ell)$ in $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ is $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$. The conditions of the poison subgroup lemma (Lemma 4.4) are thus satisfied, so using it we conclude that $\mathrm{Sp}_{2g_1}(\mathbb{Z}/\ell) \times \mathrm{Sp}_{2g_2}(\mathbb{Z}/\ell)$ is universally mixed in $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$. $\quad\square$

5. A criterion for algebraicity III: proof of algebraicity criterion

We finally use our results to prove our algebraicity criterion (Theorem 3.1):

Theorem 3.1 (Algebraicity criterion). *Let $\mathbf{k}$ be a field of characteristic 0 and let $g \geq 3$. Let $\mathbf{V}_{g-2}$ be an $\mathrm{Sp}_{2(g-2)}(\mathbb{Z})$-representation over $\mathbf{k}$ and $\mathbf{V}_g$ be an $\mathrm{Sp}_{2g}(\mathbb{Z})$-representation over $\mathbf{k}$. Let $f \colon \mathbf{V}_{g-2} \boxtimes \mathbf{k} \to \mathbf{V}_g$ be an $\mathrm{Sp}_{2(g-2)}(\mathbb{Z}) \times \mathrm{Sp}_4(\mathbb{Z})$-equivariant map. Assume:*

(a) the representation $\mathbf{V}_g$ is finite dimensional; and
(b) the cokernel of f is an algebraic representation of $\mathrm{Sp}_{2(g-2)}(\mathbb{Z})$.

Then $\mathbf{V}_g$ is an algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$.

Proof. Whether an $\mathrm{Sp}_{2g}(\mathbb{Z})$-representation is algebraic is unchanged under field extensions, so we can assume that $\mathbf{k}$ is algebraically closed. We now appeal to a theorem of Lubotzky [28] that says the following:[23]

- the finite-dimensional representations of $\mathrm{Sp}_{2g}(\mathbb{Z})$ over $\mathbf{k}$ are semisimple, i.e., they decompose as direct sums of irreducible representations; and
- the finite-dimensional irreducible representations of $\mathrm{Sp}_{2g}(\mathbb{Z})$ over $\mathbf{k}$ are precisely those of the form $\mathbf{U} \otimes \mathbf{W}$, where $\mathbf{U}$ is an irreducible algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$ and $\mathbf{W}$ is an irreducible representation of[24] $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ for some $\ell \geq 2$. Here $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts on $\mathbf{W}$ via the surjection $\mathrm{Sp}_{2g}(\mathbb{Z}) \twoheadrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$.

Applying this to $\mathbf{V}_g$, we can decompose it as a direct sum of irreducible representations. By projecting everything onto an irreducible subrepresentation of $\mathbf{V}_g$, we can assume that $\mathbf{V}_g$ is an irreducible representation, and thus is of the form $\mathbf{U} \otimes \mathbf{W}$ with $\mathbf{U}$ an irreducible

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[23]This uses the fact that $g \geq 2$. See [49] for an expository account of the related case of $\mathrm{SL}_n(\mathbb{Z})$.

[24]For $g \geq 2$, the congruence subgroup property [31] says that all finite quotients of $\mathrm{Sp}_{2g}(\mathbb{Z})$ factor through $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ for some $\ell \geq 2$.

algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$ and $\mathbf{W}$ an irreducible representation of $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ for some $\ell \geq 2$. Our goal is to prove that $\mathbf{W}$ is a trivial representation.

We will prove below that the restriction of $\mathbf{W}$ to $\mathrm{Sp}_{2(g-2)}(\mathbb{Z}/\ell) \times \mathrm{Sp}_4(\mathbb{Z}/\ell)$ is unmixed. Assuming this, since by Proposition 4.8 the subgroup $\mathrm{Sp}_{2(g-2)}(\mathbb{Z}/\ell) \times \mathrm{Sp}_4(\mathbb{Z}/\ell)$ is universally mixed in $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ we deduce that the restriction of $\mathbf{W}$ to $\mathrm{Sp}_{2(g-2)}(\mathbb{Z}/\ell) \times \mathrm{Sp}_4(\mathbb{Z}/\ell)$ trivial. Since the normal closure of $\mathrm{Sp}_{2(g-2)}(\mathbb{Z}/\ell) \times \mathrm{Sp}_4(\mathbb{Z}/\ell)$ in $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ is $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$, this implies that $\mathbf{W}$ is a trivial representation of $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$, as desired.

It remains to prove that the restriction of $\mathbf{W}$ to $\mathrm{Sp}_{2(g-2)}(\mathbb{Z}/\ell) \times \mathrm{Sp}_4(\mathbb{Z}/\ell)$ is unmixed. Let

$$\mathrm{Res}^{\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)}_{\mathrm{Sp}_{2(g-2)}(\mathbb{Z}/\ell) \times \mathrm{Sp}_4(\mathbb{Z}/\ell)} \mathbf{W} = \bigoplus_{i=1}^n \mathbf{W}_i$$

be a decomposition into irreducible representations of $\mathrm{Sp}_{2(g-2)}(\mathbb{Z}/\ell) \times \mathrm{Sp}_4(\mathbb{Z}/\ell)$ and let

$$\mathrm{Res}^{\mathrm{Sp}_{2g}(\mathbb{Z})}_{\mathrm{Sp}_{2(g-2)}(\mathbb{Z}) \times \mathrm{Sp}_4(\mathbb{Z})} \mathbf{U} = \bigoplus_{j=1}^m \mathbf{U}_j$$

be a decomposition into irreducible algebraic representations of $\mathrm{Sp}_{2(g-2)}(\mathbb{Z}) \times \mathrm{Sp}_4(\mathbb{Z})$. We thus have

$$\mathrm{Res}^{\mathrm{Sp}_{2g}(\mathbb{Z})}_{\mathrm{Sp}_{2(g-2)}(\mathbb{Z}) \times \mathrm{Sp}_4(\mathbb{Z})} \mathbf{V}_g = \bigoplus_{i=1}^n \bigoplus_{j=1}^m \mathbf{U}_j \otimes \mathbf{W}_i.$$

Each $\mathbf{U}_j \otimes \mathbf{W}_i$ is an irreducible representation[25] of $\mathrm{Sp}_{2(g-2)}(\mathbb{Z}) \times \mathrm{Sp}_4(\mathbb{Z}/\ell)$.

For each $1 \leq i \leq n$ and $1 \leq j \leq m$, assumption (b) implies that one of the following two things happens:

- $\mathbf{U}_j \otimes \mathbf{W}_i$ is in the image of $f \colon \mathbf{V}_{g-2} \boxtimes \mathbf{k} \to \mathbf{V}_g$, so $1 \times \mathrm{Sp}_4(\mathbb{Z})$ acts trivially on $\mathbf{U}_j \otimes \mathbf{W}_i$ and hence on $\mathbf{W}_i$; or
- $\mathbf{U}_j \otimes \mathbf{W}_i$ survives in $\mathrm{coker}(f)$, so its restriction to $\mathrm{Sp}_{2(g-2)}(\mathbb{Z})$ is an algebraic representation of $\mathrm{Sp}_{2(g-2)}(\mathbb{Z})$. This implies that $\mathrm{Sp}_{2(g-2)}(\mathbb{Z}) \times 1$ acts trivially on $\mathbf{W}_i$.

Let $\mathbf{W}'$ be the direct sum of the $\mathbf{W}_i$ such that $1 \times \mathrm{Sp}_4(\mathbb{Z})$ acts trivially on $\mathbf{W}_i$ and let $\mathbf{W}''$ be the direct sum of the $\mathbf{W}_i$ such that $\mathrm{Sp}_{2(g-2)}(\mathbb{Z}) \times 1$ acts trivially on $\mathbf{W}_i$. The restriction of $\mathbf{W}$ to $\mathrm{Sp}_{2(g-2)}(\mathbb{Z}/\ell) \times \mathrm{Sp}_4(\mathbb{Z}/\ell)$ decomposes as $\mathbf{W}' \oplus \mathbf{W}''$, showing that it is unmixed. $\square$

Part 2. Homology of Torelli, step 1: reduction to curve stabilizers

Recall from §1 that the Torelli group $\mathcal{I}^b_{g,p}$ on a genus g surface $\Sigma^b_{g,p}$ with p marked points and b boundary components is the kernel of the action of the mapping class group $\mathrm{Mod}^b_{g,p}$ on $\mathrm{H}_1(\Sigma_g)$. Each $\mathrm{H}_d(\mathcal{I}^b_{g,p};\mathbb{Q})$ is a representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$. Our goal is to prove Theorem B, which says that $\mathrm{H}_2(\mathcal{I}^b_{g,p};\mathbb{Q})$ is finite dimensional for $g \geq 5$ and an algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$ for $g \geq 6$. In this part of the paper, we reduce this to a theorem about curve stabilizers. This reduction occurs in §7, which is preceded by the preliminary §6.

6. Step 1.1: preliminary results about the homology of Torelli

This section contains some preliminary results about the homology of the Torelli group.

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[25] This follows from the Jacobson density theorem (cf. the proof of the first Claim of [49, Theorem C]).

6.1. **Deleting boundary components and marked points.** If $f \colon \mathbf{V} \to \mathbf{W}$ is an equivariant map between $\mathrm{Sp}_{2g}(\mathbb{Z})$-representations $\mathbf{V}$ and $\mathbf{W}$ over $\mathbb{Q}$, then say that f is an *isomorphism mod fin dim alg reps* if both $\ker(f)$ and $\mathrm{coker}(f)$ are finite-dimensional algebraic representations of $\mathrm{Sp}_{2g}(\mathbb{Z})$. If this holds, then $\mathbf{V}$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$ if and only if $\mathbf{W}$ is.

The following two lemmas imply that for a fixed $g \geq 3$, to prove that $\mathrm{H}_2(\mathcal{I}_{g,p}^b; \mathbb{Q})$ is either finite-dimensional or an algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$ for all $p, b \geq 0$ it is enough to prove this for any single choice of p or b. This will reduce us to only considering $\mathrm{H}_2(\mathcal{I}_g^1; \mathbb{Q})$.

Lemma 6.1 (Cap boundary). *Let $b, p \geq 0$ and $g \geq 3$. Let ∂ be a component of $\partial\Sigma_{g,p}^{b+1}$ and let $\mathcal{I}_{g,p}^{b+1} \to \mathcal{I}_{g,p+1}^b$ be the map that glues a disc containing a marked point to ∂ and extends mapping classes in $\mathcal{I}_{g,p}^{b+1}$ over it by the identity. Then the induced map $\mathrm{H}_2(\mathcal{I}_{g,p}^{b+1}; \mathbb{Q}) \to \mathrm{H}_2(\mathcal{I}_{g,p+1}^b; \mathbb{Q})$ is an isomorphism mod fin dim alg representations.*

The proofs of this and many other results will use the following fact:

(♠) the collection of finite-dimensional algebraic representations of $\mathrm{Sp}_{2g}(\mathbb{Z})$ is a Serre class, i.e., it is closed under subquotients and extensions.

Proof. By [10, Proposition 3.19], there is a central extension

$$(6.1) \qquad 1 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Mod}_{g,p}^{b+1} \xrightarrow{\ f\ } \mathrm{Mod}_{g,p+1}^b \longrightarrow 1,$$

where the central $\mathbb{Z}$ is generated by the Dehn twist T_∂ and f glues a disc containing a marked point to ∂ and extends mapping classes over it by the identity. Since the action of $\mathrm{Mod}_{g,p}^{b+1}$ on $\mathrm{H}_1(\Sigma_g)$ factors through $\mathrm{Mod}_{g,p+1}^b$, an element $\phi \in \mathrm{Mod}_{g,p}^{b+1}$ acts trivially on $\mathrm{H}_1(\Sigma_g)$ if and only if $f(\phi)$ does. It follows that (6.1) restricts to a similar central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{I}_{g,p}^{b+1} \longrightarrow \mathcal{I}_{g,p+1}^b \longrightarrow 1$$

of Torelli groups. The Hochschild–Serre spectral sequence of this extension induces a long exact Gysin sequence that contains the segment

$$\mathrm{H}_1(\mathcal{I}_{g,p+1}^b; \mathbb{Q}) \longrightarrow \mathrm{H}_2(\mathcal{I}_{g,p}^{b+1}; \mathbb{Q}) \longrightarrow \mathrm{H}_2(\mathcal{I}_{g,p+1}^b; \mathbb{Q}) \longrightarrow \mathrm{H}_0(\mathcal{I}_{g,p+1}^b; \mathbb{Q})$$

It follows from Johnson's work [24] that $\mathrm{H}_1(\mathcal{I}_{g,p+1}^b; \mathbb{Q})$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$ for $g \geq 3$. Also, $\mathrm{H}_0(\mathcal{I}_{g,p+1}^n; \mathbb{Q}) = \mathbb{Q}$ is an algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$. The lemma now follows from (♠). $\square$

Lemma 6.2 (Delete marked point). *Let $b, p \geq 0$ and $g \geq 3$. Let p_0 be a marked point of $\Sigma_{g,p+1}^b$ and let $\mathcal{I}_{g,p+1}^b \to \mathcal{I}_{g,p}^b$ be the map that deletes p_0. Then the induced map $\mathrm{H}_2(\mathcal{I}_{g,p+1}^b; \mathbb{Q}) \to \mathrm{H}_2(\mathcal{I}_{g,p}^b; \mathbb{Q})$ is an isomorphism mod fin dim alg reps.*

Proof. There is a Birman exact sequence [10, Theorem 4.6]

$$(6.2) \qquad 1 \longrightarrow \pi_1(\Sigma_{g,p}^b, p_0) \longrightarrow \mathrm{Mod}_{g,p+1}^b \xrightarrow{\ f\ } \mathrm{Mod}_{g,p}^b \longrightarrow 1,$$

where[26] $\pi_1(\Sigma_{g,p}^b, p_0)$ is the point-pushing subgroup of $\mathrm{Mod}_{g,p+1}^b$ and f deletes p_0. Since the action of $\mathrm{Mod}_{g,p+1}^b$ on $\mathrm{H}_1(\Sigma_g)$ factors through $\mathrm{Mod}_{g,p}^b$, an element $\phi \in \mathrm{Mod}_{g,p+1}^b$ acts trivially

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[26]Here by $\pi_1(\Sigma_{g,p}^b, p_0)$ we mean the fundamental group of a genus g surface with p punctures (not marked points) and b boundary components. Throughout this proof, we will continue to let context indicate whether p means "punctures" or "marked points".

on $H_1(\Sigma_g)$ if and only if $f(\phi)$ does. It follows that (6.2) restricts to an exact sequence

$$1 \longrightarrow \pi_1(\Sigma_{g,p}^b, p_0) \longrightarrow \mathcal{I}_{g,p+1}^b \xrightarrow{\ f\ } \mathcal{I}_{g,p}^b \longrightarrow 1.$$

The associated Hochschild–Serre spectral sequence takes the form

(6.3) $$\mathrm{E}_{pq}^2 = \mathrm{H}_p(\mathcal{I}_{g,p}^b; \mathrm{H}_q(\pi_1(\Sigma_{g,p}^b); \mathbb{Q})) \Rightarrow \mathrm{H}_{p+q}(\mathcal{I}_{g,p+1}^b; \mathbb{Q}).$$

The terms of this spectral sequence are representations of $\mathrm{Sp}_{2g}(\mathbb{Z})$, and the differentials are $\mathrm{Sp}_{2g}(\mathbb{Z})$-equivariant. We will use this spectral sequence to prove that the kernel and cokernel of $\mathrm{H}_2(\mathcal{I}_{g,p+1}^b; \mathbb{Q}) \to \mathrm{H}_2(\mathcal{I}_{g,p}^b; \mathbb{Q})$ are finite-dimensional algebraic representations of $\mathrm{Sp}_{2g}(\mathbb{Z})$.

We start with the cokernel. We have $\mathrm{E}_{20}^2 = \mathrm{H}_2(\mathcal{I}_{g,p}^b; \mathbb{Q})$, and the image of the map $\mathrm{H}_2(\mathcal{I}_{g,p+1}^b; \mathbb{Q}) \to \mathrm{H}_2(\mathcal{I}_{g,p}^b; \mathbb{Q})$ is

$$\mathrm{E}_{20}^\infty = \mathrm{E}_{20}^3 = \ker(\mathrm{E}_{20}^2 \to \mathrm{E}_{01}^2) = \ker(\mathrm{H}_2(\mathcal{I}_{g,p}^b; \mathbb{Q}) \to \mathrm{E}_{01}^2).$$

This implies that cokernel of $\mathrm{H}_2(\mathcal{I}_{g,p+1}^b; \mathbb{Q}) \to \mathrm{H}_2(\mathcal{I}_{g,p}^b; \mathbb{Q})$ embeds into E_{01}^2. Using (♠), to prove that the cokernel of $\mathrm{H}_2(\mathcal{I}_{g,p+1}^b; \mathbb{Q}) \to \mathrm{H}_2(\mathcal{I}_{g,p}^b; \mathbb{Q})$ is a finite dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$ it is enough to prove that

$$\mathrm{E}_{01}^2 = \mathrm{H}_0(\mathcal{I}_{g,p}^b; \mathrm{H}_1(\Sigma_{g,p}^b; \mathbb{Q})) = \mathrm{H}_1(\Sigma_{g,p}^b; \mathbb{Q})_{\mathcal{I}_{g,p}^b}$$

is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$, where the subscript indicates that we are taking coinvariants. Though $\mathcal{I}_{g,p}^b$ acts trivially on $\mathrm{H}_1(\Sigma_g; \mathbb{Q})$, it might not act trivially on $\mathrm{H}_1(\Sigma_{g,p}^b; \mathbb{Q})$. However, letting $m = \max(p+q-1, 0)$ we do have an extension

(6.4) $$0 \longrightarrow \mathbb{Q}^m \longrightarrow \mathrm{H}_1(\Sigma_{g,p}^b; \mathbb{Q}) \longrightarrow \mathrm{H}_1(\Sigma_g; \mathbb{Q}) \longrightarrow 0$$

with $\mathcal{I}_{g,p}^b$ acting trivially on the kernel and cokernel. This induces a right-exact sequence

$$\mathbb{Q}^m \longrightarrow \mathrm{H}_1(\Sigma_{g,p}^b; \mathbb{Q})_{\mathcal{I}_{g,p}^b} \longrightarrow \mathrm{H}_1(\Sigma_g; \mathbb{Q}) \longrightarrow 0.$$

Since $\mathbb{Q}^m$ is a trivial representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$ and $\mathrm{H}_1(\Sigma_g; \mathbb{Q})$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$, using (♠) it follows that $\mathrm{E}_{01}^2 = \mathrm{H}_1(\Sigma_{g,p}^b; \mathbb{Q})_{\mathcal{I}_{g,p}^b}$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$, as desired.

We now handle the kernel. In terms of the spectral sequence (6.3), the kernel of the map $\mathrm{H}_2(\mathcal{I}_{g,p+1}^b; \mathbb{Q}) \to \mathrm{H}_2(\mathcal{I}_{g,p}^b; \mathbb{Q})$ has a filtration whose associated graded terms are E_{02}^∞ and E_{11}^∞. These are subquotients of E_{02}^2 and E_{11}^2. Using (♠), it is enough to prove that E_{02}^2 and E_{11}^2 are finite-dimensional algebraic representations of $\mathrm{Sp}_{2g}(\mathbb{Z})$.

The term E_{02}^2 is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$ since

$$\mathrm{E}_{02}^2 = \mathrm{H}_0(\mathcal{I}_{g,p}^b; \mathrm{H}_2(\Sigma_{g,p}^b; \mathbb{Q})) = \mathrm{H}_2(\Sigma_{g,p}^b; \mathbb{Q})_{\mathcal{I}_{g,p}^b} = \begin{cases} \mathbb{Q} & \text{if } p = b = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $\mathrm{E}_{11}^2 = \mathrm{H}_1(\mathcal{I}_{g,p}^b; \mathrm{H}_1(\Sigma_{g,p}^b; \mathbb{Q}))$, note that the long exact sequence in homology associated to the extension (6.4) of $\mathcal{I}_{g,p}^b$-representations contains the segment

$$\mathrm{H}_1(\mathcal{I}_{g,p}^b; \mathbb{Q}^m) \longrightarrow \mathrm{H}_1(\mathcal{I}_{g,p}^b; \mathrm{H}_1(\Sigma_{g,p}^b; \mathbb{Q})) \longrightarrow \mathrm{H}_1(\mathcal{I}_{g,p}^b; \mathrm{H}_1(\Sigma_g; \mathbb{Q})).$$

This can be rewritten as

$$\mathrm{H}_1(\mathcal{I}_{g,p}^b; \mathbb{Q})^m \longrightarrow \mathrm{E}_{11}^2 \longrightarrow \mathrm{H}_1(\mathcal{I}_{g,p}^b; \mathbb{Q}) \otimes \mathrm{H}_1(\Sigma_g; \mathbb{Q}).$$

It follows from Johnson's work [24] that $H_1(\mathcal{I}_{g,p+1}^b; \mathbb{Q})$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$ for $g \geq 3$. Using ($\spadesuit$) it follows that E_{11}^2 is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$, as desired. $\qquad\square$

6.2. **Curve stabilizers.** Embed Σ_{g-1}^1 in Σ_g and let γ be an oriented nonseparating simple closed curve on Σ_g that is disjoint from Σ_{g-1}^1:
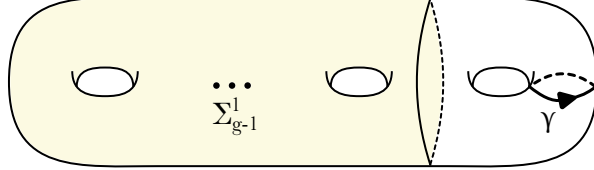
Let $(\mathcal{I}_g)_\gamma$ be the $\mathcal{I}_g$-stabilizer of the isotopy class of γ. The image of the map $\mathcal{I}_{g-1}^1 \hookrightarrow \mathcal{I}_g$ induced by our embedding lies in $(\mathcal{I}_g)_\gamma$. Then:

Lemma 6.3. *Let $g \geq 5$. Embed Σ_{g-1}^1 in Σ_g and let γ be a nonseparating simple closed curve on Σ_g that is disjoint from Σ_{g-1}^1. Let $h\colon H_2(\mathcal{I}_{g-1}^1; \mathbb{Q}) \to H_2((\mathcal{I}_g)_\gamma; \mathbb{Q})$ be the map induced by the inclusion $\mathcal{I}_{g-1}^1 \hookrightarrow (\mathcal{I}_g)_\gamma$. Regard h as a map of $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$-representations. Then h is an isomorphism mod fin dim alg reps.*

The proof uses a result that we will also use several other times. For a subsurface S of Σ_g, let $\mathcal{I}_g(S)$ be the subgroup of $\mathcal{I}_g$ consisting of mapping classes supported on S. For instance, if $\gamma_1, \ldots, \gamma_k$ are simple closed curves on Σ_g that intersect transversely, then $(\mathcal{I}_g)_{\gamma_1,\ldots,\gamma_k} = \mathcal{I}_g(S)$ for S the complement of an open regular neighborhood of $\gamma_1 \cup \cdots \cup \gamma_k$. Then:

Theorem 6.4 (Putman, [46, Theorem B]). *Let $S \subset \Sigma_g$ be a connected subsurface of genus at least 3. Then the map $H_1(\mathcal{I}_g(S); \mathbb{Q}) \to H_1(\mathcal{I}_g; \mathbb{Q})$ is injective.*

Proof of Lemma 6.3. Let ∂_1 and ∂_2 be the components of $\partial\Sigma_{g-1}^2$. Embed Σ_{g-1}^1 in Σ_{g-1}^2:
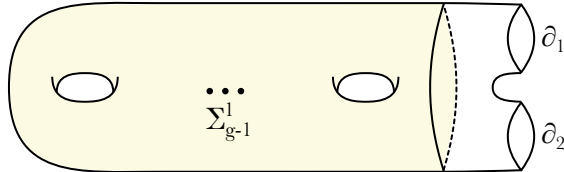
Let $\Sigma_{g-1}^2 \to \Sigma_g$ be the map that glues ∂_1 to ∂_2 and maps those components to γ and Σ_{g-1}^1 to Σ_{g-1}^1. Let $\phi\colon \mathrm{Mod}_{g-1}^2 \to \mathrm{Mod}_g$ be the induced map on mapping class groups, so $\phi(T_{\partial_1}) = \phi(T_{\partial_2}) = T_\gamma$. The image of ϕ is $(\mathrm{Mod}_g)_\gamma$, and ϕ takes $\mathrm{Mod}_{g-1}^1 < \mathrm{Mod}_{g-1}^2$ isomorphically onto $\mathrm{Mod}_{g-1}^1 < \mathrm{Mod}_g$. Define $\widetilde{\mathcal{I}}_{g-1}^2 = \phi^{-1}(\mathcal{I}_g)$. Be warned that this is a proper subgroup of $\mathcal{I}_{g-1}^2$; see [40]. The map h factors as

$$ H_2(\mathcal{I}_{g-1}^1; \mathbb{Q}) \xrightarrow{\ f\ } H_2(\widetilde{\mathcal{I}}_{g-1}^2; \mathbb{Q}) \xrightarrow{\ \widetilde{\phi}\ } H_2((\mathcal{I}_g)_\gamma; \mathbb{Q}), $$

where f is induced by the inclusion $\mathcal{I}_{g-1}^1 \hookrightarrow \widetilde{\mathcal{I}}_{g-1}^2$ and $\widetilde{\phi}$ is induced by the restriction of $\phi\colon \mathrm{Mod}_{g-1}^2 \to (\mathrm{Mod}_g)_\gamma$ to $\widetilde{\mathcal{I}}_{g-1}^2$. To prove that h is an isomorphism mod fin dim alg reps, it is enough to prove this for f and $\widetilde{\phi}$:

Step 1. *The map $\widetilde{\phi}\colon H_2(\widetilde{\mathcal{I}}_{g-1}^2; \mathbb{Q}) \longrightarrow H_2((\mathcal{I}_g)_\gamma; \mathbb{Q})$ is an isomorphism mod fin dim alg reps.*

Since $\phi(T_{\partial_1}) = \phi(T_{\partial_2}) = T_\gamma$, we have $T_{\partial_1} T_{\partial_2}^{-1} \in \ker(\phi)$. In fact, we have a central extension

$$ 1 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Mod}_{g-1}^2 \xrightarrow{\ \phi\ } (\mathrm{Mod}_g)_\gamma \longrightarrow 1 $$

whose central $\mathbb{Z}$ is generated by $T_{\partial_1} T_{\partial_2}^{-1}$. We have $T_{\partial_1} T_{\partial_2}^{-1} \in \widetilde{\mathcal{I}}_{g-1}^2$, so this restricts to a central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathcal{I}}_{g-1}^2 \longrightarrow (\mathcal{I}_g)_\gamma \longrightarrow 1.$$

The Hochschild–Serre spectral sequence of this extension induces a long exact Gysin sequence that contains the segment

$$\mathrm{H}_1((\mathcal{I}_g)_\gamma; \mathbb{Q}) \longrightarrow \mathrm{H}_2(\widetilde{\mathcal{I}}_{g-1}^2; \mathbb{Q}) \xrightarrow{\widetilde{\phi}} \mathrm{H}_2((\mathcal{I}_g)_\gamma; \mathbb{Q}) \longrightarrow \mathrm{H}_0((\mathcal{I}_g)_\gamma; \mathbb{Q}).$$

Since $\mathrm{H}_0((\mathcal{I}_g)_\gamma; \mathbb{Q}) = \mathbb{Q}$ is an algebraic representation of $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$, by ($\spadesuit$) it is enough to prove that $\mathrm{H}_1((\mathcal{I}_g)_\gamma; \mathbb{Q})$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$.

Theorem 6.4 implies that the map $\mathrm{H}_1((\mathcal{I}_g)_\gamma; \mathbb{Q}) \to \mathrm{H}_1(\mathcal{I}_g; \mathbb{Q})$ is injective. Johnson [24] proved that $\mathrm{H}_1(\mathcal{I}_g; \mathbb{Q})$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$. Its restriction to $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$ is also algebraic, so by ($\spadesuit$) its subrepresentation $\mathrm{H}_1((\mathcal{I}_g)_\gamma; \mathbb{Q})$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$, as desired.

Step 2. *The map $f \colon \mathrm{H}_2(\mathcal{I}_{g-1}^1; \mathbb{Q}) \to \mathrm{H}_2(\widetilde{\mathcal{I}}_{g-1}^2; \mathbb{Q})$ is an isomorphism mod fin dim alg reps.*

Putman [40] constructed a version of the Birman exact sequence for[27] $\widetilde{\mathcal{I}}_{g-1}^2$. Letting $\pi = \pi_1(\Sigma_{g-1}^1)$, it takes the form

(6.5) $$1 \longrightarrow [\pi, \pi] \longrightarrow \widetilde{\mathcal{I}}_{g-1}^2 \longrightarrow \mathcal{I}_{g-1}^1 \longrightarrow 1.$$

Here $\widetilde{\mathcal{I}}_{g-1}^2 \to \mathcal{I}_{g-1}^1$ is the map induced by gluing a disc to ∂_1 and extending mapping classes by the identity, and $[\pi, \pi]$ is an appropriate subgroup of the "disc pushing group". The extension (6.5) splits via the map $\mathcal{I}_{g-1}^1 \to \widetilde{\mathcal{I}}_{g-1}^2$ induced by the embedding $\Sigma_{g-1}^1 \hookrightarrow \Sigma_{g-1}^2$ discussed above.

Since $[\pi, \pi]$ is a free group, the Hochschild–Serre spectral sequence of (6.5) has two rows, and since this extension is split all the differentials coming out of the bottom row of this spectral sequence vanish. We deduce that this spectral sequence degenerates to give an extension

$$0 \longrightarrow \mathrm{H}_1(\mathcal{I}_{g-1}^1; \mathrm{H}_1([\pi, \pi]; \mathbb{Q})) \longrightarrow \mathrm{H}_2(\widetilde{\mathcal{I}}_{g-1}^2; \mathbb{Q}) \longrightarrow \mathrm{H}_2(\mathcal{I}_{g-1}^1; \mathbb{Q}) \longrightarrow 0.$$

The $f \colon \mathrm{H}_2(\mathcal{I}_{g-1}^1; \mathbb{Q}) \to \mathrm{H}_2(\widetilde{\mathcal{I}}_{g-1}^2; \mathbb{Q})$ we are trying to prove is an isomorphism mod fin dim alg reps is the splitting of this exact sequence coming from the splitting of (6.5). We conclude that $\ker(f) = 1$ and that $\mathrm{coker}(f) \cong \mathrm{H}_1(\mathcal{I}_{g-1}^1; \mathrm{H}_1([\pi, \pi]; \mathbb{Q}))$. For $g \geq 5$, the authors proved in [36][28] that $\mathrm{H}_1(\mathcal{I}_{g-1}^1; \mathrm{H}_1([\pi, \pi]; \mathbb{Q}))$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$. The step follows. $\qquad\square$

7. STEP 1.1: REDUCTION TO CURVE STABILIZERS

Let α and β be the following curves on Σ_g:

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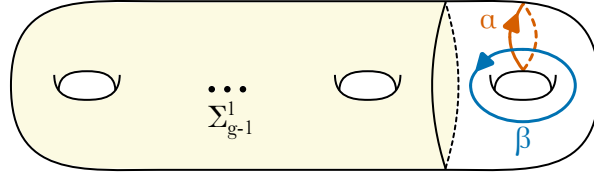
[27]In the notation of [40], the group $\widetilde{\mathcal{I}}_{g-1}^2$ is $\mathcal{I}(\Sigma_{g-1}^2, \{\{\partial_1, \partial_2\}\})$.

[28]Earlier Putman [42] proved an analogous result for the level-ℓ subgroup of Mod_{g-1}^1. This played an important role in Putman's computation of the second homology of the level-ℓ subgroup in [43, 44].

Let
$$\lambda\colon \operatorname{H}_2((\mathcal{I}_g)_\alpha;\mathbb{Q}) \oplus \operatorname{H}_2((\mathcal{I}_g)_\beta;\mathbb{Q}) \longrightarrow \operatorname{H}_2(\mathcal{I}_g;\mathbb{Q})$$
be the sum of the maps induced by the inclusions $(\mathcal{I}_g)_\alpha \hookrightarrow \mathcal{I}_g$ and $(\mathcal{I}_g)_\beta \hookrightarrow \mathcal{I}_g$ and let Λ_g be the cokernel of λ. The group Mod_g does not act on Λ_g since it does not fix α and β, but the subgroup $\operatorname{Mod}_{g-1}^1$ of Mod_g does act on Λ_g. This factors through $\operatorname{Sp}_{2(g-1)}(\mathbb{Z})$, making Λ_g into a representation of $\operatorname{Sp}_{2(g-1)}(\mathbb{Z})$. The rest of this paper is devoted to the proof of:

Theorem B′. *The representation Λ_g is a finite-dimensional algebraic representation of $\operatorname{Sp}_{2(g-1)}(\mathbb{Z})$ for $g \geq 5$.*

Here we will assume Theorem B′ and use it to prove Theorem B.

Theorem B. *Let $b, p \geq 0$. Then $\operatorname{H}_2(\mathcal{I}_{g,p}^b;\mathbb{Q})$ is finite dimensional for $g \geq 5$ and an algebraic representation of $\operatorname{Sp}_{2g}(\mathbb{Z})$ for $g \geq 6$.*

Proof of Theorem B, assuming Theorem B′. By Lemmas 6.1 and 6.2, it is enough to prove Theorem B for $\operatorname{H}_2(\mathcal{I}_g^1;\mathbb{Q})$. Embed Σ_{g-1}^1 into Σ_g^1 as in the following figure:
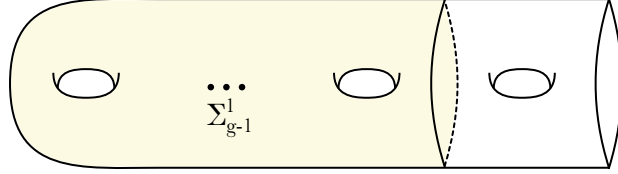
Extending mapping classes in $\mathcal{I}_{g-1}^1$ to Σ_g^1 by the identity, we get an induced map $\mathcal{I}_{g-1}^1 \to \mathcal{I}_g^1$. Passing to H_2 gives a coherent sequence[29]
$$\operatorname{H}_2(\mathcal{I}_1^1;\mathbb{Q}) \xrightarrow{f_1} \operatorname{H}_2(\mathcal{I}_2^1;\mathbb{Q}) \xrightarrow{f_2} \operatorname{H}_2(\mathcal{I}_3^1;\mathbb{Q}) \xrightarrow{f_3} \cdots$$
of representations of $\operatorname{Sp}_{2g}(\mathbb{Z})$.

We would like to apply our stability theorem (Theorem D) to this coherent sequence with $g_0 = 5$. If we can do this, that theorem will imply that $\operatorname{H}_2(\mathcal{I}_g^1;\mathbb{Q})$ is finite dimensional for $g \geq g_0$ and an algebraic representation of $\operatorname{Sp}_{2g}(\mathbb{Z})$ for $g \geq g_0 + 1$, just like we want. Theorem D has two hypotheses (i) and (ii) we must verify:

Step 1. *Hypothesis (i) holds:* $\operatorname{coker}(f_{g-1})$ *is a finite-dimensional algebraic representation of $\operatorname{Sp}_{2(g-1)}(\mathbb{Z})$ for $g \geq g_0 = 5$.*

Let $f_{g-1}'\colon \operatorname{H}_2(\mathcal{I}_{g-1}^1;\mathbb{Q}) \to \operatorname{H}_2(\mathcal{I}_g;\mathbb{Q})$ be the composition
$$\operatorname{H}_2(\mathcal{I}_{g-1}^1;\mathbb{Q}) \xrightarrow{f_{g-1}} \operatorname{H}_2(\mathcal{I}_g^1;\mathbb{Q}) \xrightarrow{\pi} \operatorname{H}_2(\mathcal{I}_g;\mathbb{Q}),$$
where π is the map induced by gluing a disc to $\partial\Sigma_g^1$ and extending mapping classes by the identity. We can factor π as
$$\operatorname{H}_2(\mathcal{I}_g^1;\mathbb{Q}) \longrightarrow \operatorname{H}_2(\mathcal{I}_{g,1};\mathbb{Q}) \longrightarrow \operatorname{H}_2(\mathcal{I}_g;\mathbb{Q}).$$

Lemmas 6.1 and 6.2 say that these two maps are isomorphisms mod fin dim alg reps for $g \geq 3$, so using (♠) the map π is as well. Again using (♠), this implies that to prove that $\operatorname{coker}(f_{g-1})$ is a finite-dimensional algebraic representation of $\operatorname{Sp}_{2(g-1)}(\mathbb{Z})$, it is enough to prove the same result for $\operatorname{coker}(f_{g-1}')$.

Let α and β be the curves on Σ_g from Theorem B′. Let
$$h_\alpha\colon \operatorname{H}_2(\mathcal{I}_{g-1}^1;\mathbb{Q}) \longrightarrow \operatorname{H}_2((\mathcal{I}_g)_\alpha;\mathbb{Q}) \quad \text{and} \quad h_\beta\colon \operatorname{H}_2(\mathcal{I}_{g-1}^1;\mathbb{Q}) \longrightarrow \operatorname{H}_2((\mathcal{I}_g)_\beta;\mathbb{Q})$$

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[29]See §1.4 for the definition of a coherent sequence.

be the maps induced by the inclusions $\mathcal{I}^1_{g-1} \hookrightarrow (\mathcal{I}_g)_\alpha$ and $\mathcal{I}^1_{g-1} \hookrightarrow (\mathcal{I}_g)_\beta$ and let

$$\lambda_\alpha \colon H_2((\mathcal{I}_g)_\alpha; \mathbb{Q}) \longrightarrow H_2(\mathcal{I}_g; \mathbb{Q}) \quad \text{and} \quad \lambda_\beta \colon H_2((\mathcal{I}_g)_\beta; \mathbb{Q}) \longrightarrow H_2(\mathcal{I}_g; \mathbb{Q})$$

be the maps induced by the inclusions $(\mathcal{I}_g)_\alpha \hookrightarrow \mathcal{I}_g$ and $(\mathcal{I}_g)_\beta \hookrightarrow \mathcal{I}_g$. Consider the sum of two copies of f'_{g-1}:

$$f'_{g-1} + f'_{g-1} \colon H_2(\mathcal{I}^1_{g-1}; \mathbb{Q}) \oplus H_2(\mathcal{I}^1_{g-1}; \mathbb{Q}) \longrightarrow H_2(\mathcal{I}_g; \mathbb{Q}).$$

We can factor $f'_{g-1} + f'_{g-1}$ as

$$(7.1) \quad H_2(\mathcal{I}^1_{g-1}; \mathbb{Q}) \oplus H_2(\mathcal{I}^1_{g-1}; \mathbb{Q}) \xrightarrow{h_\alpha \oplus h_\beta} H_2((\mathcal{I}_g)_\alpha; \mathbb{Q}) \oplus H_2((\mathcal{I}_g)_\beta; \mathbb{Q}) \xrightarrow{\lambda_\alpha + \lambda_\beta} H_2(\mathcal{I}_g; \mathbb{Q}).$$

Lemma 6.3 says that h_α and h_β are both isomorphisms mod fin dim alg reps for $g \geq 5$, so by ($\spadesuit$) the map $h_\alpha \oplus h_\beta$ is as well. Theorem B$'$ says that $\Lambda_g = \operatorname{coker}(\lambda_\alpha + \lambda_\beta)$ is a finite-dimensional algebraic representation of $\operatorname{Sp}_{2(g-1)}(Z)$ for $g \geq 5$. Using the factorization (7.1) and ($\spadesuit$), we deduce that $\operatorname{coker}(f'_{g-1} + f'_{g-1})$ is also a finite-dimensional algebraic representation of $\operatorname{Sp}_{2(g-1)}(\mathbb{Z})$ for $g \geq 5$. Since f'_{g-1} and $f'_{g-1} + f'_{g-1}$ have the same image, the same is true for $\operatorname{coker}(f'_{g-1})$, as desired.

Step 2. *Hypothesis (ii) holds: the coinvariants* $H_2(\mathcal{I}^1_g; \mathbb{Q})_{\operatorname{Sp}_{2g}(\mathbb{Z})}$ *are finite-dimensional for* $g \geq g_0 = 5$. *In fact, this holds for* $g \geq 3$.

One quick way to see this is to appeal to [25], which says that $H_2(\mathcal{I}^1_g; \mathbb{Q})$ is finitely generated as a module over the group ring $\mathbb{Q}[\operatorname{Sp}_{2g}(\mathbb{Z})]$. Another more elementary approach is to use the Hochschild–Serre spectral sequence of the extension

$$1 \longrightarrow \mathcal{I}^1_g \longrightarrow \operatorname{Mod}^1_g \longrightarrow \operatorname{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1,$$

which takes the form

$$E^2_{pq} = H_p(\operatorname{Sp}_{2g}(\mathbb{Z}); H_q(\mathcal{I}^1_g; \mathbb{Q})) \Rightarrow H_{p+q}(\operatorname{Mod}^1_g; \mathbb{Q}).$$

We want to show that

$$E^2_{02} = H_0(\operatorname{Sp}_{2g}(\mathbb{Z}); H_2(\mathcal{I}^1_g; \mathbb{Q})) = H_2(\mathcal{I}^1_g; \mathbb{Q})_{\operatorname{Sp}_{2g}(\mathbb{Z})}$$

is finite-dimensional. Since the homology groups of Mod^1_g are all finite-dimensional, we know that E^∞_{02} is finite dimensional. To go from E^2_{02} to E^∞_{02}, we must kill the images of differentials coming from

$$E^2_{21} = H_2(\operatorname{Sp}_{2g}(\mathbb{Z}); H_1(\mathcal{I}^1_g; \mathbb{Q})) \quad \text{and}$$
$$E^3_{30} \subset E^2_{30} = H_3(\operatorname{Sp}_{2g}(\mathbb{Z}); \mathbb{Q}).$$

In light of Johnson's theorem [24] saying that $H_1(\mathcal{I}^1_g; \mathbb{Q})$ is a finite-dimensional algebraic representation of $\operatorname{Sp}_{2g}(\mathbb{Z})$ and Borel–Serre's theorem [5] saying that $\operatorname{Sp}_{2g}(\mathbb{Z})$ has a finite-index subgroup with a compact classifying space, these are both finite-dimensional. It follows that E^2_{02} is finite-dimensional, as desired. □

Part 3. Homology of Torelli, step 2: generators for cokernel

It remains to prove Theorem B$'$, which says that Λ_g is a finite-dimensional algebraic representation of $\operatorname{Sp}_{2(g-1)}(\mathbb{Z})$ for $g \geq 5$. In this part of the paper, we find generators for Λ_g. In §8 we introduce equivariant homology, in §9 we introduce the handle complex $\mathcal{C}_{ab}(\Sigma_g)$, and then in §10 we find our generators.

8. Step 2.1: Preliminaries on equivariant homology

We start with some preliminaries on (Borel) equivariant homology. See [7, §VII.7] for a textbook reference. Fix a group G and a commutative ring $\mathbf{k}$.

8.1. **Basic definitions.** A *G-CW complex* is a CW complex equipped with a cellular action of G. Fix a contractible G-CW complex EG on which G acts freely. For a G-CW complex X, the group G acts freely on $EG \times X$. Define

$$EG \times_G X = (EG \times X)/G.$$

This is known as the *Borel construction*. The *G-equivariant homology groups* of X, denoted $\mathrm{H}_\bullet^G(X; \mathbf{k})$, are $\mathrm{H}_\bullet(EG \times_G X; \mathbf{k})$. They do not depend on the choice of EG and are functorial under G-equivariant cellular maps. They are also functorial under group homomorphisms in the following sense: if $f \colon G_1 \to G_2$ is a group homomorphism and $\phi \colon X_1 \to X_2$ is a cellular map from a G_1-CW complex X_1 to a G_2-CW complex X_2 such that

$$\phi(g{\cdot}x) = f(g){\cdot}\phi(x) \quad \text{for all } g \in G_1 \text{ and } x \in X_1,$$

then there is an induced map $(f, \phi)_* \colon \mathrm{H}_\bullet^{G_1}(X_1; \mathbf{k}) \to \mathrm{H}_\bullet^{G_2}(X_2; \mathbf{k})$.

8.2. **Relation to group homology.** If X is a contractible G-CW complex, then $EG \times X$ is also contractible. Since G acts freely on $EG \times X$, we deduce that $EG \times_G X$ is a $K(G, 1)$, so by definition[30] $\mathrm{H}_\bullet^G(X; \mathbf{k}) = \mathrm{H}_\bullet(G; \mathbf{k})$. A basic example is $X = \mathrm{pt}$ equipped with the trivial G-action, so

$$\mathrm{H}_\bullet^G(\mathrm{pt}; \mathbf{k}) = \mathrm{H}_\bullet(G; \mathbf{k}).$$

For an arbitrary G-CW complex X, the map $X \to \mathrm{pt}$ induces a canonical map

(8.1) $$\mathrm{H}_\bullet^G(X; \mathbf{k}) \longrightarrow \mathrm{H}_\bullet^G(\mathrm{pt}; \mathbf{k}) = \mathrm{H}_\bullet(G; \mathbf{k}).$$

We then have the following:

Lemma 8.1. *Let G be a group and let X be an n-connected G-CW complex. For all commutative rings $\mathbf{k}$, the map $\mathrm{H}_d^G(X; \mathbf{k}) \to \mathrm{H}_d(G; \mathbf{k})$ from* (8.1) *is an isomorphism for $d \le n$ and a surjection for $d = n + 1$.*

Proof. We can build a contractible G-CW complex Y from X by equivariantly attaching cells of dimension at least $n + 2$. The CW complex $EG \times_G Y$ is then built from $EG \times_G X$ by attaching cells of dimension at least $n + 2$. Letting $\iota \colon X \to Y$ be the inclusion, the map we are studying factors as

$$\mathrm{H}_d^G(X; \mathbf{k}) \xrightarrow{\iota_*} \mathrm{H}_d^G(Y; \mathbf{k}) \longrightarrow \mathrm{H}_d(G; \mathbf{k}).$$

Since Y is contractible the canonical map $\mathrm{H}_d^G(Y; \mathbf{k}) \to \mathrm{H}_d(G; \mathbf{k})$ is an isomorphism for all d, and by construction ι_* is an isomorphism for $d \le n$ and a surjection for $d = n + 1$. The lemma follows. $\square$

8.3. **Spectral sequence.** Let X be a G-CW complex. Assume that G acts on X *without rotations*, i.e., for all cells σ of X the stabilizer G_σ fixes c pointwise. This ensures that X/G is a CW complex whose p-cells are in bijection with the G-orbits of p-cells of X.

For a p-cell $\sigma \in (X/G)^{(p)}$, let $\widetilde{\sigma}$ be a lift of σ to X. Consider $\mathrm{H}_q(G_{\widetilde{\sigma}}; \mathbf{k})$. This appears to depend on the choice of $\widetilde{\sigma}$. However, if $\widetilde{\sigma}'$ is another lift of σ to X then there exists some $g \in G$ with $g{\cdot}\widetilde{\sigma} = \widetilde{\sigma}'$ and hence $gG_{\widetilde{\sigma}}g^{-1} = G_{\widetilde{\sigma}'}$. Conjugation by g thus gives an isomorphism

(8.2) $$\mathrm{H}_q(G_{\widetilde{\sigma}}; \mathbf{k}) \cong \mathrm{H}_q(G_{\widetilde{\sigma}'}; \mathbf{k}).$$

---

[30]We use $=$ rather than $\cong$ to indicate that this isomorphism is canonical.

This isomorphism does not depend on the choice of g; indeed, any other choice of $g \in G$ with $g \cdot \widetilde{\sigma} = \widetilde{\sigma}'$ is of the form gh with $h \in G_{\widetilde{\sigma}}$, and conjugation by h induces the trivial automorphism of $H_q(G_{\widetilde{\sigma}}; \mathbf{k})$. Since the isomorphism (8.2) is canonical, we can unambiguously write $H_q(G_{\widetilde{\sigma}}; \mathbf{k})$ for $\sigma \in (X/G)^{(p)}$. With this convention, we have:

Proposition 8.2 ([7, VII.7.7]). *Let G be a group, let X be a G-CW complex, and let $\mathbf{k}$ be a commutative ring. Assume that G acts on X without rotations. There is then a functorial first quadrant spectral sequence*

$$E_{pq}^1 = \bigoplus_{\sigma \in (X/G)^{(p)}} H_q(G_{\widetilde{\sigma}}; \mathbf{k}) \Rightarrow H_{p+q}^G(X; \mathbf{k}).$$

In the rest of this section, we will fix a G-CW complex X on which G acts without rotations and discuss properties of the spectral sequence E given by Proposition 8.2.

8.4. **Left column.** Since 0-cells are vertices, we will denote them with v instead of σ. For a fixed q, consider the composition

$$(8.3) \qquad \bigoplus_{v \in (X/G)^{(0)}} H_q(G_{\widetilde{v}}; \mathbf{k}) = E_{0q}^1 \longrightarrow\!\!\!\!\!\rightarrow E_{0q}^\infty \hookrightarrow H_q^G(X; \mathbf{k}) \longrightarrow H_q(G; \mathbf{k})$$

whose maps are as follows:
- the surjection $E_{0q}^1 \twoheadrightarrow E_{0q}^\infty$ comes from the fact that there are no nonzero differentials coming out of E_{0q}^1; and
- the inclusion $E_{0q}^\infty \hookrightarrow H_q^G(X; \mathbf{k})$ comes from the fact that E_{0q}^∞ is the first term in the filtration of $H_q^G(X; \mathbf{k})$ coming from our spectral sequence; and
- the map $H_q^G(X; \mathbf{k}) \to H_q(G; \mathbf{k})$ is the canonical map.

We claim that (8.3) equals the sum of the maps induced by the inclusions $G_{\widetilde{v}} \hookrightarrow G$ of vertex stabilizers. This can be proved easily from the construction of the spectral sequence in [7, VII.7.7], but we prefer the following less computational proof.

Consider the G-equivariant map $X \to \text{pt}$. Letting F be the spectral sequence obtained by applying Proposition 8.2 to pt, we get a map $E \to F$ of spectral sequences converging to the canonical map

$$H_\bullet^G(X; \mathbf{k}) \longrightarrow H_\bullet^G(\text{pt}; \mathbf{k}) = H_\bullet(G; \mathbf{k}).$$

The spectral sequence F degenerates at F^1, which is of the form

$$F_{pq}^1 = \begin{cases} H_q(G; \mathbf{k}) & \text{if } p = 0, \\ 0 & \text{if } p \neq 0. \end{cases}$$

Identifying E_{0q}^1 with

$$\bigoplus_{v \in (X/G)^{(0)}} H_q(G_{\widetilde{v}}; \mathbf{k}),$$

the map $E_{0q}^1 \to F_{0q}^1 = H_q(G; \mathbf{k})$ is exactly the sum of the maps induced by the inclusions $G_{\widetilde{v}} \hookrightarrow G$ of vertex stabilizers. The claim follows.

8.5. **Bottom row.** We will next need a description of the differentials of our spectral sequence in two special case. The first is when $q = 0$. Observe that

$$E_{p0}^1 = \bigoplus_{\sigma \in (X/G)^{(p)}} H_0(G_{\widetilde{\sigma}}; \mathbf{k}) = \bigoplus_{\sigma \in (X/G)^{(p)}} \mathbf{k} = C_p(X/G; \mathbf{k}),$$

where $C_p(X/G; \mathbf{k})$ is the pth term of the cellular chain complex for X/G. The E^1-differentials $E^1_{p0} \to E^1_{p-1,0}$ thus fit into a chain complex of the form

$$C_0(X/G; \mathbf{k}) \longleftarrow C_1(X/G; \mathbf{k}) \longleftarrow C_2(X/G; \mathbf{k}) \longleftarrow \cdots.$$

This is exactly the cellular chain complex of X/G; see [7, §VII.8]. It follows that $E^2_{p0} = H_p(X/G; \mathbf{k})$.

8.6. 1-cell differentials.
We also need a description of the differentials when $p = 1$. By our description of the E^1-page, these differentials $\partial \colon E^1_{1q} \to E^1_{0q}$ are of the form

$$\bigoplus_{e \in (X/G)^{(1)}} H_q(G_{\widetilde{e}}; \mathbf{k}) \xrightarrow{\ \partial\ } \bigoplus_{v \in (X/G)^{(0)}} H_q(G_{\widetilde{v}}; \mathbf{k}).$$

Consider some $e \in (X/G)^{(1)}$. For our differential, we must fix some (arbitrary) orientation on e. Let $\widetilde{e} \in X^{(1)}$ be our lift to X, which has an orientation coming from the orientation on e. Let $w_0 \in X^{(0)}$ and $w_1 \in X^{(0)}$ be the initial and terminal vertices of $\widetilde{e}$, respectively. We have inclusions $G_{\widetilde{e}} \hookrightarrow G_{w_0}$ and $G_{\widetilde{e}} \hookrightarrow G_{w_1}$. On the summand $H_q(G_{\widetilde{e}}; \mathbf{k})$ of E^1_{1q}, the differential ∂ is then the difference between the two induced maps

$$(8.4) \qquad H_q(G_{\widetilde{e}}; \mathbf{k}) \longrightarrow H_q(G_{w_1}; \mathbf{k}) \hookrightarrow \bigoplus_{v \in (X/G)^{(0)}} H_q(G_{\widetilde{v}}; \mathbf{k})$$

and

$$(8.5) \qquad H_q(G_{\widetilde{e}}; \mathbf{k}) \longrightarrow H_q(G_{w_0}; \mathbf{k}) \hookrightarrow \bigoplus_{v \in (X/G)^{(0)}} H_q(G_{\widetilde{v}}; \mathbf{k}).$$

See [7, §VII.8] for a proof.

If e is a loop, then w_0 and w_1 are in the same G-orbit, so there exists some $s \in G$ with $s(w_0) = w_1$. The terms $H_q(G_{w_1}; \mathbf{k})$ and $H_q(G_{w_0}; \mathbf{k})$ in (8.4) and (8.5) go to the same term in the indicated direct sum, and $H_q(G_{w_1}; \mathbb{Q})$ is identified with $H_q(G_{w_0}; \mathbf{k})$ via conjugation by s.

Consider the special case $q = 1$. For $g \in G_{w_0}$, let $\overline{g} \in H_1(G_{w_0}; \mathbf{k})$ be its homology class. Our differential is the composition

$$H_1(G_{\widetilde{e}}; \mathbf{k}) \longrightarrow H_1(G_{w_0}; \mathbf{k}) \hookrightarrow \bigoplus_{v \in (X/G)^{(0)}} H_1(G_{\widetilde{v}}; \mathbf{k}),$$

where the first map take the homology class of $g \in G_{\widetilde{e}}$ to

$$\overline{s^{-1}gs} - \overline{g} = \overline{g^{-1}s^{-1}gs} = \overline{[g,s]} \in H_1(G_{w_0}; \mathbf{k}).$$

9. STEP 2.2: THE HANDLE COMPLEX

We now introduce a space on which $\mathcal{I}_g$ acts.

9.1. Complex of homologous curves.
Let $v \in H_1(\Sigma_g)$ be a primitive element, i.e., one that is only divisible by ± 1. Let $\mathcal{C}_v(\Sigma_g)$ be the following simplicial complex:

- **vertices**: isotopy classes of oriented simple closed curves γ on Σ_g with $[\gamma] = v$.
- **p-simplices**: sets $\sigma = \{\gamma_0, \ldots, \gamma_p\}$ of distinct vertices such that the γ_i can be isotoped to be pairwise disjoint.

For instance, the following is a 2-simplex of $\mathcal{C}_v(\Sigma_g)$ for $v = [\gamma_0]$:[31]
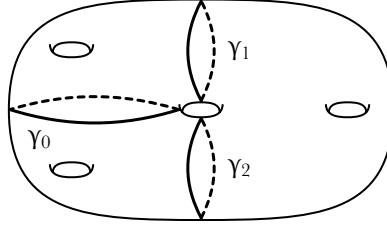

Putman [41] introduced $\mathcal{C}_v(\Sigma_g)$ and proved it was connected for $g \geq 3$. Hatcher–Margalit [18] gave an alternate proof of this that Minahan [34] generalized to show:[32]

Theorem 9.1 (Minahan, [34]). *Let $g \geq 2$ and let $v \in \mathrm{H}_1(\Sigma_g)$ be a primitive element. Then $\mathcal{C}_v(\Sigma_g)$ is $(g-3)$-acyclic, i.e., $\widetilde{\mathrm{H}}_k(\mathcal{C}_v(\Sigma_g)) = 0$ for $k \leq g-3$.*

9.2. **Handle complex.** Let $a, b \in \mathrm{H}_1(\Sigma_g)$ be primitive elements with algebraic intersection number 1. An *a-curve* (resp. a *b-curve*) is an oriented simple closed curve γ with $[\gamma] = a$ (resp. $[\gamma] = b$). The *handle complex*, denoted $\mathcal{C}_{ab}(\Sigma_g)$, is the following simplicial complex:

- **vertices**: isotopy classes of a-curves and b-curves.
- **p-simplices**: sets $\sigma = \{\gamma_0, \ldots, \gamma_p\}$ of distinct vertices such that either:
  - σ is a p-simplex of $\mathcal{C}_a(\Sigma_g)$ or $\mathcal{C}_b(\Sigma_g)$; or
  - for some $\gamma_{i_0} \in \sigma$, the set $\sigma \setminus \{\gamma_{i_0}\}$ is a $(p-1)$-simplex of $\mathcal{C}_a(\Sigma_g)$ and γ_{i_0} is a b-curve that can be isotoped to intersect each curve in $\sigma \setminus \{\gamma_{i_0}\}$ once; or
  - for some $\gamma_{i_0} \in \sigma$, the set $\sigma \setminus \{\gamma_{i_0}\}$ is a $(p-1)$-simplex of $\mathcal{C}_b(\Sigma_g)$ and γ_{i_0} is an a-curve that can be isotoped to intersect each curve in $\sigma \setminus \{\gamma_{i_0}\}$ once.

We will call the simplices of $\mathcal{C}_a(\Sigma_g)$ and $\mathcal{C}_b(\Sigma_g)$ the *pure simplices* of $\mathcal{C}_{ab}(\Sigma_g)$ and the other simplices the *mixed simplices.* If in the following figure the orange curves are a-curves and the blue curves are b-curves,[33] then the indicated curves form mixed simplices of $\mathcal{C}_{ab}(\Sigma_g)$:

 and
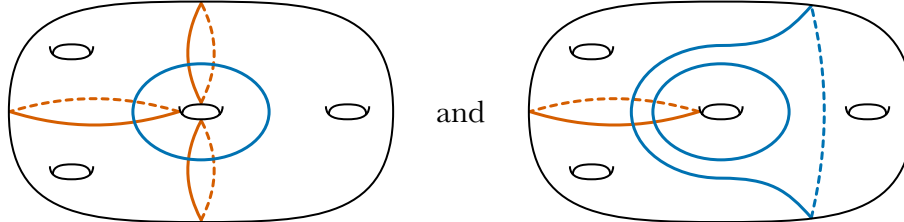
The 1-skeleton of $\mathcal{C}_{ab}(\Sigma_g)$ is the *handle graph* defined by Putman [45], who proved it is connected for $g \geq 3$. Proposition 9.9 below says that $\mathcal{C}_{ab}(\Sigma_g)$ is 1-acyclic[34] for $g \geq 4$.

9.3. **Rotations.** The group $\mathcal{I}_g$ acts on $\mathcal{C}_{ab}(\Sigma_g)$. This action is without rotations:

Lemma 9.2. *Let $g \geq 1$ and let $a, b \in \mathrm{H}_1(\Sigma_g)$ be primitive elements with algebraic intersection number 1. Then $\mathcal{I}_g$ acts on $\mathcal{C}_{ab}(\Sigma_g)$ without rotations.*

Proof. This is immediate from the fact that every simplex of $\mathcal{C}_{ab}(\Sigma_g)$ is the join of a simplex of $\mathcal{C}_a(\Sigma_g)$ and a simplex of $\mathcal{C}_b(\Sigma_g)$ along with the fact that $\mathcal{I}_g$ acts on $\mathcal{C}_a(\Sigma_g)$ and $\mathcal{C}_b(\Sigma_g)$ without rotations (see [19, Theorem 1.2] for a more general result). $\square$

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[31]To avoid cluttering our figures, they will often not indicate the orientations on the curves.

[32]It is not known if $\pi_1(\mathcal{C}_v(\Sigma_g)) = 1$ for $g \geq 4$, which would let us conclude it is $(g-3)$-connected.

[33]We will use this coloring convention in the rest of the paper.

[34]Presumably it could be made more highly acyclic by allowing mixed simplices that contain more a- and b-curves. We defined $\mathcal{C}_{ab}(\Sigma_g)$ like we did to ensure that $\mathcal{C}_{ab}(\Sigma_g)/\mathcal{I}_g$ is contractible; see Proposition 9.7.

9.4. **Mixing pure simplices.** The following shows that every pure simplex of $\mathcal{C}_{ab}(\Sigma_g)$ can be extended to a mixed simplex, and this extension is unique up to the action of $\mathcal{I}_g$:

Lemma 9.3. *Let $g \geq 1$ and let $a, b \in \mathrm{H}_1(\Sigma_g)$ be primitive elements with algebraic intersection number 1. Let σ be a pure simplex of $\mathcal{C}_{ab}(\Sigma_g)$. Then there is a vertex δ of $\mathcal{C}_{ab}(\Sigma_g)$ such that $\sigma \cup \{\delta\}$ is a mixed simplex. Moreover, if δ' is another vertex of $\mathcal{C}_{ab}(\Sigma_g)$ such that $\sigma \cup \{\delta'\}$ is a mixed simplex, then there exists $f \in \mathcal{I}_g$ with $f(\sigma) = \sigma$ and $f(\delta) = \delta'$.*

Proof. We will give the proof for σ a simplex of $\mathcal{C}_a(\Sigma_g)$. The case where it is a simplex of $\mathcal{C}_b(\Sigma)$ is identical. Let $\sigma = \{\gamma_0, \ldots, \gamma_p\}$, so the γ_i are disjoint a-curves. They divide Σ_g into $(p+1)$ components $T_0, \ldots, T_p$ with $T_i \cong \Sigma_{g_i}^2$ for some $g_i \geq 1$ with $1 + \sum g_i = g$:

As in this figure, we order the γ_i and T_i such that $\partial T_i = \gamma_i \sqcup \gamma_{i+1}$, where the indices are taken modulo p. We now divide the proof into two steps.

Step 1. *There exists a b-curve δ that intersects each γ_i once, so $\sigma \cup \{\delta\}$ is a mixed simplex.*

Let ζ be an arbitrary oriented simple closed curve that intersects each γ_i once such that the intersection number of γ_i with ζ is $+1$. Let $S_0, \ldots, S_p$ be the components of the complement of a regular neighborhood of $\zeta \cup \gamma_0 \cup \cdots \cup \gamma_p$, ordered such that $S_i \subset T_i$:
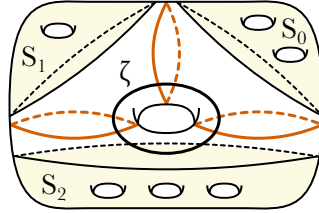
We then have

$$\mathrm{H}_1(\Sigma_g) = \langle [\gamma_0], [\zeta] \rangle \oplus \bigoplus_{i=0}^{p} \mathrm{H}_1(S_i).$$

Write

$$b = c[\gamma_0] + d[\zeta] + \sum_{i=0}^{p} x_i \quad \text{with } c, d \in \mathbb{Z} \text{ and } x_i \in \mathrm{H}_1(S_i).$$

Since the algebraic intersection numbers of b and $[\zeta]$ with $a = [\gamma_0]$ are 1, we have $d = 1$. Replacing ζ with $T_{\gamma_0}^{-c}(\zeta)$, we can also assume that $c = 0$, so $b = [\zeta] + \sum_{i=0}^{p} x_i$. For $0 \leq i \leq p$, let q_i be the intersection point of ζ with γ_i and let ζ_i be the subarc of ζ lying in T_i, so ζ_i goes from q_i to q_{i+1}. By [45, Lemma 3.2], there exist a properly embedded arc δ_i in T_i going from q_i to q_{i+1} such that in the relative homology group $\mathrm{H}_1(T_i, \{q_i, q_{i+1}\})$, we have $[\delta_i] = [\zeta_i] + x_i$. We can then take δ to be the loop made up of the δ_i.

Step 2. *Let δ and δ' be b-curves such that $\sigma \cup \{\delta\}$ and $\sigma \cup \{\delta'\}$ are mixed simplices. Then there exists $f \in \mathcal{I}_g$ with $f(\sigma) = \sigma$ and $f(\delta) = \delta'$.*

Let $S_0, \ldots, S_p$ be the components of the complement of a regular neighborhood of $\delta \cup \gamma_0 \cup \cdots \cup \gamma_p$, ordered such that $S_i \subset T_i$. The span of $\mathrm{H}_1(S_i)$ and $a = [\gamma_i]$ equals $\mathrm{H}_1(T_i)$, so $\mathrm{H}_1(S_i)$ is the intersection of $\mathrm{H}_1(T_i)$ with the orthogonal complement of $b = [\delta]$.

Similarly, let $S'_1, \ldots, S'_p$ be the components of the complement of a regular neighborhood of $\delta' \cup \gamma_0 \cup \cdots \cup \gamma_p$, ordered such that $S'_i \subset T_i$. Just like above, $H_1(S'_i)$ is the intersection of $H_1(T_i)$ with the orthogonal complement of $b = [\delta']$. In other words, as subgroups of $H_1(\Sigma_g)$ we have $H_1(S_i) = H_1(S'_i)$ for $0 \leq i \leq p$. Let $V_i = H_1(S_i) = H_1(S'_i)$.

By the change of coordinates principle from [10], we can find $\phi \in \text{Mod}_g$ with $\phi(\delta) = \delta'$ and $\phi(\gamma_i) = \gamma_i$ for $0 \leq i \leq p$. By construction ϕ fixes $a = [\gamma_0]$ and $b = [\delta]$, and it takes $V_i = H_1(S_i)$ to $V_i = H_1(S'_i)$ for $0 \leq i \leq p$. Since mapping classes on the 1-holed surface S_i can realize any symplectic automorphism of $V_i = H_1(S_i)$, we can find some $\psi_i \in \text{Mod}_g$ supported on S_i such that ψ_i induces $\phi_*|_{V_i} \colon V_i \to V_i$ on $V_i = H_1(S_i)$. Define

$$f = \phi \psi_0^{-1} \cdots \psi_p^{-1} \in \text{Mod}_g.$$

We have $f(\sigma) = \sigma$ and $f(\delta) = \delta'$. By construction f fixes $a = [\gamma_0]$ and $b = [\delta]$ as well as each V_i. Since these span $H_1(\Sigma_g)$, we conclude that f acts trivially on $H_1(\Sigma_g)$, i.e., $f \in \mathcal{I}_g$. □

9.5. **Description of action.** We now prove several results about the action of $\mathcal{I}_g$ on $\mathcal{C}_{ab}(\Sigma)$.

Lemma 9.4. *Let $g \geq 1$ and let $a, b \in H_1(\Sigma_g)$ be primitive elements with algebraic intersection number 1. Then $\mathcal{I}_g$ acts transitively on:*

(i) *vertices of $\mathcal{C}_{ab}(\Sigma_g)$ that are a-curves; and*
(ii) *vertices of $\mathcal{C}_{ab}(\Sigma_g)$ that are b-curves; and*
(iii) *mixed 1-simplices of $\mathcal{C}_{ab}(\Sigma_g)$.*

Proof. Johnson [22, Lemma 5] proved that for any oriented nonseparating simple closed curves γ and γ' on Σ_g with $[\gamma] = [\gamma']$, there exists $f \in \mathcal{I}_g$ with $f(\gamma) = \gamma'$. This implies that $\mathcal{I}_g$ acts transitively on a-curves and b-curves, as in (i) and (ii).

To prove (iii), for $i = 1, 2$ let e_i be a mixed 1-simplex joining an a-curve α_i to a b-curve β_i. By the previous paragraph, there exists $f \in \mathcal{I}_g$ with $f(\alpha_1) = \alpha_2$. Both $f(\beta_1)$ and β_2 are b-curves intersecting $f(\alpha_1) = \alpha_2$ once, so by Lemma 9.3 there exists $f' \in \mathcal{I}_g$ with $f'(f(\alpha_1)) = f(\alpha_1) = \alpha_2$ and $f'(f(\beta_1)) = \beta_2$. It follows that $f'f$ takes e_1 to e_2, as desired. □

This immediately implies:

Corollary 9.5. *Let $g \geq 1$ and let $a, b \in H_1(\Sigma_g)$ be primitive elements with algebraic intersection number 1. Then the 1-skeleton of $\mathcal{C}_{ab}(\Sigma_g)/\mathcal{I}_g$ consists of:*

- *two vertices v_a and v_b, with v_a (resp. v_b) the image of any a-curve (resp. b-curve).*
- *loops based at v_a and v_b, each the image of a pure 1-simplex.*
- *a single 1-simplex e_{ab} joining v_a and v_b, with e_{ab} the image of any mixed 1-simplex.*

The following puts edges of $\mathcal{C}_{ab}(\Sigma_g)$ into a normal form up to the action of $\mathcal{I}_g$:

Lemma 9.6. *Let $g \geq 1$, let $a, b \in H_1(\Sigma_g)$ be primitive elements with algebraic intersection number 1, and let $\{\alpha, \beta\}$ be a mixed 1-simplex of $\mathcal{C}_{ab}(\Sigma_g)$ with $[\alpha] = a$ and $[\beta] = b$. Then:*

(i) *for all oriented 1-simplices e of $\mathcal{C}_a(\Sigma_g)$, there exists some $f, B \in \mathcal{I}_g$ with $B(\beta) = \beta$ such that $f(e)$ goes from α to $B(\alpha)$; and*
(ii) *for all oriented 1-simplices e of $\mathcal{C}_b(\Sigma_g)$, there exists some $f, A \in \mathcal{I}_g$ with $A(\alpha) = \alpha$ such that $f(e)$ goes from β to $A(\beta)$.*
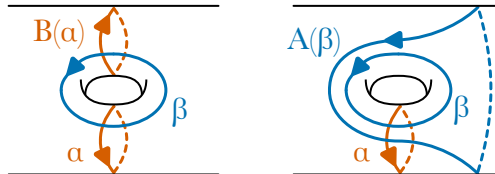
Proof. The proofs of (i) and (ii) are similar, so we prove (i) and leave (ii) to the reader. Write $e = \{\alpha_1, \alpha_2\}$. Our goal is to find $f, B \in \mathcal{I}_g$ such that $B(\beta) = \beta$ and $f(e) = \{f(\alpha_1), f(\alpha_2)\} = \{\alpha, B(\alpha)\}$.

By Lemma 9.4, there is an $f_1 \in \mathcal{I}_g$ with $f_1(\alpha_1) = \alpha$, so $f_1(e) = \{\alpha, f_1(\alpha_2)\}$. By Lemma 9.3, there is a b-curve β' with $\{\alpha, f_1(\alpha_2), \beta'\}$ a mixed simplex. Since $\{\alpha, \beta\}$ and $\{\alpha, \beta'\}$ are mixed 1-simplices, by Lemma 9.4 there is an $f_2 \in \mathcal{I}_g$ with $f_2(\alpha) = \alpha$ and $f_2(\beta') = \beta$. Let $f = f_2 f_1$, so $f(e) = \{\alpha, f(\alpha_2)\}$ and $\{\alpha, f(\alpha_2), \beta\}$ is a mixed simplex. Since $\{\alpha, \beta\}$ and $\{f(\alpha_2), \beta\}$ are mixed 1-simplices, by Lemma 9.3 there is a $B \in \mathcal{I}_g$ with $B(\alpha) = f(\alpha_2)$ and $B(\beta) = \beta$, so $f(e) = \{\alpha, B(\alpha)\}$, as desired. $\square$

9.6. Contractability of quotient.
Below we will prove that $\mathcal{C}_{ab}(\Sigma_g)$ is 1-acyclic. First, however, we note that our results quickly imply that $\mathcal{C}_{ab}(\Sigma_g)/\mathcal{I}_g$ is contractible:

Proposition 9.7. *Let $g \geq 1$ and let $a, b \in \mathrm{H}_1(\Sigma_g)$ be primitive elements with algebraic intersection number 1. Then $\mathcal{C}_{ab}(\Sigma_g)/\mathcal{I}_g$ is contractible.*

Proof. Let $X = \mathcal{C}_{ab}(\Sigma_g)/\mathcal{I}_g$. Let A and B be the images in X of $\mathcal{C}_a(\Sigma_g)$ and $\mathcal{C}_b(\Sigma_g)$, respectively. Let v_a and v_b be the vertices of $\mathcal{C}_{ab}(\Sigma_g)/\mathcal{I}_g$ discussed in Corollary 9.5, so $v_a \in A$ and $v_b \in B$. Let e_{ab} be the 1-simplex of X connecting v_a to v_b from Corollary 9.5. Lemma 9.3 implies that for every cell σ of A, there is a unique cell of X obtained by coning σ off with v_b. It follows that X contains the cone $\mathrm{Cone}_{v_b}(A)$ of A with cone point v_b. Similarly, X also contains the cone $\mathrm{Cone}_{v_a}(B)$ of B with cone point v_a. Thus

$$X = \mathrm{Cone}_{v_a}(B) \cup \mathrm{Cone}_{v_b}(A) \quad \text{and} \quad \mathrm{Cone}_{v_a}(B) \cap \mathrm{Cone}_{v_b}(A) = e_{ab}.$$

Since $\mathrm{Cone}_{v_a}(B)$ and $\mathrm{Cone}_{v_b}(A)$ and e_{ab} are contractible, X is also contractible. $\square$

9.7. Strips.
To understand the topology of $\mathcal{C}_{ab}(\Sigma_g)$ itself, we need some preliminaries. For mixed 1-simplices e_0 and e_1 of $\mathcal{C}_{ab}(\Sigma_g)$, an *ab-strip* connecting e_0 to e_1 is a triangulation S of $[0,1]^2$ equipped with a simplicial map $f : S \to \mathcal{C}_{ab}(\Sigma_g)$ such that:

- for $i = 0, 1$, the subspace $i \times [0,1]$ of S is a 1-simplex mapping to e_i; and
- f maps each vertex in $[0,1] \times 0$ to an a-curve and each vertex in $[0,1] \times 1$ to a b-curve.
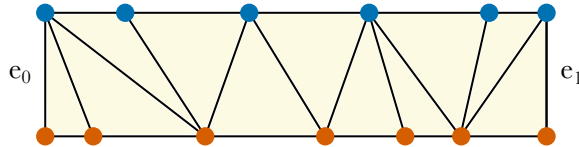
Every two mixed 1-simplices can be connected by an *ab*-strip:

Lemma 9.8. *Let $g \geq 3$ and let $a, b \in \mathrm{H}_1(\Sigma_g)$ be primitive elements with algebraic intersection number 1. Let e_0 and e_1 be mixed 1-simplices of $\mathcal{C}_{ab}(\Sigma_g)$. Then there exists an ab-strip $f : S \to \mathcal{C}_{ab}(\Sigma_g)$ connecting e_0 to e_1.*

Proof. [35] For mixed 1-simplices e_0' and e_1' of $\mathcal{C}_{ab}(\Sigma_g)$, write $e_0' \sim e_1'$ if there exists an *ab*-strip $f : S \to \mathcal{C}_{ab}(\Sigma_g)$ connecting e_0' and e_1'. This is an equivalence relation, and our goal is to prove that $e_0 \sim e_1$. Using the change of coordinates principle from [10], we can assume that e_0 is the edge connecting the curves α and β in the following:
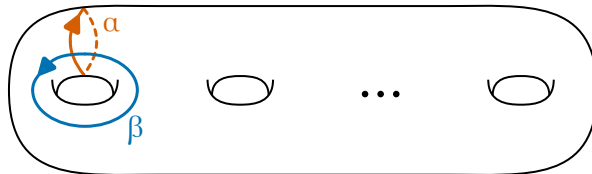
Lemma 9.4 says that the group $\mathcal{I}_g$ acts transitively on the set of mixed 1-simplices of $\mathcal{C}_{ab}(\Sigma_g)$, so we can find $f \in \mathcal{I}_g$ with $f(e_0) = e_1$.

Let $\Delta \subset \mathcal{I}_g$ be the finite generating set for $\mathcal{I}_g$ constructed by Johnson [23]. Using the notation from [23], this generating set is defined using a set of curves[36] $\{c_1, \ldots, c_{2g}, c_\beta\}$, and matching up our figures these satisfy $c_1 = \alpha$ and $c_2 = \beta$. All the other curves in $\{c_1, \ldots, c_{2g}, c_\beta\}$ are disjoint from α and β. We will say more about Δ during the proof of the claim below.

Below we will prove that for $s \in \Delta^{\pm 1}$ we have $e_0 \sim s(e_0)$. This implies the lemma via the trick from the second author's paper [41]. Here are more details: writing $f = s_1 \cdots s_k$ with $s_i \in \Delta^{\pm 1}$, the fact that $e_0 \sim s_i(e_0)$ for all $1 \leq i \leq k$ implies that

$$e_0 \sim s_1(e_0) \sim s_1 s_2(e_0) \sim s_1 s_2 s_3(e_0) \sim \cdots \sim s_1 \cdots s_k(e_0) = e_1.$$

Here we are using the fact that $\sim$ is invariant under $\mathcal{I}_g$, and thus for instance if $e_0 \sim s_2(e_0)$ then $s_1(e_0) \sim s_1 s_2(e_0)$. It remains to prove:

**Claim.** *For $s \in \Delta^{\pm 1}$, we have $e_0 \sim s(e_0)$.*

The generators for $\mathcal{I}_g$ from [23] are all bounding pair maps, i.e., products $T_x T_y^{-1}$ with x and y disjoint simple closed curves on Σ_g such that $x \cup y$ separates Σ_g. Since the inverse of a bounding pair map is a bounding pair map, we can therefore write $s = T_x T_y^{-1}$ with x and y as above. If $x \cup y$ is disjoint from $\alpha \cup \beta$, then $s(e_0) = e_0$ and there is nothing to prove. We can therefore assume that $x \cup y$ intersects $\alpha \cup \beta$.

Let T be a regular neighborhood of $\alpha \cup \beta$, so T is a torus with one boundary component. Since α and β are the curves pictured above and $x \cup y$ intersects $\alpha \cup \beta$, it is immediate from the construction in [23] that:

- both x and y intersect T in single arcs x_1 and y_1; and
- both x_1 and y_1 intersect α at most once and β at most once.

Since x is homologous to y, it follows[37] that x_1 is homologous to y_1 in $H_1(T, \partial T)$. This implies that x_1 and y_1 are parallel properly embedded arcs in T. There thus exists an embedded $U \hookrightarrow T$ with $U \cong [0,1]^2$ such that ∂U consists of $x_1 \cup y_1$ along with two subarcs of ∂T.

There are now two possibilities. The first is that x and y intersect one of α or β and are disjoint from the other. For concreteness, assume that they intersect α. We can then find a small ball around these intersections that looks like the following figure:
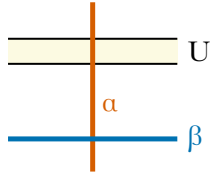
Depending on whether x is the top or bottom arc of the depicted portion of U, the loop $s(\alpha) = T_x T_y^{-1}(\alpha)$ looks like the following figure:
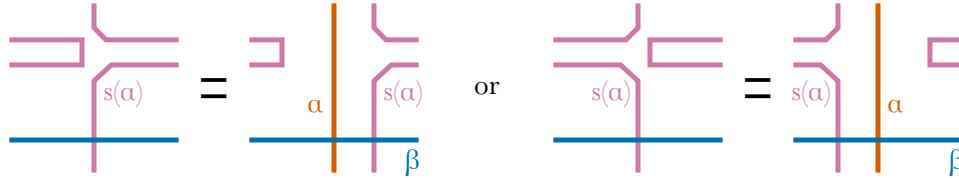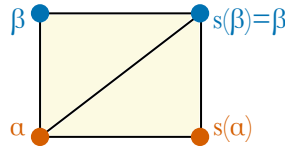

In either case, $s(\alpha)$ is disjoint from α and intersects β once. Using this, we can then use the

---

[36]Be warned that c_β has nothing to do with our curve β.
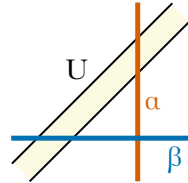
[37]This uses the fact that ∂T is a separating curve, and in particular is null-homologous.
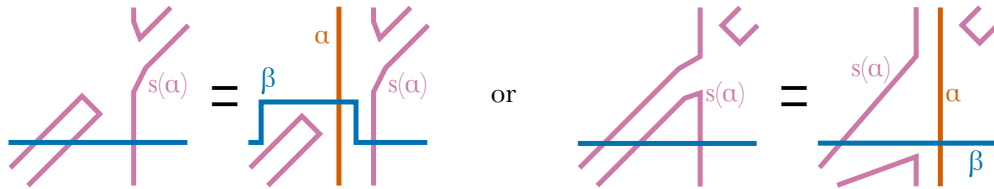
ab-strip

to show that $e_0 \sim s(e_0)$.

The second possibility is that x and y intersect both α and β. We can then find a small open ball in T that looks like the following figure:[38]

Depending on whether x is the top or bottom arc of the depicted portion of U, the loop $s(\alpha) = T_x T_y^{-1}(\alpha)$ looks like the following figure:

In either case, $s(\alpha)$ is disjoint from α and intersects β once. Similarly, $s(\beta)$ is disjoint from β, and since α and β intersect once, $s(\alpha)$ intersects $s(\beta)$ once. The ab-strip
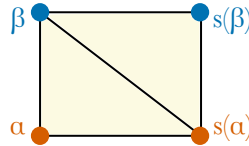
then witnesses the fact that $e_0 \sim s(e_0)$. $\qquad\square$

9.8. **Connectivity.** We can now prove:

Proposition 9.9. *Let $g \geq 4$ and let $a, b \in \mathrm{H}_1(\Sigma_g)$ be primitive elements with algebraic intersection number 1. Then $\mathcal{C}_{ab}(\Sigma_g)$ is 1-acyclic.*

Proof. Let γ be a loop in $\mathcal{C}_{ab}(\Sigma_g)$. We must prove that γ is homologous to a constant loop. Homotoping γ, we can assume that it is a simplicial loop in the 1-skeleton. Theorem 9.1 says that $\mathcal{C}_a(\Sigma_g)$ is 1-acyclic, so it is enough to prove that γ is homologous to a loop lying in $\mathcal{C}_a(\Sigma_g)$, i.e., to a loop taking all vertices to a-curves.

Assume that some vertices of γ are mapped to b-curves. If two adjacent vertices are mapped to b-curves β and β', then by Lemma 9.3 we can find an a-curve α that intersects β and β' once. This can be used to homotope γ to make β and β' the images of non-adjacent vertices:
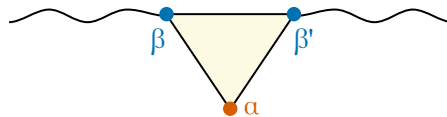
Repeating this, we can can ensure that no adjacent vertices of γ are mapped to b-curves.

---

[38]This picture might not preserve orientations, but this does not matter for our argument.

To complete the proof, we must show how to eliminate vertices mapping to b-curves. Consider such a vertex, and let e_0 and e_1 be the images of the edges on either side of it. Both e_0 and e_1 are mixed 1-simplices, so by Lemma 9.8 there exists an ab-strip connecting e_0 to e_1. Attach this strip to γ as follows:
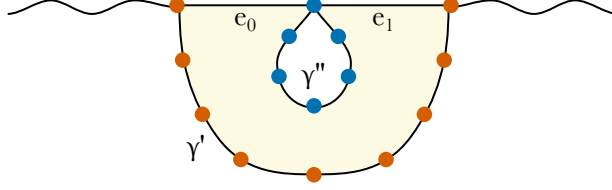
As is shown in this figure, the result is that γ is homologous to a sum of two loops: a loop γ' with one fewer vertex mapping to a b-curve, and another loop γ'' mapping entirely to $\mathcal{C}_b(\Sigma_g)$. Another application of Theorem 9.1 shows that γ'' is null-homologous, so we deduce that γ is homologous to γ', as desired.   □

10. Step 2.3: generators for cokernel

We first recall some notation. Let α and β be these curves on Σ_g:
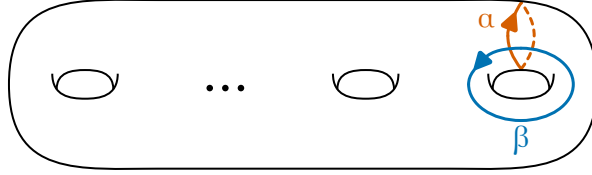
Let $\lambda\colon \mathrm{H}_2((\mathcal{I}_g)_\alpha; \mathbb{Q}) \oplus \mathrm{H}_2((\mathcal{I}_g)_\beta; \mathbb{Q}) \to \mathrm{H}_2(\mathcal{I}_g; \mathbb{Q})$ be the sum of the maps induced by the inclusions $(\mathcal{I}_g)_\alpha \hookrightarrow \mathcal{I}_g$ and $(\mathcal{I}_g)_\beta \hookrightarrow \mathcal{I}_g$ and let $\Lambda_g = \mathrm{coker}(\lambda)$. Recall that our goal is to prove Theorem B′, which says that Λ_g is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$ for $g \geq 5$. This section constructs generators for Λ_g.

10.1. **Commutator and Dehn twist conventions.** For a group G and $x, y \in G$, our conventions are $[x, y] = x^{-1}y^{-1}xy$ and $x^y = y^{-1}xy$. For a simple closed curve η on a surface, T_η denotes the right Dehn twist about η.

10.2. **Surface relations.** To construct generators for Λ_g, we need a formalism for describing elements of Λ_g. Let G be a group. A *surface relation* in G is a relation of the form

$$[x_1, y_1] \cdots [x_k, y_k] = 1 \quad \text{with } x_1, y_1, \ldots, x_k, y_k \in G.$$

Write this $r = [x_1, y_1] \cdots [x_k, y_k]$. We emphasize that r is a formal product of commutators, not an element of G. Let $\phi_r\colon \pi_1(\Sigma_k) \to G$ be the map taking the standard generators of $\pi_1(\Sigma_k)$ to the x_i and y_i. Define $\mathfrak{h}(r) \in \mathrm{H}_2(G; \mathbb{Q})$ to be the image under $(\phi_r)_*\colon \mathrm{H}_2(\pi_1(\Sigma_k); \mathbb{Q}) \to \mathrm{H}_2(G; \mathbb{Q})$ of the fundamental class of $\mathrm{H}_2(\pi_1(\Sigma_k); \mathbb{Z}) = \mathbb{Z}$.

Writing $G = F/R$ for a free group F and $R \lhd F$, Hopf's formula says that

(10.1) $$\mathrm{H}_2(G; \mathbb{Z}) = \frac{[F, F] \cap R}{[R, F]}.$$

Each element in the numerator of this gives a surface relation r in G, and $\mathfrak{h}(r) \in \mathrm{H}_2(G; \mathbb{Q})$ is the associated element of homology. See [48] for a discussion of Hopf's formula in these terms. From (10.1), we see that the $\mathfrak{h}(r)$ satisfy several basic identities. These will be used repeatedly throughout the remainder of the paper without citation.

First, consider surface relations $r = [x_1, y_1] \cdots [x_k, y_k]$ and $r' = [x'_1, y'_1] \cdots [x'_{k'}, y'_{k'}]$. Define

$$rr' = [x_1, y_1] \cdots [x_k, y_k][x'_1, y'_1] \cdots [x'_{k'}, y'_{k'}].$$

We then have $\mathfrak{h}(rr') = \mathfrak{h}(r) + \mathfrak{h}(r')$.

Since surface relations are formal product of commutators, it does not make literal sense to include terms in them like $[x, y]^{-1}$. In the context of surface relations, we therefore let $[x, y]^{-1}$ denote the commutator[39] $[y, x]$. If r is a surface relation and r' is the surface relation obtained from r by deleting an adjacent pair of inverse commutators $[x, y][x, y]^{-1}$, then $\mathfrak{h}(r) = \mathfrak{h}(r')$. Consequently, if $r = [x_1, y_1] \cdots [x_k, y_k]$ is a surface relation and we define $r^{-1} = [x_k, y_k]^{-1} \cdots [x_1, y_1]^{-1}$, then $\mathfrak{h}(r^{-1}) = -\mathfrak{h}(r)$.

For $x, y, z \in G$, we let $[x, y]^z$ denote the commutator $[x^z, y^z]$. If r is a surface relation containing two adjacent terms $[x_i, y_i][x_{i+1}, y_{i+1}]$ and r' is the surface relation obtained by replacing these with $[x_{i+1}, y_{i+1}][x_i, y_i]^{[x_{i+1}, y_{i+1}]}$, then $\mathfrak{h}(r) = \mathfrak{h}(r')$. Also, if r is a surface relation containing a commutator $[xz, y]$ and r' is obtained by expanding this to $[x, y]^z[z, y]$, then $\mathfrak{h}(r) = \mathfrak{h}(r')$, and similarly if r contains $[x, zy]$.
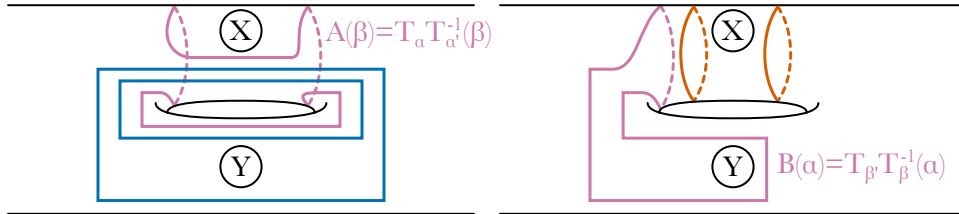
10.3. **Elements in cokernel.** How might elements in Λ_g arise? Assume that we have

- $A \in (\mathcal{I}_g)_\alpha$ and $B \in (\mathcal{I}_g)_\beta$ such that $[A, B]$ fixes either α or β.

Example 10.1. Let $\alpha, \alpha', \beta, \beta'$ be as follows, where X and Y are connected subsurfaces of Σ_g:
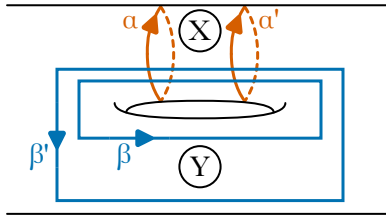
Letting $A = T_\alpha T_{\alpha'}^{-1}$ and $B = T_{\beta'} T_\beta^{-1}$, the commutator $[A, B]$ fixes both α and β. To see this, note that $A(\beta)$ is disjoint from $\beta \cup \beta'$ and $B(\alpha)$ is disjoint from $\alpha \cup \alpha'$:

This implies that

$$[A, B](\alpha) = A^{-1}B^{-1}AB(\alpha) = A^{-1}B^{-1}B(\alpha) = A^{-1}(\alpha) = \alpha,$$

and similarly that $[A, B](\beta) = \beta$. □

We define $\langle\!\langle A, B \rangle\!\rangle \in \Lambda_g$ as follows. To simplify our notation, we will assume that $[A, B]$ fixes α. The case where it fixes β is similar. Let $\pi \colon \mathrm{H}_2(\mathcal{I}_g; \mathbb{Q}) \to \Lambda_g$ be the projection. The element $[A, B] \in (\mathcal{I}_g)_\alpha$ vanishes in the abelianization of $\mathcal{I}_g$. Theorem 6.4 says[40] that the map $\mathrm{H}_1((\mathcal{I}_g)_\alpha; \mathbb{Q}) \to \mathrm{H}_1(\mathcal{I}_g; \mathbb{Q})$ is injective for $g \geq 4$, so $[A, B]$ vanishes in $\mathrm{H}_1((\mathcal{I}_g)_\alpha; \mathbb{Q})$. This implies that there exists some $n \geq 1$ such that $[A, B]^n$ vanishes in the abelianization of $(\mathcal{I}_g)_\alpha$, so we can write $[A, B]^n \mathfrak{c} = 1$ for some product $\mathfrak{c}$ of commutators in $(\mathcal{I}_g)_\alpha$. Define

$$\langle\!\langle A, B \rangle\!\rangle = \pi\left(\frac{1}{n}\mathfrak{h}\left([A, B]^n \mathfrak{c}\right)\right) \in \Lambda_g.$$

---

[39]The reason for this is that $[x, y][y, x] = 1$, so $[y, x]$ is the inverse to $[x, y]$.

[40]In unpublished work, Putman has also proved that the map $\mathrm{H}_1((\mathcal{I}_g)_\alpha; \mathbb{Z}) \to \mathrm{H}_1(\mathcal{I}_g; \mathbb{Z})$ is injective for $g \geq 4$. Using this would allow us to take $n = 1$ in the argument below, simplifying several parts of our proof. To avoid a dependence on unpublished work, we do not use this integral statement.

This appears to depend on the choice of n and $\mathfrak{c}$, but the following claim shows that it is well-defined:

Claim. *This does not depend the choice of n and $\mathfrak{c}$.*

Proof of claim. If $m \geq 1$ and $\mathfrak{d}$ is another product of commutators in $(\mathcal{I}_g)_\alpha$ such that $[A, B]^m \mathfrak{d} = 1$, then

$$
\begin{aligned}
\frac{1}{n}\mathfrak{h}\left([A,B]^n\mathfrak{c}\right) - \frac{1}{m}\mathfrak{h}\left([A,B]^m\mathfrak{d}\right) &= \frac{1}{nm}\left(\mathfrak{h}\left(([A,B]^n\mathfrak{c})^m\right) + \mathfrak{h}\left(([A,B]^m\mathfrak{d})^{-n}\right)\right) \\
&= \frac{1}{nm}\mathfrak{h}\left([A,B]^{nm}\mathfrak{c}'\mathfrak{d}'[A,B]^{-nm}\right) \\
&= \frac{1}{nm}\mathfrak{h}\left([A,B]^{nm}[A,B]^{-nm}\mathfrak{c}''\mathfrak{d}''\right) \\
&= \frac{1}{nm}\mathfrak{h}\left(\mathfrak{c}''\mathfrak{d}''\right) \in \operatorname{Im}\left(\mathrm{H}_2\left((\mathcal{I}_g)_\alpha;\mathbb{Q}\right) \to \mathrm{H}_2\left(\mathcal{I}_g;\mathbb{Q}\right)\right).
\end{aligned}
$$

Here $\mathfrak{c}'$ and $\mathfrak{c}''$ are products of commutators in $(\mathcal{I}_g)_\alpha$ obtained by commuting terms of the form $[A, B]^{\pm 1}$ past terms in $\mathfrak{c}$, and similarly for $\mathfrak{d}'$ and $\mathfrak{d}''$. The claim follows. $\qquad\square$

There is one remaining ambiguity: if $[A, B]$ fixes α and β, then our recipe gives two potentially different definitions of $\langle\!\langle A, B\rangle\!\rangle \in \Lambda_g$. However, if $[A, B]$ fixes α and β, then $[A, B] \in (\mathcal{I}_g)_{\alpha,\beta} \cong \mathcal{I}_{g-1}^1$. In the above procedure, we can therefore choose $\mathfrak{c}$ to be a product of commutators lying in $(\mathcal{I}_g)_{\alpha,\beta} \cong \mathcal{I}_{g-1}^1$ whether we are considering $[A, B]$ as an element of $(\mathcal{I}_g)_\alpha$ or of $(\mathcal{I}_g)_\beta$.

10.4. **Interlude: twisted surface groups.** For $n, k \geq 1$, let $\Gamma_{n,k}$ be the following group:

$$
\Gamma_{n,k} = \langle z_0, w_0, \ldots, z_k, w_k \mid [z_0, w_0]^n[z_1, w_1]\cdots[z_k, w_k] = 1\rangle.
$$

These arise naturally in the construction of the elements $\langle\!\langle A, B\rangle\!\rangle$ above, which are multiples of the homology classes associated to surface relators

$$
[A, B]^n[c_1, d_1]\cdots[c_k, d_k] = 1 \quad \text{with } A, B, c_i, d_i \in \mathcal{I}_g.
$$

The associated maps $\pi_1(\Sigma_{n+k}) \to \mathcal{I}_g$ factor through $\Gamma_{n,k}$. The homology of $\Gamma_{n,k}$ is given by:

Lemma 10.2. *For $n, k \geq 1$, we have $\mathrm{H}_1(\Gamma_{n,k}) = \mathbb{Z}^{2k+2}$ and $\mathrm{H}_2(\Gamma_{n,k}) = \mathbb{Z}$ and $\mathrm{H}_d(\Gamma_{n,k}) = 0$ for $d \geq 3$.*

Proof. The single relation in $\Gamma_{n,k}$ can be rewritten as

$$
[z_0, w_0]^{-n} = [z_1, w_1]\cdots[z_k, w_k].
$$

Letting $F(S)$ denote the free group on a set S, this implies that $\Gamma_{n,k}$ can be decomposed as an amalgamated free product[41]

$$
\Gamma_{n,k} = F(z_0, w_0) *_{\mathbb{Z}} F(z_1, w_1, \ldots, z_k, w_k),
$$

where the infinite cyclic group $\mathbb{Z}$ is identified with the cyclic subgroups generated by $[z_0, w_0]^{-n} \in F(z_0, w_0)$ and $[z_1, w_1]\cdots[z_k, w_k] \in F(z_1, w_1, \ldots, z_k, w_k)$. Since $\mathrm{H}_1(\mathbb{Z}) = \mathbb{Z}$, the lemma follows from the associated Mayer-Vietoris sequence in group homology whose nonzero terms are

$$
0 \longrightarrow \mathrm{H}_2(\Gamma_{n,k}) \longrightarrow \mathrm{H}_1(\mathbb{Z}) \xrightarrow{0} \mathrm{H}_1(F(z_0, w_0)) \oplus \mathrm{H}_1(F(z_1, w_1, \ldots, z_k, w_k)) \longrightarrow \mathrm{H}_1(\Gamma_{n,k}) \longrightarrow 0. \quad \square
$$

---

[41]The Freiheitsatz [29, 47] for one-relator groups implies that $\{z_0, w_0\}$ and $\{z_1, w_1, \ldots, z_k, w_k\}$ generate free subgroups of $\Gamma_{n,k}$. This general result is unnecessary since our relation can be written as $r_1 = r_2$ with $r_1 \in F(z_0, w_0)$ and $r_2 \in F(z_1, w_1, \ldots, z_k, w_k)$, giving this decomposition as a free product with amalgamation.

To identify our elements $\langle\!\langle A, B \rangle\!\rangle \in \mathrm{H}_2(\mathcal{I}_g; \mathbb{Q})$ with terms we will construct using equivariant homology, we need a space for $\Gamma_{n,k}$ to act on. For this, define

$$\Gamma'_{n,k} = \left\langle z_0, u_0, z_1, w_1, \ldots, z_k, w_k \mid (z_0^{-1} u_0)^n [z_1, w_1] \cdots [z_k, w_k] = 1 \right\rangle.$$

There is a homomorphism $\Gamma'_{n,k} \to \Gamma_{n,k}$ taking u_0 to $z_0^{w_0} = w_0^{-1} z_0 w_0$. The following implies that this is injective, so henceforth we can identify $\Gamma'_{n,k}$ with a subgroup of $\Gamma_{n,k}$:

Lemma 10.3. *The group $\Gamma_{n,k}$ is an HNN extension of $\Gamma'_{n,k}$ over an infinite cyclic subgroup with stable letter w_0 conjugating z_0 to u_0.*

Proof. This HNN extension can be written

$$\left\langle w_0, z_0, u_0, z_1, w_1, \ldots, z_k, w_k \mid (z_0^{-1} u_0)^n [z_1, w_1] \cdots [z_k, w_k] = 1, \ w_0^{-1} z_0 w_0 = u_0 \right\rangle.$$

A Tietze transformation now eliminates u_0 and turns this into the presentation for $\Gamma_{n,k}$. $\square$

By Bass–Serre theory [50], the following is an immediate corollary of Lemma 10.3:

Corollary 10.4. *For $n, k \geq 1$, the group $\Gamma_{n,k}$ acts without rotations on a tree T such that:*
- *$\Gamma_{n,k}$ acts transitively on the vertices and edges of T, so $T/\Gamma_{n,k} \cong S^1$; and*
- *there is an edge τ of T going from a vertex τ_0 to a vertex τ_1 such that $(\Gamma_{n,k})_{\tau_0} = \Gamma'_{n,k}$ and $(\Gamma_{n,k})_\tau = \langle z_0 \rangle \cong \mathbb{Z}$ and $w_0 \cdot \tau_0 = \tau_1$.*

We will also need the homology of $\Gamma'_{n,k}$:

Lemma 10.5. *For $n, k \geq 1$, we have $\mathrm{H}_1(\Gamma'_{n,k}) \cong \mathbb{Z}^{2k+1} \oplus \mathbb{Z}/n$ and $\mathrm{H}_d(\Gamma'_{n,k}) = 0$ for $d \geq 2$.*

Proof. Just like in the proof of Lemma 10.2, write $\Gamma'_{n,k}$ as an amalgamated free product

$$\Gamma'_{n,k} = F(z_0, u_0) *_{\mathbb{Z}} F(z_1, w_1, \ldots, z_k, w_k),$$

where the infinite cyclic group $\mathbb{Z}$ is identified with the cyclic subgroups generated by $(z_0^{-1} u_0)^{-n} \in F(z_0, u_0)$ and $[z_1, w_1] \cdots [z_k, w_k] \in F(z_1, w_1, \ldots, z_k, w_k)$. Since $\mathrm{H}_1(\mathbb{Z}) = \mathbb{Z}$, the lemma follows from the associated Mayer-Vietoris sequence in group homology whose nonzero terms are

$$0 \longrightarrow \mathrm{H}_2(\Gamma'_{n,k}) \longrightarrow \mathrm{H}_1(\mathbb{Z}) \overset{f}{\longrightarrow} \mathrm{H}_1(F(u_0, w_0)) \oplus \mathrm{H}_1(F(z_1, w_1, \ldots, z_k, w_k)) \longrightarrow \mathrm{H}_1(\Gamma_{n,k}) \longrightarrow 0.$$

Here f is the injective map taking the generator of $\mathrm{H}_1(\mathbb{Z}) = \mathbb{Z}$ to

$$(n[u_0] - n[z_0], 0) \in \mathrm{H}_1(F(u_0, w_0)) \oplus \mathrm{H}_1(F(z_1, w_1, \ldots, z_k, w_k)). \qquad \square$$

10.5. **Generators.** We now return to the $\langle\!\langle A, B \rangle\!\rangle$. How can we find $A \in (\mathcal{I}_g)_\alpha$ and $B \in (\mathcal{I}_g)_\beta$ such that $[A, B]$ fixes either α or β? One way for this to hold is for A to fix $B(\alpha)$, in which case $[A, B]$ will fix α. Similarly, if B fixes $A(\beta)$ then $[A, B]$ will fix β. For instance, both of these hold in Example 10.1. The following says that the $\langle\!\langle A, B \rangle\!\rangle$ coming from certain A and B of this form generate Λ_g:

Proposition 10.6. *Let $g \geq 5$, and let α and β be the curves discussed above. Then Λ_g is spanned by elements of the form $\langle\!\langle A, B \rangle\!\rangle$, where $A \in (\mathcal{I}_g)_\alpha$ and $B \in (\mathcal{I}_g)_\beta$ satisfy either:[42]*
- *$A(\beta)$ is disjoint from β, and B fixes $A(\beta)$; or*
- *$B(\alpha)$ is disjoint from α, and A fixes $B(\alpha)$.*

---

[42]The condition that $A(\beta)$ is disjoint from β is equivalent to saying that $\{\beta, A(\beta)\}$ is a simplex of $\mathcal{C}_{ab}(\Sigma_g)$, and similarly when $B(\alpha)$ is disjoint from α.

Proof. In this proof, all homology is taken with $\mathbb{Q}$ coefficients.[43] Let $\Lambda'_g < \Lambda_g$ be the span of the purported generators. Our goal is to prove that $\Lambda'_g = \Lambda_g$. Let $a = [\alpha]$ and $b = [\beta]$, and consider the $\mathcal{I}_g$-equivariant homology of the handle complex $\mathcal{C}_{ab}(\Sigma_g)$ from §9. Proposition 9.9 says that $\mathcal{C}_{ab}(\Sigma_g)$ is 1-acyclic, so by Lemma 8.1 the canonical map

$$\mathrm{H}_2^{\mathcal{I}_g}(\mathcal{C}_{ab}(\Sigma_g)) \longrightarrow \mathrm{H}_2(\mathcal{I}_g)$$

is surjective. Lemma 9.2 says that $\mathcal{I}_g$ acts on $\mathcal{C}_{ab}(\Sigma_g)$ without rotations, so we can study $\mathrm{H}_2^{\mathcal{I}_g}(\mathcal{C}_{ab}(\Sigma_g))$ using the spectral sequence from Proposition 8.2:

$$\mathrm{E}^1_{pq} = \bigoplus_{\sigma \in (\mathcal{C}_{ab}(\Sigma_g)/\mathcal{I}_g)^{(p)}} \mathrm{H}_q((\mathcal{I}_g)_{\widetilde{\sigma}}) \Rightarrow \mathrm{H}^{\mathcal{I}_g}_{p+q}(\mathcal{C}_{ab}(\Sigma_g)).$$

As was discussed in §8.5, the terms E^1_{p0} are exactly the cellular chain complex of $\mathcal{C}_{ab}(\Sigma_g)/\mathcal{I}_g$. As for the other entries, Corollary 9.5 says that the 1-skeleton of $\mathcal{C}_{ab}(\Sigma_g)/\mathcal{I}_g$ consists of:

  (i) vertices v_a and v_b, with v_a and v_b the images of α and β, respectively.
  (ii) loops based at v_a and v_b, each the image of a pure 1-simplex.
  (iii) a 1-simplex e_{ab} joining v_a and v_b, with e_{ab} the image of the edge $\{\alpha, \beta\}$.

It follows that the E^1-page of our spectral sequence is

$$
\begin{array}{llll}
\mathrm{H}_2((\mathcal{I}_g)_\alpha) \oplus \mathrm{H}_2((\mathcal{I}_g)_\beta) \leftarrow & * & \leftarrow & * \\[4pt]
\mathrm{H}_1((\mathcal{I}_g)_\alpha) \oplus \mathrm{H}_1((\mathcal{I}_g)_\beta) \leftarrow \mathrm{H}_1((\mathcal{I}_g)_{\alpha,\beta}) \oplus \displaystyle\bigoplus_{e \in (\mathcal{C}_a(\Sigma_g)/\mathcal{I}_g)^{(1)}} \mathrm{H}_1((\mathcal{I}_g)_{\widetilde{e}}) \oplus \displaystyle\bigoplus_{e \in (\mathcal{C}_b(\Sigma_g)/\mathcal{I}_g)^{(1)}} \mathrm{H}_1((\mathcal{I}_g)_{\widetilde{e}}) \leftarrow & & * \\[4pt]
C_0(\mathcal{C}_{ab}(\Sigma_g)/\mathcal{I}_g) \qquad \leftarrow & C_1(\mathcal{C}_{ab}(\Sigma_g)/\mathcal{I}_g) & \leftarrow & C_2(\mathcal{C}_{ab}(\Sigma_g)/\mathcal{I}_g)
\end{array}
$$

Proposition 9.7 says that $\mathcal{C}_{ab}(\Sigma_g)/\mathcal{I}_g$ is contractible, so

$$(10.2) \qquad\qquad \mathrm{E}^\infty_{p0} = \mathrm{E}^2_{p0} = 0 \quad \text{for } p \geq 1.$$

Moreover, as we discussed in §8.4 the composition

$$(10.3) \qquad \mathrm{H}_2((\mathcal{I}_g)_\alpha) \oplus \mathrm{H}_2((\mathcal{I}_g)_\beta) = \mathrm{E}^1_{02} \longrightarrow\!\!\!\!\!\rightarrow \mathrm{E}^\infty_{02} \lhook\joinrel\longrightarrow \mathrm{H}_2^{\mathcal{I}_g}(\mathcal{C}_{ab}(\Sigma_g)) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{H}_2(\mathcal{I}_g)$$

is the sum of the maps induced by the inclusions $(\mathcal{I}_g)_\alpha \hookrightarrow \mathcal{I}_g$ and $(\mathcal{I}_g)_\beta \hookrightarrow \mathcal{I}_g$. It follows that the cokernel of (10.3) is Λ_g. Combining this with (10.2), we deduce that the surjection $\mathrm{H}_2^{\mathcal{I}_g}(\mathcal{C}_{ab}(\Sigma_g)) \twoheadrightarrow \mathrm{H}_2(\mathcal{I}_g)$ induces a surjection

$$(10.4) \qquad\qquad \mathrm{E}^2_{11} = \mathrm{E}^\infty_{11} \longrightarrow\!\!\!\!\!\rightarrow \Lambda_g.$$

To prove that $\Lambda'_g = \Lambda_g$, it is enough to prove that the image of (10.4) is contained in Λ'_g. The vector space E^2_{11} is a quotient of the kernel of the following differential $\partial \colon \mathrm{E}^1_{11} \to \mathrm{E}^1_{01}$:

$$\partial \colon \mathrm{H}_1((\mathcal{I}_g)_{\alpha,\beta}) \oplus \bigoplus_{e \in (\mathcal{C}_a(\Sigma_g)/\mathcal{I}_g)^{(1)}} \mathrm{H}_1((\mathcal{I}_g)_{\widetilde{e}}) \oplus \bigoplus_{e \in (\mathcal{C}_b(\Sigma_g)/\mathcal{I}_g)^{(1)}} \mathrm{H}_1((\mathcal{I}_g)_{\widetilde{e}}) \longrightarrow \mathrm{H}_1((\mathcal{I}_g)_\alpha) \oplus \mathrm{H}_1((\mathcal{I}_g)_\beta).$$

To prove that the image of (10.4) lies in Λ'_g, it is enough to prove the following two claims:

Claim 1. *The kernel of the differential* $\partial \colon \mathrm{E}^1_{11} \to \mathrm{E}^1_{01}$ *is*

$$(10.5) \qquad\qquad \bigoplus_{e \in (\mathcal{C}_a(\Sigma_g)/\mathcal{I}_g)^{(1)}} \mathrm{H}_1((\mathcal{I}_g)_{\widetilde{e}}) \oplus \bigoplus_{e \in (\mathcal{C}_b(\Sigma_g)/\mathcal{I}_g)^{(1)}} \mathrm{H}_1((\mathcal{I}_g)_{\widetilde{e}}).$$

---

[43]This is just to simplify our notation in this proof. It does not represent a change in our general conventions for writing homology groups.

We must prove ∂ vanishes on (10.5) and that the restriction of ∂ to $\mathrm{H}_1((\mathcal{I}_g)_{\alpha,\beta})$ is an injection. We start with the latter fact. From the description of the differentials in §8.6, we see that on $\mathrm{H}_1((\mathcal{I}_g)_{\alpha,\beta})$ our differential is the difference between the maps

$$\mathrm{H}_1((\mathcal{I}_g)_{\alpha,\beta}) \longrightarrow \mathrm{H}_1((\mathcal{I}_g)_{\alpha}) \hookrightarrow \mathrm{H}_1((\mathcal{I}_g)_{\alpha}) \oplus \mathrm{H}_1((\mathcal{I}_g)_{\beta})$$

and

$$\mathrm{H}_1((\mathcal{I}_g)_{\alpha,\beta}) \longrightarrow \mathrm{H}_1((\mathcal{I}_g)_{\beta}) \hookrightarrow \mathrm{H}_1((\mathcal{I}_g)_{\alpha}) \oplus \mathrm{H}_1((\mathcal{I}_g)_{\beta})$$

The fact that this difference is injective follows from Theorem 6.4, which implies that $\mathrm{H}_1((\mathcal{I}_g)_{\alpha,\beta}) \to \mathrm{H}_1((\mathcal{I}_g)_{\alpha})$ and $\mathrm{H}_1((\mathcal{I}_g)_{\alpha,\beta}) \to \mathrm{H}_1((\mathcal{I}_g)_{\beta})$ are both injective.[44]

We now prove that ∂ vanishes on (10.5). We will handle the terms coming from $\mathcal{C}_a(\Sigma_g)$; the other case is similar. Consider a 1-cell e of $\mathcal{C}_a(\Sigma_g)/\mathcal{I}_g$. Corollary 9.5 says that e is a loop. Using Lemma 9.6, we can lift e to an edge $\widetilde{e}$ of $\mathcal{C}_a(\Sigma_g)$ going from α to $B(\alpha)$ for some $B \in (\mathcal{I}_g)_{\beta}$. As we discussed in §8.6, on $\mathrm{H}_1((\mathcal{I}_g)_{\alpha,B(\alpha)})$ our differential is the composition

$$(10.6) \qquad \mathrm{H}_1((\mathcal{I}_g)_{\alpha,B(\alpha)}) \longrightarrow \mathrm{H}_1((\mathcal{I}_g)_{\alpha}) \hookrightarrow \mathrm{H}_1((\mathcal{I}_g)_{\alpha}) \oplus \mathrm{H}_1((\mathcal{I}_g)_{\beta})$$

where the first map takes the homology class of $x \in (\mathcal{I}_g)_{\alpha,B(\alpha)}$ to the homology class of $[x,B] \in (\mathcal{I}_g)_{\alpha}$. Theorem 6.4 says that the inclusion $(\mathcal{I}_g)_{\alpha} \hookrightarrow \mathcal{I}_g$ induces an injection $\mathrm{H}_1((\mathcal{I}_g)_{\alpha}) \hookrightarrow \mathrm{H}_1(\mathcal{I}_g)$. Since in $\mathrm{H}_1(\mathcal{I}_g)$ the homology class of the commutator $[x,B]$ vanishes, the same is true in $\mathrm{H}_1((\mathcal{I}_g)_{\alpha})$. We conclude that (10.6) is zero, as desired.

Claim 2. *Let e be a 1-simplex of either $\mathcal{C}_a(\Sigma_g)/\mathcal{I}_g$ or $\mathcal{C}_b(\Sigma_g)/\mathcal{I}_g$. Then the image of the composition $\mathrm{H}_1((\mathcal{I}_g)_{\widetilde{e}}) \to \mathrm{E}^2_{11} \twoheadrightarrow \Lambda_g$ is contained in Λ'_g.*

We will assume that e is a 1-simplex of $\mathcal{C}_a(\Sigma_g)/\mathcal{I}_g$. The other case is similar. By Lemma 9.6, we can construct a lift $\widetilde{e}$ of e to $\mathcal{C}_{ab}(\Sigma_g)$ going from α to $\alpha' = B(\alpha)$ for some $B \in (\mathcal{I}_g)_{\beta}$. Consider $A \in (\mathcal{I}_g)_{\alpha,B(\alpha)}$. The element $\langle\!\langle A, B \rangle\!\rangle \in \Lambda_g$ is one of our generators for Λ'_g, so it is enough to prove that the composition

$$\mathrm{H}_1((\mathcal{I}_g)_{\alpha,B(\alpha)}) \longrightarrow \mathrm{E}^2_{11} \longrightarrow\!\!\!\!\!\to \Lambda_g$$

takes the homology class of A to a multiple of $\langle\!\langle A, B \rangle\!\rangle$. We divide the proof of this into 3 steps. We reiterate that during this proof all homology has $\mathbb{Q}$-coefficients.

Step 2.1. *For some $n, k \geq 1$, we construct a homomorphism[45] $f\colon \Gamma_{n,k} \to \mathcal{I}_g$ such that the image of the composition*

$$(10.7) \qquad\qquad \mathrm{H}_2(\Gamma_{n,k}) \xrightarrow{\ f_*\ } \mathrm{H}_2(\mathcal{I}_g) \longrightarrow \Lambda_g$$

is the $\mathbb{Q}$-span of $\langle\!\langle A, B \rangle\!\rangle$.

Recall that $\langle\!\langle A, B \rangle\!\rangle$ is the image in Λ_g of the element of $\mathrm{H}_2(\mathcal{I}_g)$ constructed as follows. We have $[A, B] \in (\mathcal{I}_g)_{\alpha}$, and we can find $n \geq 1$ and $x_1, y_2, \ldots, x_k, y_k \in (\mathcal{I}_g)_{\alpha}$ such that

$$[A, B]^n [x_1, y_1] \cdots [x_k, y_k] = 1.$$

Then $\langle\!\langle A, B \rangle\!\rangle$ is the image in Λ_g of the homology class

$$\frac{1}{n}\mathfrak{h}([A, B]^n [x_1, y_1] \cdots [x_k, y_k]) \in \mathrm{H}_2(\mathcal{I}_g).$$

Let $f\colon \Gamma_{n,k} \to \mathcal{I}_g$ be the map defined by

$$f(z_0) = A \text{ and } f(w_0) = B \text{ and } f(z_i) = x_i \text{ and } f(w_i) = y_i \text{ for } 1 \leq i \leq k.$$

---

[44]In fact, we only need that one of them is injective.

[45]Here $\Gamma_{n,k} = \langle z_0, w_0, \ldots, z_k, w_k \mid [z_0, w_0]^n [z_1, w_1] \cdots [z_k, w_k] = 1 \rangle$ is the group defined in §10.4.

Lemma 10.2 says that $H_2(\Gamma_{n,k}) = \mathbb{Q}$, and the image of $f_* \colon H_2(\Gamma_{n,k}) \to H_2(\mathcal{I}_g)$ is the $\mathbb{Q}$-span of $\mathfrak{h}([A, B]^n[x_1, y_1] \cdots [x_k, y_k])$. It follows that the image of (10.7) is the $\mathbb{Q}$-span of $\langle\!\langle A, B \rangle\!\rangle$.

Step 2.2. *We construct a tree T on which $\Gamma_{n,k}$ acts without rotations and a commutative diagram in equivariant homology*

$$
\begin{array}{ccc}
H_2^{\Gamma_{n,k}}(T) & \longrightarrow & H_2^{\mathcal{I}_g}(\mathcal{C}_{ab}(\Sigma_g)) \\
\downarrow{\scriptstyle\cong} & & \downarrow \\
H_2(\Gamma_{n,k}) & \xrightarrow{\ f_* \ } & H_2(\mathcal{I}_g).
\end{array}
$$

Recall that $\Gamma'_{n,k}$ is the subgroup of $\Gamma_{n,k}$ generated by $\{z_0, w_0^{-1} z_0 w_0, z_1, w_1, \ldots, z_k, w_k\}$. Corollary 10.4 gives a tree T on which $\Gamma_{n,k}$ acts without rotations such that:

- $\Gamma_{n,k}$ acts transitively on the vertices and edges of T, so $T/\Gamma_{n,k} \cong S^1$; and
- there is an edge τ of T going from a vertex τ_0 to a vertex τ_1 such that

(10.8) $\qquad (\Gamma_{n,k})_{\tau_0} = \Gamma'_{n,k} \quad \text{and} \quad (\Gamma_{n,k})_\tau = \langle z_0 \rangle \cong \mathbb{Z} \quad \text{and} \quad w_0 \cdot \tau_0 = \tau_1.$

Since T is contractible, the canonical map $H_d^{\Gamma_{n,k}}(T) \to H_d(\Gamma_{n,k})$ is an isomorphism for $d \geq 0$.

We claim that $f(\Gamma'_{n,k}) \subset (\mathcal{I}_g)_\alpha$. This can be checked on generators:

- $f(z_0) = A$ fixes the initial point α and the endpoint $\alpha' = B(\alpha)$ of $\widetilde{e}$; and
- $f(w_0^{-1} z_0 w_0) = B^{-1}AB$ fixes α since A fixes α and $B(\alpha)$; and
- for $1 \leq i \leq k$, the elements $f(z_i) = x_i$ and $f(w_i) = y_i$ fix α by construction.

In light of (10.8) and the fact that $\Gamma_{n,k}$ acts transitively on the vertices and edges of T, we can therefore define a cellular map $\phi \colon T \to \mathcal{C}_{ab}(\Sigma_g)$ via the formula

$$
\phi(v\tau) = f(v)\widetilde{e} \quad \text{and} \quad \phi(v\tau_0) = f(v)\alpha \quad \text{for all } v \in \Gamma_{n,k}.
$$

By the functoriality of equivariant homology discussed in §8.1, the maps f and ϕ induce a map $H_2^{\Gamma_{n,k}}(T) \to H_2^{\mathcal{I}_g}(\mathcal{C}_{ab}(\Sigma_g))$ fitting into the claimed commutative diagram.

Step 2.3. *We use these constructions to prove our goal: that under the map*

$$
H_1((\mathcal{I}_g)_{\alpha, B(\alpha)}) \longrightarrow E_{11}^2 \longrightarrow\!\!\!\!\!\rightarrow \Lambda_g
$$

the homology class of A maps to a multiple of $\langle\!\langle A, B \rangle\!\rangle \in \Lambda_g$.

Proposition 8.2 gives a spectral sequence

$$
F_{pq}^1 = \bigoplus_{\sigma \in (T/\Gamma_{n,k})^{(p)}} H_q((\Gamma_{n,k})_{\widetilde{\sigma}}) \Rightarrow H_{p+q}^{\Gamma_{n,k}}(T).
$$

Using Lemma 10.5 to identify $H_\bullet(\Gamma'_{n,k})$, the nonzero terms of the F^1-page are

$$
\left|
\begin{array}{l}
H_1((\Gamma_{n,k})_{\tau_0}) \leftarrow H_1((\Gamma_{n,k})_\tau) \\
H_0((\Gamma_{n,k})_{\tau_0}) \leftarrow H_0((\Gamma_{n,k})_\tau)
\end{array}
\right|
\cong
\left|
\begin{array}{l}
H_1(\Gamma'_{n,k}) \leftarrow H_1(\langle z_0 \rangle) \\
H_0(\Gamma'_{n,k}) \leftarrow H_0(\langle z_0 \rangle)
\end{array}
\right|
\cong
\left|
\begin{array}{l}
\mathbb{Q}^{2k+1} \leftarrow \mathbb{Q} \\
\mathbb{Q} \quad\ \leftarrow \mathbb{Q}
\end{array}
\right|
$$

This converges to the homology of $\Gamma_{n,k}$, which by Lemma 10.2 has $H_1(\Gamma_{n,k}) \cong \mathbb{Q}^{2k+2}$ and $H_2(\Gamma_{n,k}) = \mathbb{Q}$. The differentials on F^1 are thus all 0, and F degenerates at F^1. In particular,

$$
H_2^{\Gamma_{n,k}}(T) = F_{11}^\infty = F_{11}^1.
$$

The spectral sequence from Proposition 8.2 is functorial, so there is a map of spectral sequences $F \to E$ converging to the map $H_\bullet^{\Gamma_{n,k}}(T) \to H_\bullet^{\mathcal{I}_g}(\mathcal{C}_{ab}(\Sigma_g))$ induced by f and ϕ.

Since $f(z_0) = A \in (\mathcal{I}_g)_{\alpha, B(\alpha)}$, we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{H}_1(\langle z_0 \rangle) & \xrightarrow{\ \cong\ } & \mathrm{F}_{11}^2 \\
\downarrow & & \downarrow \\
\mathrm{H}_1((\mathcal{I}_g)_{\alpha, B(\alpha)}) & \longrightarrow & \mathrm{E}_{11}^2
\end{array}
$$

and the image of $\mathrm{F}_{11}^2 \to \mathrm{E}_{11}^2$ is the $\mathbb{Q}$-span of the image of the homology class of $A \in (\mathcal{I}_g)_{\alpha, B(\alpha)}$. As we discussed in Step 2.1, the image of

$$\mathrm{F}_{11}^2 = \mathrm{H}_2^{\Gamma_{n,k}}(T) = \mathrm{H}_2(\Gamma_{n,k}) \cong \mathbb{Q}$$

in Λ_g is the line spanned by $\langle\!\langle A, B \rangle\!\rangle$. The step follows. $\qquad\square$

Part 4. Homology of Torelli, step 3: algebraicity of cokernel

We close by proving Theorem B′, whose statement we recall in §11. To avoid having to constantly impose genus hypotheses, we make the following standing assumption:

Assumption 10.7. Throughout Part 4, we fix some $g \geq 5$. $\qquad\square$

11. Introduction to step 3

After fixing some notation, we outline the rest of the paper.

11.1. **Recollection of goal.** Let α and β be the following curves on Σ_g:
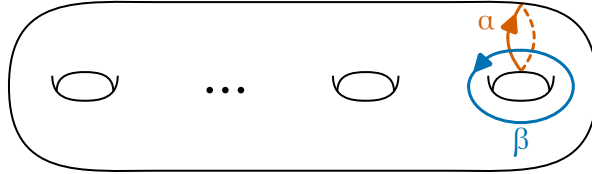
Recall that we are trying to prove Theorem B′, which says that the cokernel Λ_g of the map $\lambda \colon \mathrm{H}_2((\mathcal{I}_g)_\alpha; \mathbb{Q}) \oplus \mathrm{H}_2((\mathcal{I}_g)_\beta; \mathbb{Q}) \to \mathrm{H}_2(\mathcal{I}_g; \mathbb{Q})$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$.

11.2. **Homology.** The action of $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$ on Λ_g is induced by the conjugation action of $(\mathrm{Mod}_g)_{\alpha,\beta}$ on $\mathcal{I}_g$. Set $a = [\alpha]$ and $b = [\beta]$. Let $H_{\mathbb{Z}}$ be the orthogonal complement in $\mathrm{H}_1(\Sigma_g; \mathbb{Z})$ of a and b with respect to the algebraic intersection form. We can identify $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$ with $\mathrm{Sp}(H_{\mathbb{Z}})$. Similarly, let H be the orthogonal complement in $\mathrm{H}_1(\Sigma_g; \mathbb{Q})$ of a and b.

11.3. **Outline.** Proposition 10.6 gives generators $\langle\!\langle A, B \rangle\!\rangle$ for Λ_g. Roughly speaking, we will find enough relations between the $\langle\!\langle A, B \rangle\!\rangle$ to force Λ_g to be a subquotient of $(\wedge^2 H)^{\otimes 2}$. There are six steps:

- §12 establishes some terminology and notation for the $\langle\!\langle A, B \rangle\!\rangle$.
- §13 and §14 show how to interpret the A and B in $\langle\!\langle A, B \rangle\!\rangle$ in terms of H.
- §15 refines our generating set for Λ_g.
- §16 identifies some redundancies in our refined generating set.
- §17 introduces a subquotient of $(\wedge^2 H)^{\otimes 2}$ called the *symmetric kernel*, and discusses a presentation of it from the authors' recent paper [37]. We then use this presentation to prove that Λ_g is a quotient of the symmetric kernel. This implies that Λ_g is a finite-dimensional algebraic representation of $\mathrm{Sp}(H_{\mathbb{Z}})$, as claimed by Theorem B′

11.4. **Standing notation.** We already fixed $g \geq 5$ in Assumption 10.7. To avoid having to re-introduce notation in each section, we fix the following notation once and for all as above: α, β, a, b, H, $H_{\mathbb{Z}}$.

12. Step 3.1: notation for generators

We introduce some terminology for the generators of Λ_g given by Proposition 10.6.

12.1. **Shifters and compatibility.** The generators come in two families. Say that $A \in \mathcal{I}_g$ is a β-*shifter* if A fixes α and $A(\beta)$ is disjoint from β. In that case, say that $B \in \mathcal{I}_g$ is *compatible* with A if B fixes β and $A(\beta)$. We then have a generator $\langle\!\langle A, B \rangle\!\rangle \in \Lambda_g$, which for clarity we will denote $\langle\!\langle A, B \rangle\!\rangle_\beta$. Similarly, say that $B \in \mathcal{I}_g$ is an α-*shifter* if B fixes β and $B(\alpha)$ is disjoint from α. In that case, say that $A \in \mathcal{I}_g$ is *compatible* with B if A fixes α and $B(\alpha)$. We then have a generator $\langle\!\langle A, B \rangle\!\rangle \in \Lambda_g$, which for clarity we will denote $\langle\!\langle A, B \rangle\!\rangle_\alpha$.

Remark 12.1. It is possible for both $\langle\!\langle A, B \rangle\!\rangle_\alpha$ and $\langle\!\langle A, B \rangle\!\rangle_\beta$ to be defined, in which case they are identical elements of Λ_g. $\square$

12.2. **Left and right sides.** Let A be a β-shifter, so β and $A(\beta)$ are disjoint. Since we always consider curves up to isotopy, it is possible for $A(\beta) = \beta$. Assume that $A(\beta) \neq \beta$, in which case we will call A a *nontrivial β-shifter*. The curves $\beta \cup A(\beta)$ divide Σ_g into two subsurfaces T and T'. Order these such that T lies to the left of β and T' lies to the right:[46]
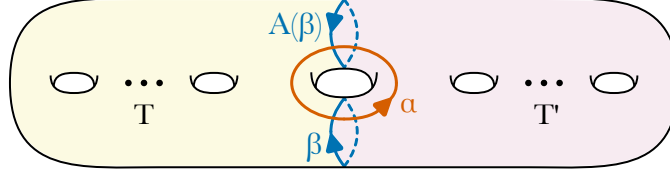
We call T the *left side* of $\beta \cup A(\beta)$ and T' the *right side*. An element $B \in \mathcal{I}_g$ is *left-compatible* with A if B is supported on T and is *right-compatible* with A if B is supported on T'.

Remark 12.2. If B is compatible with a β-shifter A, then we can write $B = B_1 B_2$ with B_1 left-compatible with A and B_2 right-compatible with A. $\square$

Similarly, let B be an α-shifter, so α and $B(\alpha)$ are disjoint. Assume that $B(\alpha) \neq \alpha$, in which case we call B a *nontrivial α-shifter*. In that case, $\alpha \cup B(\alpha)$ divides Σ_g into two subsurfaces T and T', ordered such that T is to the left of α and T' is to the right. We call T the *left side* of $\alpha \cup B(\alpha)$ and T' the *right side*. We say that $A \in \mathcal{I}_g$ is *left-compatible* with B if A is supported on T and is *right-compatible* with B if A is supported on T'.

12.3. **Symplectic summands and splittings.** Recall that $a = [\alpha] \in H_1(\Sigma_g)$ and $b = [\beta] \in H_1(\Sigma_g)$, and $H_{\mathbb{Z}} = \langle a, b \rangle^\perp$. A *symplectic summand* of $H_{\mathbb{Z}}$ is a subgroup $V < H_{\mathbb{Z}}$ such that $H_{\mathbb{Z}} = V \oplus V^\perp$, in which case $V \cong \mathbb{Z}^{2h}$ for some h called the *genus* of V. If V is a symplectic summand of $H_{\mathbb{Z}}$, then $V^\perp$ is too. A *symplectic splitting* of $H_{\mathbb{Z}}$ is a splitting of the form $H_{\mathbb{Z}} = V \oplus V^\perp$. The terms V and $V^\perp$ are ordered, so $H_{\mathbb{Z}} = V^\perp \oplus V$ is a different symplectic splitting. We call $H_{\mathbb{Z}} = V \oplus V^\perp$ a *nontrivial* symplectic splitting if $V, V^\perp \neq 0$.

12.4. **Induced splittings.** Let A be a nontrivial β-shifter. Let T and T' be the left and right sides of $\beta \cup A(\beta)$, respectively. Let S and S' be the components of the complement of a regular neighborhood of $\alpha \cup \beta \cup A(\beta)$, ordered such that $S \subset T$ and $S' \subset T'$. This is all depicted in the following figure:

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[46]If $A(\beta) = \beta$, then you can slide $A(\beta)$ over β and exchange which side is the left and the right.
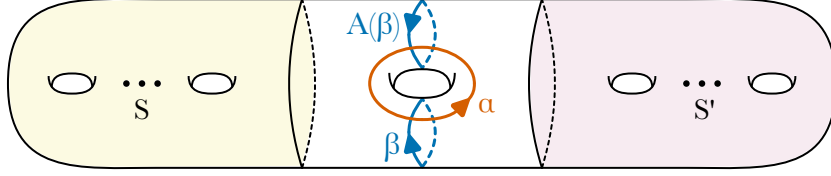
We have $H_1(S) \subset H_{\mathbb{Z}}$ and $H_1(S') \subset H_{\mathbb{Z}}$, and $H_{\mathbb{Z}} = H_1(S) \oplus H_1(S')$. This is a nontrivial symplectic splitting of $H_{\mathbb{Z}}$ that we call the symplectic splitting *induced* by A. We call $H_1(S)$ and $H_1(S')$ the *left-summand* and *right-summand* of A, respectively.

Similarly, let B be a nontrivial α-shifter. Let T and T' be the left and right sides of $\alpha \cup B(\alpha)$, respectively. Let S and S' be the components of the complement of a regular neighborhood of $\beta \cup \alpha \cup B(\alpha)$, ordered such that $S \subset T$ and $S' \subset T'$. We then have $H_{\mathbb{Z}} = H_1(S) \oplus H_1(S')$, and we call this the symplectic splitting *induced* by B. We call $H_1(S)$ and $H_1(S')$ the *left-summand* and *right-summand* of B, respectively.

13. Step 3.2: homological interpretation of compatible elements

Our next goal is to give a homological interpretation of the compatible elements B in $\langle\!\langle A, B \rangle\!\rangle_\beta$ and A in $\langle\!\langle A, B \rangle\!\rangle_\alpha$. This requires first proving some relations in Λ_g.

Remark 13.1. In our proofs, we will freely use the identities between homology classes of surface relations from §10.2 □

13.1. Linearity. Our first relation is:

Lemma 13.2. *The following hold:*

- *Let A be a β-shifter and let B_1 and B_2 be compatible with A. Then $\langle\!\langle A, B_1 B_2 \rangle\!\rangle_\beta = \langle\!\langle A, B_1 \rangle\!\rangle_\beta + \langle\!\langle A, B_2 \rangle\!\rangle_\beta$.*
- *Let B be an α-shifter and let A_1 and A_2 be compatible with B. Then $\langle\!\langle A_1 A_2, B \rangle\!\rangle_\alpha = \langle\!\langle A_1, B \rangle\!\rangle_\alpha + \langle\!\langle A_2, B \rangle\!\rangle_\alpha$.*

Proof.[47] Both are proved the same way, so we will give the details for the first. We will show that it follows from the commutator identity[48] $[A, B_1 B_2] = [A, B_2][A, B_1]^{B_2}$. Recall that for $i = 1, 2$ we have $[A, B_i] \in (\mathcal{I}_g)_\beta$, and there is some $n_i \geq 1$ and a product of commutators $\mathfrak{c}_i$ in $(\mathcal{I}_g)_\beta$ such that $[A, B_i]^{n_i} \mathfrak{c} = 1$ and $\langle\!\langle A, B_i \rangle\!\rangle_\beta$ is the image in Λ_g of

$$\frac{1}{n_i} \mathfrak{h}([A, B_i]^{n_i} \mathfrak{c}_i) \in H_2(\mathcal{I}_g; \mathbb{Q}).$$

Set $n = n_1 n_2$. The product $n \langle\!\langle A, B_i \rangle\!\rangle_\beta$ is the image in Λ_g of

$$\frac{n}{n_i} \mathfrak{h}([A, B_i]^{n_i} \mathfrak{c}_i) = \mathfrak{h}(([A, B_i]^{n_i} \mathfrak{c}_i)^{n/n_i}) = \mathfrak{h}([A, B_i]^n \mathfrak{c}_i'),$$

where $\mathfrak{c}_i'$ is the product of commutators in $(\mathcal{I}_g)_\beta$ obtained by commuting all the $[A, B_i]^{n_i}$ factors to the left. Since inner automorphisms act trivially on homology, we can conjugate our expression for $\langle\!\langle A, B_2 \rangle\!\rangle_\beta$ by B_1 and see that $n \langle\!\langle A, B_2 \rangle\!\rangle_\beta$ is also the image in Λ_g of $\mathfrak{h}(([A, B_2]^{B_1})^n \mathfrak{c}_2'')$ for some product of commutators $\mathfrak{c}_2''$ in $(\mathcal{I}_g)_\beta$. We then have that $n(\langle\!\langle A, B_1 \rangle\!\rangle_\beta + \langle\!\langle A, B_2 \rangle\!\rangle_\beta)$ is the image in Λ_g of

$$\mathfrak{h}([A, B_1]^n \mathfrak{c}_1'([A, B_2]^{B_1})^n \mathfrak{c}_2'') = \mathfrak{h}(([A, B_1][A, B_2]^{B_1})^n \mathfrak{c}''') = \mathfrak{h}([A, B_1 B_2]^n \mathfrak{c}'''),$$

where $\mathfrak{c}'''$ is a product of commutators in $(\mathcal{I}_g)_\beta$. This maps to $n \langle\!\langle A, B_1 B_2 \rangle\!\rangle_\beta$, as claimed. □

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[47] This almost follows from the proof of Claim 2 of Proposition 10.6, but it takes extra work to see that this does not e.g. give an identity of the form $\langle\!\langle A, B_1 B_2 \rangle\!\rangle_\beta = \lambda_1 \langle\!\langle A, B_1 \rangle\!\rangle_\beta + \lambda_2 \langle\!\langle B_2 \rangle\!\rangle_\beta$ for some $\lambda_1, \lambda_2 \in \mathbb{Q}$, so we give a direct proof. Similar comments apply to many of our other relations. The most important relations that cannot be derived from the proof of Proposition 10.6 even with additional work are those in §16 below.

[48] Recall that our conventions are $[x, y] = x^{-1} y^{-1} xy$ and $x^y = y^{-1} xy$.

13.2. **Vanishing.** We next prove:

Lemma 13.3. *The following hold:*

- *Let A be a β-shifter and let B be compatible with A. Assume that either $A(\beta) = \beta$ or $B(\alpha) = \alpha$. Then $\langle\!\langle A, B \rangle\!\rangle_\beta = 0$.*
- *Let B be an α-shifter and let A be compatible with B. Assume that either $B(\alpha) = \alpha$ or $A(\beta) = \beta$. Then $\langle\!\langle A, B \rangle\!\rangle_\alpha = 0$.*

Proof. Both are proved the same way, so we will give the details for the first. Recall that $[A, B] \in (\mathcal{I}_g)_\beta$, and there exists $n \geq 1$ and a product $\mathfrak{c}$ of commutators in $(\mathcal{I}_g)_\beta$ such that $[A, B]^n \mathfrak{c} = 1$ and $\langle\!\langle A, B \rangle\!\rangle_\beta$ is the image in Λ_g of

$$\frac{1}{n}\mathfrak{h}([A, B]^n \mathfrak{c}) \in \mathrm{H}_2(\mathcal{I}_g; \mathbb{Q}).$$

Assume first that $A(\beta) = \beta$. Then $[A, B]$ is a commutator of two elements of $(\mathcal{I}_g)_\beta$, so $\mathfrak{h}([A, B]^n \mathfrak{c})$ is in the image of the map

$$\mathrm{H}_2((\mathcal{I}_g)_\beta; \mathbb{Q}) \longrightarrow \mathrm{H}_2(\mathcal{I}_g; \mathbb{Q}).$$

By the definition of Λ_g, the image of $\mathfrak{h}([A, B]^n \mathfrak{c})$ in Λ_g therefore vanishes.

Assume next that $B(\alpha) = \alpha$. The commutator $[A, B]$ then fixes both α and β, and therefore as we noted at the end of §10.3 we can make our choices such that $\mathfrak{c}$ is a product of commutators in $(\mathcal{I}_g)_{\alpha,\beta} \cong \mathcal{I}_{g-1}^1$. The fact that B fixes α also implies that $[A, B]$ is a commutator in $(\mathcal{I}_g)_\alpha$. We conclude that $[A, B]^n \mathfrak{c}$ is a product of commutators in $(\mathcal{I}_g)_\alpha$, and thus that $\mathfrak{h}([A, B]^n \mathfrak{c})$ lies in the image of the map

$$\mathrm{H}_2((\mathcal{I}_g)_\alpha; \mathbb{Q}) \longrightarrow \mathrm{H}_2(\mathcal{I}_g; \mathbb{Q}).$$

By the definition of Λ_g, the image of $\mathfrak{h}([A, B]^n \mathfrak{c})$ in Λ_g therefore vanishes. $\square$

13.3. **Compatible quotient.** Let A be a nontrivial β-shifter and let X be either the left or the right side of $\beta \cup A(\beta)$. Recall from §6.2 that $\mathcal{I}_g(X)$ denotes the subgroup of $\mathcal{I}_g$ consisting of mapping classes supported on X. Each $B \in \mathcal{I}_g(X)$ is either left- or right-compatible with A depending on which side T is on. For $B \in \mathcal{I}_g(X)$, we thus have $\langle\!\langle A, B \rangle\!\rangle_\beta \in \Lambda_g$. By Lemma 13.2, this only depends on the image of B in the abelianization of $\mathcal{I}_g(X)$. In fact, since Λ_g is a $\mathbb{Q}$-vector space it only depends on the image of B in $\mathrm{H}_1(\mathcal{I}_g(X); \mathbb{Q})$.

Lemma 13.3 says that $\langle\!\langle A, B \rangle\!\rangle_\beta = 0$ for $B \in \mathcal{I}_g(X)$ with $B(\alpha) = \alpha$. We thus define:

- For a β-shifter A and X either the left or the right side of $\beta \cup A(\beta)$, define $\Omega_\beta(A, X)$ to be the quotient of $\mathrm{H}_1(\mathcal{I}_g(X); \mathbb{Q})$ by the subspace spanned by the homology classes of elements of $\mathcal{I}_g(X)$ that fix α.

For $\kappa \in \Omega_\beta(A, X)$, the discussion above implies that we have a well-defined $\langle\!\langle A, \kappa \rangle\!\rangle_\beta \in \Lambda_g$.

Similarly, for B a nontrivial α-shifter and X either the left or right side of $\alpha \cup B(\alpha)$, define $\Omega_\alpha(B, X)$ to be the quotient of $\mathrm{H}_1(\mathcal{I}_g(X); \mathbb{Q})$ by the subspace spanned by the homology classes of elements of $\mathcal{I}_g(X)$ that fix β. For $\kappa \in \Omega_\alpha(B, X)$, we have a well-defined $\langle\!\langle \kappa, B \rangle\!\rangle_\alpha \in \Lambda_g$.

13.4. **Identification of compatible quotient.** The above has the following description. For a subgroup V of $H_\mathbb{Z}$, let $V_\mathbb{Q} = V \otimes \mathbb{Q}$ be the corresponding subspace of H.

Lemma 13.4. *The following hold:*

- *Let A be a nontrivial β-shifter, let X be either the left or right side of $\beta \cup A(\beta)$, and let V be the summand of A on the same side[49] as X. Assume that the genus of X is at least 3. Then $\Omega_\beta(A, X) \cong \wedge^2 V_\mathbb{Q}$.*

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[49] In other words, V is the left-summand if X is the left side and the right-summand if X is the right side.

- *Let B be a nontrivial α-shifter, let X be either the left or right side of $\alpha \cup B(\alpha)$, and let V be the summand of B on the same side as X. Assume that the genus of X is at least 3. Then $\Omega_\alpha(B, X) \cong \wedge^2 V_\mathbb{Q}$.*

Proof. Both bullet points are proved the same way, so we will give details for the first. Let $h \geq 3$ be the genus of X. Using the fact that $h \geq 3$, Putman [46] proved that

$$(13.1) \qquad \mathrm{H}_1(\mathcal{I}_g(X); \mathbb{Q}) \cong \wedge^3 \mathrm{H}_1(X; \mathbb{Q}).$$

This isomorphism is given by the Johnson homomorphism, which we will say more about in §13.6 below. Let $(\mathcal{I}_g(X))_\alpha$ be the stabilizer of α in $\mathcal{I}_g(X)$. By definition, $\Omega_\beta(A, X)$ is the quotient of $\mathrm{H}_1(\mathcal{I}_g(X); \mathbb{Q})$ by the image of $\mathrm{H}_1((\mathcal{I}_g(X))_\alpha; \mathbb{Q})$.

Let S be the component of the complement of a regular neighborhood of $\alpha \cup \beta \cup A(\beta)$ that is contained in X, so $S \cong \Sigma_h^1$:
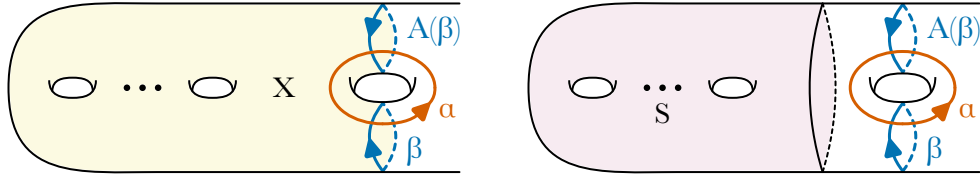
By definition, we have $\mathrm{H}_1(S) = V$. We have

$$\mathrm{H}_1(X; \mathbb{Q}) = \mathrm{H}_1(S; \mathbb{Q}) \oplus \langle [\beta] \rangle = V_\mathbb{Q} \oplus \langle [\beta] \rangle,$$

so

$$(13.2) \qquad \wedge^3 \mathrm{H}_1(X; \mathbb{Q}) \cong \left(\wedge^3 V_\mathbb{Q} \right) \oplus \left((\wedge^2 V_\mathbb{Q}) \wedge [\beta] \right).$$

It follows from the calculations in [46] that under the isomorphism (13.1), the image of $\mathrm{H}_1((\mathcal{I}_g(X))_\alpha; \mathbb{Q})$ is the term $\wedge^3 V_\mathbb{Q}$ from (13.2). We conclude that

$$\Omega_\beta(A, X) \cong (\wedge^2 V_\mathbb{Q}) \wedge [\beta] \cong \wedge^2 V_\mathbb{Q}. \qquad \square$$

13.5. **Notation.** In light of Lemma 13.4, if A is a nontrivial β-shifter, V is either the left- or right-summand of A, and the genus of V is at least 3, then for $\kappa \in \wedge^2 V_\mathbb{Q}$ we have a well-defined $\langle\!\langle A, \kappa \rangle\!\rangle_\beta \in \Lambda_g$. Similarly, if B is a nontrivial α-shifter, V is either the left- or right-summand of B, and the genus of V is at least 3, then for $\kappa \in \wedge^2 V_\mathbb{Q}$ we have a well-defined $\langle\!\langle \kappa, B \rangle\!\rangle_\alpha \in \Lambda_g$. These elements satisfy the following linearity relations:

Lemma 13.5. *The following hold:*

- *Let A be a nontrivial β-shifter and let V be either the left- or right-summand of A. Assume that the genus of V is at least 3. Then for $\kappa_1, \kappa_2 \in \wedge^2 V_\mathbb{Q}$ and $\lambda_1, \lambda_2 \in \mathbb{Q}$ we have*

$$\langle\!\langle A, \lambda_1 \kappa_1 + \lambda_2 \kappa_2 \rangle\!\rangle_\beta = \lambda_1 \langle\!\langle A, \kappa_1 \rangle\!\rangle_\beta + \lambda_2 \langle\!\langle A, \kappa_2 \rangle\!\rangle_\beta.$$

- *Let B be a nontrivial α-shifter and let V be either the left- or right-summand of B. Assume that the genus of V is at least 3. Then for $\kappa_1, \kappa_2 \in \wedge^2 V_\mathbb{Q}$ and $\lambda_1, \lambda_2 \in \mathbb{Q}$ we have*

$$\langle\!\langle \lambda_1 \kappa_1 + \lambda_2 \kappa_2, B \rangle\!\rangle_\alpha = \lambda_1 \langle\!\langle \kappa_1, B \rangle\!\rangle_\alpha + \lambda_2 \langle\!\langle \kappa_2, B \rangle\!\rangle_\alpha.$$

Proof. Immediate from the linearity relations in Lemma 13.2 along with the definitions. $\square$

13.6. **Johnson homomorphism.** For later use, we give a computation involving the Johnson homomorphism used in the proof of Lemma 13.4. To set it up, let X be a subsurface of Σ_g with the following properties:

- X is a connected subsurface with two boundary components; and
- the complement $\Sigma_g \setminus \text{Int}(X)$ is connected and has positive genus.

In this context, the Johnson homomorphism is a homomorphism

$$\tau \colon \mathcal{I}_g(X) \longrightarrow \wedge^3 \text{H}_1(X; \mathbb{Q})$$

that was defined by Putman [46] based on work of Johnson [20] for closed surfaces.

Consider a bounding pair $T_\delta T_\lambda^{-1}$ in $\mathcal{I}_g(X)$, so δ and λ are disjoint nonseparating curves on Σ_g such that $\delta, \lambda \subset X$ and such that $\delta \cup \lambda$ separates Σ_g. Let X' be the component of X cut open along $\delta \cup \lambda$ that is disjoint from ∂X. We have $X' \cong \Sigma_k^2$ for some $k \geq 0$. Let Y be a subsurface of X' with $Y \cong \Sigma_k^1$:
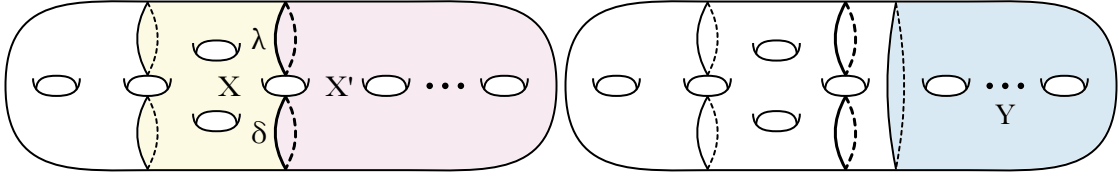
Let $\{z_1, w_1, \ldots, z_k, w_k\}$ be a symplectic basis for $\text{H}_1(Y) \cong \mathbb{Z}^{2k}$. Orient δ arbitrarily. Then

$$\tau(T_\delta T_\lambda^{-1}) = \pm(z_1 \wedge w_1 + \cdots + z_k \wedge w_k) \wedge [\delta] \in \wedge^3 \text{H}_1(X; \mathbb{Q}),$$

where the sign is $+1$ (resp. -1) if Y is to the left (resp. right) of δ. It is easy to see that this does not depend on our choices (the orientation of δ and the subsurface Y).

14. Step 3.3: homological interpretation of shifters

We now give a homological interpretation of the shifters A in $\langle\!\langle A, \kappa \rangle\!\rangle_\beta$ and B in $\langle\!\langle \kappa, B \rangle\!\rangle_\alpha$.

14.1. **Dependence on splitting.** Our main result is:

Lemma 14.1. *The following hold:*
- *Let A and A' be nontrivial β-shifters inducing the same symplectic splitting of $H_\mathbb{Z}$. Let W be either the left- or the right-summand of A and A'. Assume that W has genus at least 3, and let $\kappa \in \wedge^2 W_\mathbb{Q}$. Then $\langle\!\langle A, \kappa \rangle\!\rangle_\beta = \langle\!\langle A', \kappa \rangle\!\rangle_\beta$.*
- *Let B and B' be nontrivial α-shifters inducing the same symplectic splitting of $H_\mathbb{Z}$. Let W be either the left- or the right-summand of B and B'. Assume that W has genus at least 3, and let $\kappa \in \wedge^2 W_\mathbb{Q}$. Then $\langle\!\langle \kappa, B \rangle\!\rangle_\beta = \langle\!\langle \kappa, B' \rangle\!\rangle_\beta$.*

Proof. Both are proved the same way, so we will give the details for the first. It is enough to prove this for κ the image of an element $B \in \mathcal{I}_g$ that is either left- or right-compatible with A. The proof has two steps:

Step 1. *We have $\langle\!\langle A, \kappa \rangle\!\rangle_\beta = \langle\!\langle A', \kappa \rangle\!\rangle_\beta$ if $A(\beta) = A'(\beta)$.*

Since $A(\beta) = A'(\beta)$, the element B is also left- or right-compatible with A' and $\langle\!\langle A', \kappa \rangle\!\rangle_\beta = \langle\!\langle A', B \rangle\!\rangle_\beta$. Set $f = (A')^{-1}A$, so f fixes both α and β. The element f is a trivial β-shifter such that B is left- or right-compatible with f, so Lemma 13.3 implies that $\langle\!\langle f, B \rangle\!\rangle_\beta = 0$. Consider the commutator identity

$$[A, B] = [A'f, B] = [A', B]^f [f, B].$$

An argument like in the proof of Lemma 13.2 shows that this leads to the formula

$$\langle\!\langle A, \kappa \rangle\!\rangle_\beta = \langle\!\langle A, B \rangle\!\rangle_\beta = \langle\!\langle A', B \rangle\!\rangle_\beta + \langle\!\langle f, B \rangle\!\rangle_\beta = \langle\!\langle A', B \rangle\!\rangle_\beta = \langle\!\langle A', \kappa \rangle\!\rangle_\beta.$$

Step 2. *We have $\langle\!\langle A, \kappa \rangle\!\rangle_\beta = \langle\!\langle A', \kappa \rangle\!\rangle_\beta$ in general.*

We claim that there exists some $f \in (\mathcal{I}_g)_{\alpha,\beta}$ such that $f(A'(\beta)) = A(\beta)$. Indeed, since A and A' induce the same symplectic splitting of $H_\mathbb{Z}$, we can apply [22, Lemma 7] to find $f' \in \mathcal{I}_g$ such that

$$f'(\beta) = \beta \quad \text{and} \quad f'(A'(\beta)) = A(\beta).$$

Both $\{\alpha, \beta, A(\beta)\}$ and

$$\{f'(\alpha), f'(\beta), f'(A'(\beta))\} = \{f'(\alpha), \beta, A(\beta)\}$$

are mixed simplices of $\mathcal{C}_{ab}(\Sigma_g)$, so by Lemma 9.3 we can find some $h \in \mathcal{I}_g$ with

$$h(f'(\alpha)) = \alpha \quad \text{and} \quad h(\beta) = \beta \quad \text{and} \quad h(A(\beta)) = A(\beta).$$

The desired f is then $f = hf'$.

We have

$$A^f(\beta) = f^{-1}Af(\beta) = f^{-1}A(\beta) = A'(\beta).$$

This implies that A^f is a nontrivial β-shifter. The element B^f is left- or right-compatible with A^f, so by Step 1 and the fact that inner automorphisms act trivially on homology we have

$$\langle\!\langle A, \kappa \rangle\!\rangle_\beta = \langle\!\langle A, B \rangle\!\rangle_\beta = \langle\!\langle A^f, B^f \rangle\!\rangle_\beta = \langle\!\langle A', B^f \rangle\!\rangle_\beta = \langle\!\langle A', \kappa \rangle\!\rangle_\beta. \qquad \square$$

14.2. **Realizing symplectic splittings.** The following says that all nontrivial symplectic splittings of $H_\mathbb{Z}$ can be induced by a nontrivial β- and α-shifters.

Lemma 14.2. *Let $H_\mathbb{Z} = V \oplus V^\perp$ be a nontrivial symplectic splitting of $H_\mathbb{Z}$. Then:*
- *there exists a nontrivial β-shifter A inducing $H_\mathbb{Z} = V \oplus W$; and*
- *there exists a nontrivial α-shifter B inducing $H_\mathbb{Z} = V \oplus W$.*

Proof. Both are proved the same way, so we will give the details for the first. Let $X \cong \Sigma_{g-1}^1$ be the complement of a regular neighborhood of $\alpha \cup \beta$. In [22, Lemma 9], Johnson proved that there exists a subsurface $T \cong \Sigma_k^1$ of X with $\mathrm{H}_1(T) = V$. We can then find an b-curve β' as follows:
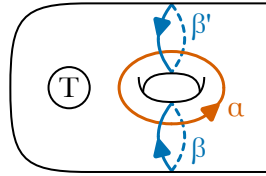
Lemma 9.4 gives an $A \in \mathcal{I}_g$ with $A(\beta) = \beta'$ and $A(\alpha) = \alpha$, i.e., a β-shifter whose left-summand is V. The right-summand of A is then $V^\perp$, so A induces the symplectic splitting $H_\mathbb{Z} = V \oplus V^\perp$. $\qquad \square$

14.3. **Notation.** In light of Lemmas 14.1 and 14.2, we introduce the following notation. Let $H_\mathbb{Z} = V \oplus V^\perp$ be a nontrivial symplectic splitting. Let W be either V or $V^\perp$, and assume that W has genus at least 3. Choose $\kappa \in \wedge^2 W_\mathbb{Q}$. Then:
- Let A be a nontrivial β-shifter inducing the symplectic splitting $H_\mathbb{Z} = V \oplus V^\perp$. Define $\langle\!\langle V, \kappa \rangle\!\rangle_\beta = \langle\!\langle A, \kappa \rangle\!\rangle_\beta$.
- Let B be a nontrivial α-shifter inducing the symplectic splitting $H_\mathbb{Z} = V \oplus V^\perp$. Define $\langle\!\langle \kappa, V \rangle\!\rangle_\alpha = \langle\!\langle \kappa, B \rangle\!\rangle_\alpha$.

Here $\langle\!\langle A, \kappa \rangle\!\rangle_\beta$ and $\langle\!\langle \kappa, B \rangle\!\rangle_\alpha$ are as defined at the end of §13.4. These elements satisfy the following linearity relations:

Lemma 14.3. *Let $H_{\mathbb{Z}} = V \oplus V^{\perp}$ be a nontrivial symplectic splitting. Let W be either V or $V^{\perp}$, and assume that W has genus at least 3. Then for $\kappa_1, \kappa_2 \in \wedge^2 W_{\mathbb{Q}}$ and $\lambda_1, \lambda_2 \in \mathbb{Q}$ we have*

$$\langle\!\langle V, \lambda_1 \kappa_1 + \lambda_2 \kappa_2 \rangle\!\rangle_\beta = \lambda_1 \langle\!\langle V, \kappa_1 \rangle\!\rangle_\beta + \lambda_2 \langle\!\langle V, \kappa_2 \rangle\!\rangle_\beta,$$
$$\langle\!\langle \lambda_1 \kappa_1 + \lambda_2 \kappa_2, V \rangle\!\rangle_\alpha = \lambda_1 \langle\!\langle \kappa_1, V \rangle\!\rangle_\alpha + \lambda_2 \langle\!\langle \kappa_2, V \rangle\!\rangle_\alpha.$$

Proof. Immediate from Lemma 13.5. $\qquad\qquad\square$

14.4. **Orthogonal complement.** The following shows how changing V to $V^{\perp}$ in the notation $\langle\!\langle V, \kappa \rangle\!\rangle_\beta$ and $\langle\!\langle \kappa, V \rangle\!\rangle_\alpha$ affects our generators:

Lemma 14.4. *Let $H_{\mathbb{Z}} = V \oplus V^{\perp}$ be a nontrivial symplectic splitting. Let W be either V or $V^{\perp}$, and assume that W has genus at least 3. Let $\kappa \in \wedge^2 W_{\mathbb{Q}}$. Then $\langle\!\langle V^{\perp}, \kappa \rangle\!\rangle_\beta = -\langle\!\langle V, \kappa \rangle\!\rangle_\beta$ and $\langle\!\langle \kappa, V^{\perp} \rangle\!\rangle_\alpha = -\langle\!\langle \kappa, V \rangle\!\rangle_\alpha$.*

Proof. Both are proved the same way, so we will give the details for the first. Let A be a nontrivial β-shifter inducing the symplectic splitting $H_{\mathbb{Z}} = V \oplus V^{\perp}$. It is enough to prove the lemma for κ the image of some B that is compatible with A, so

(14.1) $$\langle\!\langle V, \kappa \rangle\!\rangle_\beta = \langle\!\langle A, B \rangle\!\rangle_\beta.$$

We must give a similar formula for $\langle\!\langle V^{\perp}, \kappa \rangle\!\rangle_\beta$.

Since A is a β-shifter, β is disjoint from $A(\beta)$. Applying A^{-1} to this, we see that $A^{-1}(\beta)$ is disjoint from β, so A^{-1} is a β-shifter. Let T and T' be the left- and right-sides of $\beta \cup A(\beta)$:
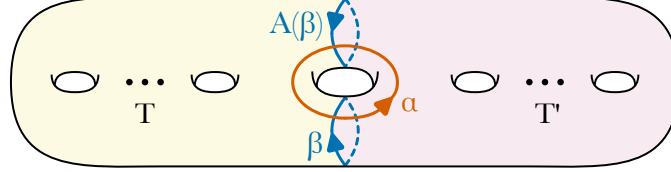
Since T lies to the right of $A(\beta)$, it follows that $A^{-1}(T)$ lies to the right of β. Similarly, $A^{-1}(T')$ lies to the left of β. We have $V \subset \mathrm{H}_1(T)$ and $V^{\perp} \subset \mathrm{H}_1(T')$, so $A^{-1}(V) \subset \mathrm{H}_1(A^{-1}(T))$ and $A^{-1}(V^{\perp}) \subset \mathrm{H}_1(A^{-1}(T'))$. Since $A \in \mathcal{I}_g$ acts trivially on $H_{\mathbb{Z}}$, we deduce that A^{-1} induces the symplectic splitting $H_{\mathbb{Z}} = V^{\perp} \oplus V$.

The fact that B is compatible with A means that B fixes β and $A(\beta)$. This implies that $B^A = A^{-1}BA$ fixes $A^{-1}(\beta)$ and β, so B^A is compatible with A^{-1}. Since $A \in \mathcal{I}_g$ fixes $\kappa \in \wedge^2 W_{\mathbb{Q}}$, we have

(14.2) $$\langle\!\langle A^{-1}, B^A \rangle\!\rangle_\beta = \langle\!\langle V^{\perp}, A^{-1}(\kappa) \rangle\!\rangle_\beta = \langle\!\langle V^{\perp}, \kappa \rangle\!\rangle_\beta.$$

In light of (14.1) and (14.2), we must prove that $\langle\!\langle A^{-1}, B^A \rangle\!\rangle_\beta = -\langle\!\langle A, B \rangle\!\rangle_\beta$. This follows from the commutator identity $[A^{-1}, B^A] = [A, B]^{-1}$ just like in the proof of Lemma 13.2. $\quad\square$

The following variant on Lemma 14.4 will also be useful:

Lemma 14.5. *The following hold:*
- *Let A be a β-shifter and let B be compatible with A. Then A^{-1} is a β-shifter, B^A is compatible with A, and $\langle\!\langle A^{-1}, B^A \rangle\!\rangle_\beta = -\langle\!\langle A, B \rangle\!\rangle_\beta$.*
- *Let B be an α-shifter and let A be compatible with B. Then B^{-1} is an α-shifter, A^B is compatible with B, and $\langle\!\langle A^B, B^{-1} \rangle\!\rangle_\alpha = -\langle\!\langle A, B \rangle\!\rangle_\alpha$.*

Proof. The proof is similar to that of Lemma 14.4, so we omit it. $\qquad\square$

15. Step 3.4: refined generating set

We now prove that only some of the $\langle\!\langle V, \kappa \rangle\!\rangle_\beta$ and $\langle\!\langle \kappa, V \rangle\!\rangle_\alpha$ are needed to generate Λ_g.

15.1. **Main result.** Let V be a genus-1 symplectic summand of $H_\mathbb{Z}$. Since $H_\mathbb{Z}$ is the orthogonal complement in $\mathrm{H}_1(\Sigma_g)$ of $\langle a, b \rangle$, it follows that $H_\mathbb{Z}$ has genus $g - 1$. This implies that $V^\perp$ has genus $g - 2$. Since $g \geq 5$ (see Assumption 10.7), we deduce that $V^\perp$ has genus at least 3. For $\kappa \in \wedge^2 V_\mathbb{Q}^\perp$, it follows that $\langle\!\langle V, \kappa \rangle\!\rangle_\beta$ and $\langle\!\langle \kappa, V \rangle\!\rangle_\alpha$ are defined. These elements generate Λ_g:

Lemma 15.1. *The vector space Λ_g is spanned by elements of the form $\langle\!\langle V, \kappa \rangle\!\rangle_\beta$ and $\langle\!\langle \kappa, V \rangle\!\rangle_\alpha$ as V ranges over genus-1 symplectic summands of $H_\mathbb{Z}$ and κ ranges over elements of $\wedge^2 V_\mathbb{Q}^\perp$.*

Proof. Lemma 13.3 implies that $\langle\!\langle A, B \rangle\!\rangle_\beta = 0$ if A is a trivial β-shifter and that $\langle\!\langle A, B \rangle\!\rangle_\alpha = 0$ if B is a trivial α-shifter. This allows us to restrict attention to nontrivial β- and α-shifters. In light of this, Λ_g is spanned by the following elements:

- $\langle\!\langle A, B \rangle\!\rangle_\beta$ with A a nontrivial β-shifter and B compatible with A; and
- $\langle\!\langle A, B \rangle\!\rangle_\alpha$ with B a nontrivial α-shifter and A compatible with B.

We must prove that each of these is a linear combination of our proposed generators. We will do this for $\langle\!\langle A, B \rangle\!\rangle_\beta$ with A a nontrivial β-shifter and B compatible with A.

For a nontrivial β-shifter A', say that the *left-genus* of A' is the genus of the left side of $\beta \cup A'(\beta)$. It is enough to prove that each $\langle\!\langle A, B \rangle\!\rangle_\beta$ is a linear combination of elements of the following form:

- $\langle\!\langle A', B' \rangle\!\rangle_\beta$ with A a β-shifter of left-genus 1 and B' right-compatible with A'.

We do this in two steps. The first step ensures that the compatible element is right-compatible, and the second that the β-shifter has left-genus 1.

Claim 1. *Let A be a nontrivial β-shifter and let B be compatible with A. Then $\langle\!\langle A, B \rangle\!\rangle_\beta$ is a linear combination of elements of the form $\langle\!\langle A', B' \rangle\!\rangle_\beta$ with A' a nontrivial β-shifter and B' right-compatible with A'.*

Let T and T' be the left and right sides of $\beta \cup A(\beta)$, respectively. Since B fixes $\beta \cup A(\beta)$ and lies in $\mathcal{I}_g$, we can write $B = B_1 B_2$ with B_1 supported on T and B_2 supported on T'. By Lemmas 13.2 and 14.5, we have

$$\langle\!\langle A, B \rangle\!\rangle_\beta = \langle\!\langle A, B_1 \rangle\!\rangle_\beta + \langle\!\langle A, B_2 \rangle\!\rangle_\beta = -\langle\!\langle A^{-1}, B_1^A \rangle\!\rangle_\beta + \langle\!\langle A, B_2 \rangle\!\rangle_\beta.$$

Since B_2 is right-compatible with A, this reduces us to showing that B_1^A is right-compatible with A^{-1}. Since T is to the left of β, it is to the right of $A(\beta)$. This implies that $A^{-1}(T)$ is to the right of β. Since B_1^A is supported on $A^{-1}(T)$, this implies that B_1^A is right-compatible with A^{-1}, as desired.

Claim 2. *Let A be a nontrivial β-shifter and let B be right-compatible with A. Then $\langle\!\langle A, B \rangle\!\rangle_\beta$ is a linear combination of elements of the form $\langle\!\langle A', B' \rangle\!\rangle_\beta$ with A' a β-shifter of left-genus 1 and B' right-compatible with A'.*

Let T and T' be the left and right sides of $\beta \cup A(\beta)$, so B is supported on T'. Let $A(\beta) = \beta_0, \beta_1, \ldots, \beta_h = \beta$ be a sequence of disjoint b-curves in T each of which intersect α once such that $\beta_{i-1} \cup \beta_i$ bounds a genus-1 subsurface of T for $1 \leq i \leq h$:
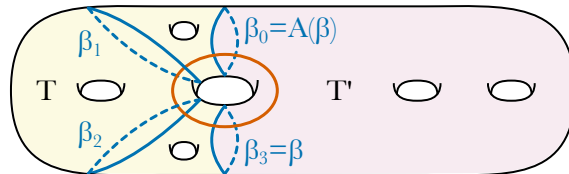
By Lemma 9.4 the group $\mathcal{I}_g$ acts transitively on mixed 1-simplices of $\mathcal{C}_{ab}(\Sigma_g)$, so we can

find $A_2, \ldots, A_h \in \mathcal{I}_g$ such that $A_i(\alpha) = \alpha$ and $A_i(\beta_i) = \beta_{i-1}$ for $2 \leq i \leq h$. Pick $A_1 \in \mathcal{I}_g$ such that $A = A_1 \cdots A_h$, so $A_1(\alpha) = \alpha$ as well. We then have

$$(15.1) \qquad\qquad A_i A_{i+1} \cdots A_h(\beta) = \beta_{i-1} \text{ for } 1 \leq i \leq h.$$

We have a commutator identity

$$(15.2) \qquad [A_1 \cdots A_h, B] = [A_1, B]^{A_2 \cdots A_h}[A_2, B]^{A_3 \cdots A_h} \cdots [A_h, B]$$
$$= [A_1^{A_2 \cdots A_h}, B^{A_2 \cdots A_h}][A_2^{A_3 \cdots A_h}, B^{A_3 \cdots A_h}] \cdots [A_h, B].$$

We then have generators $\langle\!\langle A_i^{A_{i+1} \cdots A_h}, B^{A_{i+1} \cdots A_h} \rangle\!\rangle_\beta$ associated to the terms in this identity. Indeed:

- $A_i^{A_{i+1} \cdots A_h}$ is a β-shifter since

$$(15.3) \qquad A_i^{A_{i+1} \cdots A_h}(\beta) = (A_{i+1} \cdots A_h)^{-1}(A_i \cdots A_h)(\beta) = (A_{i+1} \cdots A_h)^{-1}(\beta_{i-1})$$

  is disjoint from

$$(15.4) \qquad\qquad\qquad \beta = (A_{i+1} \cdots A_h)^{-1}(\beta_i).$$

  Both (15.3) and (15.4) use (15.1). Since $\beta_{i-1} \cup \beta_i$ bounds a genus-1 subsurface of T, it follows that $A_i^{A_{i+1} \cdots A_h}$ has left-genus 1.
- The element B fixes each β_j, so $B^{A_{i+1} \cdots A_h}$ is compatible with $A_i^{A_{i+1} \cdots A_h}$ since it fixes β (cf. (15.4)) and $A_i^{A_{i+1} \cdots A_h}(\beta)$ (cf. (15.3)). In fact, by construction B is right-compatible with $B^{A_{i+1} \cdots A_h}$.

Using (15.2), an argument similar to the one used in the proof of Lemma 13.2 shows that

$$\langle\!\langle A, B \rangle\!\rangle_\beta = \langle\!\langle A_1 A_2 \cdots A_h, B \rangle\!\rangle_\beta$$
$$= \langle\!\langle A_1^{A_2 \cdots A_h}, B^{A_2 \cdots A_h} \rangle\!\rangle_\beta + \langle\!\langle A_2^{A_3 \cdots A_h}, B^{A_3 \cdots A_h} \rangle\!\rangle_\beta + \cdots + \langle\!\langle A_h, B \rangle\!\rangle_\beta.$$

The right hand side is a sum of terms in the desired generating set, as desired. $\qquad \square$

## 16. Step 3.5: identifying generators

The generating set for Λ_g from Lemma 15.1 has some redundancies.

16.1. **Intersection form.** Identifying these redundancies requires some notation. Let W be a symplectic summand of $H_{\mathbb{Z}}$. The algebraic intersection form on W identifies W with its dual. This allows us to identify alternating bilinear forms on W with elements of $\wedge^2 W \subset \wedge^2 H_{\mathbb{Z}}$. In particular, the algebraic intersection form on W is an element ω_W of $\wedge^2 H_{\mathbb{Z}}$. If $\{a_1, b_1, \ldots, a_h, b_h\}$ is a symplectic basis for W, then $\omega_W = a_1 \wedge b_1 + \cdots + a_h \wedge b_h$.

16.2. **Redundancy.** With this notation, we have the following two lemmas:

Lemma 16.1. *Let V and W be orthogonal genus-1 symplectic summands of $H_{\mathbb{Z}}$. Then $\langle\!\langle V, \omega_W \rangle\!\rangle_\beta = -\langle\!\langle \omega_V, W \rangle\!\rangle_\alpha$.*

Proof. Since V and W are orthogonal genus-1 symplectic summands of $H_{\mathbb{Z}}$, there are disjoint subsurfaces $X \cong \Sigma_1^1$ and $Y \cong \Sigma_1^1$ of Σ_g that are disjoint from $\alpha \cup \beta$ such that $H_1(X) = V$ and $H_1(Y) = W$. Pick curves α' and β' as follows (cf. Example 10.1):
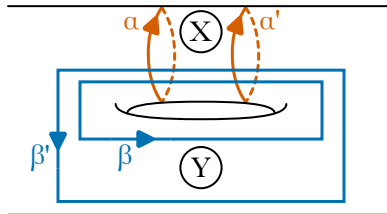
Set $A = T_\alpha T_{\alpha'}^{-1}$ and $B = T_{\beta'} T_\beta^{-1}$. Then A is a β-shifter and B is compatible with A, and also B is an α-shifter and A is compatible with B:
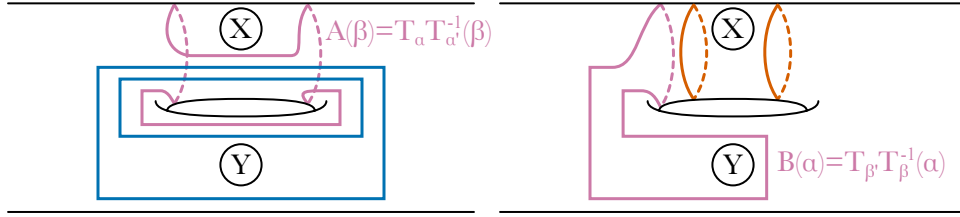
When you cut Σ_g along $\beta \cup A(\beta)$ the subsurface X is to the left of β, so A induces the symplectic splitting

$$H_{\mathbb{Z}} = \mathrm{H}_1(X) \oplus \mathrm{H}_1(X)^\perp = V \oplus V^\perp.$$

Similarly, when you cut Σ_g along $\alpha \cup B(\alpha)$ the subsurface Y is to the left of α, so B induces the symplectic splitting $H_{\mathbb{Z}} = W \oplus W^\perp$.

Examining our proof of Lemma 13.4 and using the computation of the Johnson homomorphism on a bounding pair in §13.6, we see that the fact that Y is to the left of β' implies that the image $B = T_{\beta'} T_\beta^{-1}$ in $\wedge^2 V_{\mathbb{Q}}^\perp$ is ω_W. Similarly, since X is to the right of α the image of $A = T_\alpha T_{\alpha'}^{-1}$ in $\wedge^2 W_{\mathbb{Q}}^\perp$ is $-\omega_V$. Together with the previous paragraph, this implies that (cf. Remark 12.1)

$$\langle\!\langle V, \omega_W \rangle\!\rangle_\beta = \langle\!\langle A, B \rangle\!\rangle_\beta = \langle\!\langle A, B \rangle\!\rangle_\alpha = \langle\!\langle -\omega_V, W \rangle\!\rangle_\alpha = -\langle\!\langle \omega_V, W \rangle\!\rangle_\alpha. \qquad \square$$

Lemma 16.2. *Let V be a genus-1 symplectic summand of $H_{\mathbb{Z}}$. Then $\langle\!\langle V, \omega_{V^\perp} \rangle\!\rangle_\beta = -\langle\!\langle \omega_{V^\perp}, V \rangle\!\rangle_\alpha$.*

Proof. The proof is similar to that of Lemma 16.1, but with a twist at the end. Recall that $H_{\mathbb{Z}}$ has genus $g - 1$. Pick disjoint subsurfaces $X \cong \Sigma_1^1$ and $Y \cong \Sigma_{g-2}^1$ of Σ_g that are disjoint from $\alpha \cup \beta$ such that $\mathrm{H}_1(X) = V$ and $\mathrm{H}_1(Y) = V^\perp$. Choose curves α' and β' as follows:
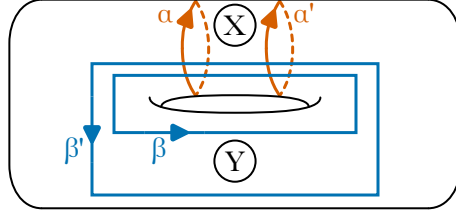
Note that unlike in the proof of Lemma 16.1, this depicts the whole surface and not just part of it.

Set $A = T_\alpha T_{\alpha'}^{-1}$ and $B = T_{\beta'} T_\beta^{-1}$. Then A is a β-shifter and B is compatible with A, and also B is an α-shifter and A is compatible with B:
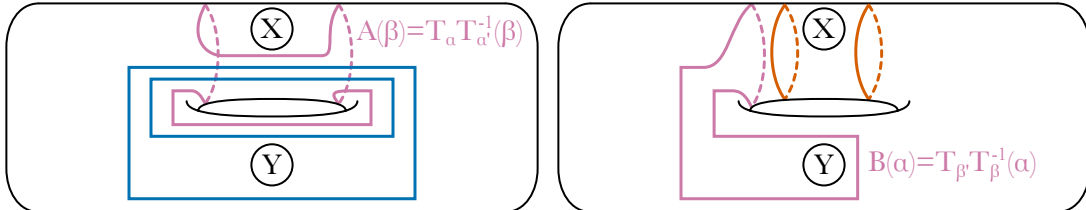
When you cut Σ_g along $\beta \cup A(\beta)$ the subsurface X is to the left of β, so A induces the symplectic splitting

$$H_{\mathbb{Z}} = \mathrm{H}_1(X) \oplus \mathrm{H}_1(X)^\perp = V \oplus V^\perp.$$

Similarly, when you cut Σ_g along $\alpha \cup B(\alpha)$ the subsurface Y is to the left of α, so B induces the symplectic splitting $H_{\mathbb{Z}} = V^\perp \oplus V$.

Examining our proof of Lemma 13.4 and using the computation of the Johnson homomorphism on a bounding pair in §13.6, we see that the fact that Y is to the left of β' implies that the image $B = T_{\beta'} T_{\beta}^{-1}$ in $\wedge^2 V^{\perp}$ is $\omega_{V^{\perp}}$. This implies that

$$(16.1) \qquad \langle\!\langle V, \omega_{V^{\perp}} \rangle\!\rangle_{\beta} = \langle\!\langle A, B \rangle\!\rangle_{\beta}.$$

After reading the proof of Lemma 16.1, you might think that $\langle\!\langle A, B \rangle\!\rangle_{\alpha}$ equals $\langle\!\langle -\omega_V, V^{\perp} \rangle\!\rangle_{\alpha}$. However, $\langle\!\langle -\omega_V, V^{\perp} \rangle\!\rangle_{\alpha}$ is not defined since V has genus 1 and $1 < 3$ (cf. Lemma 13.4). To fix this, observe that α' is homotopic to $B(\alpha)$:
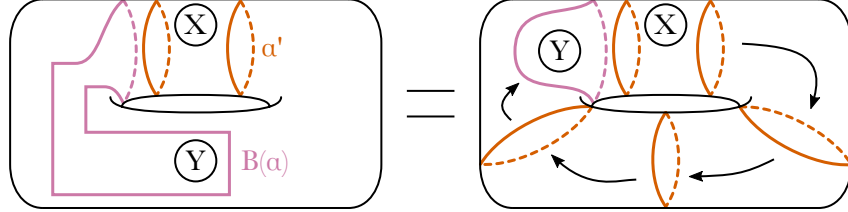
This implies that $A = T_{\alpha} T_{\alpha'}^{-1}$ can be not only be realized by a mapping class supported on the right of $\alpha \cup B(\alpha)$, but also by a mapping class supported on the left. Since Y is to the left of α, the image of $A = T_{\alpha} T_{\alpha'}^{-1}$ in $\wedge^2 V^{\perp}$ is $\omega_{V^{\perp}}$. It follows that

$$(16.2) \qquad \langle\!\langle \omega_{V^{\perp}}, V^{\perp} \rangle\!\rangle_{\beta} = \langle\!\langle A, B \rangle\!\rangle_{\beta}.$$

Combining (16.1) and (16.2) with Lemma 14.5, we see that

$$\langle\!\langle V, \omega_{V^{\perp}} \rangle\!\rangle_{\beta} = \langle\!\langle A, B \rangle\!\rangle_{\beta} = \langle\!\langle A, B \rangle\!\rangle_{\alpha} = \langle\!\langle \omega_{V^{\perp}}, V^{\perp} \rangle\!\rangle_{\alpha} = -\langle\!\langle \omega_{V^{\perp}}, V \rangle\!\rangle_{\alpha}. \qquad \square$$

17. Step 3.7: proof of Theorem B′

Theorem B′ asserts that Λ_g is a finite-dimensional algebraic representation of $\mathrm{Sp}(H_{\mathbb{Z}})$. As we will show in this final section, the generators and relations for Λ_g we constructed in the last few sections are enough to force this to hold.

17.1. **Presentation theorem.** The key is a recent theorem of the authors giving a presentation for what we call the symmetric kernel representation. Make the following definition:

Definition 17.1. Define $\mathfrak{K}(H)$ to be the vector space with the following presentation:

- **Generators**. For all genus-1 symplectic summands V of $H_{\mathbb{Z}}$ and all $\kappa \in \wedge^2 V_{\mathbb{Q}}^{\perp}$, generators $[\![V, \kappa]\!]$ and $[\![\kappa, V]\!]$.
- **Relations**. The following families of relations:
  - For all genus-1 symplectic summands V of $H_{\mathbb{Z}}$ and all $\kappa_1, \kappa_2 \in \wedge^2 V_{\mathbb{Q}}^{\perp}$ and all $\lambda_1, \lambda_2 \in \mathbb{Q}$, the linearity relations
    $$[\![V, \lambda_1 \kappa_1 + \lambda_2 \kappa_2]\!] = \lambda_1 [\![V, \kappa_1]\!] + \lambda_2 [\![V, \kappa_2]\!] \quad \text{and}$$
    $$[\![\lambda_1 \kappa_1 + \lambda_2 \kappa_2, V]\!] = \lambda_1 [\![\kappa_1, V]\!] + \lambda_2 [\![\kappa_2, V]\!].$$
  - For all orthogonal genus-1 symplectic summands V and W of $H_{\mathbb{Z}}$, the relation
    $$[\![V, \omega_W]\!] = [\![\omega_V, W]\!].$$
  - For all genus-1 symplectic summands V of $H_{\mathbb{Z}}$, the relation
    $$[\![V, \omega_{V^{\perp}}]\!] = [\![\omega_{V^{\perp}}, V]\!]. \qquad \square$$

The actions of $\mathrm{Sp}(H_{\mathbb{Z}})$ on $H_{\mathbb{Z}}$ and H induce an action of $\mathrm{Sp}(H_{\mathbb{Z}})$ on $\mathfrak{K}(H)$. We have:[50]

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[50]This theorem requires that H has genus at least 4, which follows from our assumption that $g \geq 5$ (Assumption 10.7) since H is the orthogonal complement in $\mathrm{H}_1(\Sigma_g; \mathbb{Q})$ of $\langle a, b \rangle$.

Theorem 17.2 ([37, Theorem A.6])**.** *The representation* $\mathfrak{K}(H)$ *is a finite-dimensional algebraic representation of* $\mathrm{Sp}(H_{\mathbb{Z}})$.

Remark 17.3. In fact, what [37] proves is that $\mathfrak{K}(H)$ is a subquotient of $(\wedge^2 H)^{\otimes}$. More precisely, let $(\wedge^2 H)/\mathbb{Q}$ be the quotient of $\wedge^2 H$ by the one-dimensional trivial representation spanned by the element ω representing the algebraic intersection pairing. For $\kappa \in \wedge^2 H$, let $\overline{\kappa}$ be its image in $(\wedge^2 H)/\mathbb{Q}$. Then [37] proves that the map $\Phi \colon \mathfrak{K}(H) \to ((\wedge^2 H)/\mathbb{Q})^{\otimes 2}$ defined by

$$\Phi(\llbracket V, \kappa \rrbracket) = \overline{\omega}_V \otimes \overline{\kappa} \quad \text{and} \quad \Phi(\llbracket \kappa, V \rrbracket) = \overline{\kappa} \otimes \overline{\omega}_V$$

is a well-defined isomorphism onto its image $\mathcal{K}(H)$, which [37] calls the *symmetric kernel*. It is the kernel of a contraction $((\wedge^2 H)/\mathbb{Q})^{\otimes 2} \longrightarrow \mathrm{Sym}^2(H)$. See [37] for more details. □

17.2. **The proof.** We close the paper by proving Theorem B$'$.

Theorem B$'$. *The vector space* Λ_g *is a finite-dimensional algebraic representation of* $\mathrm{Sp}(H_{\mathbb{Z}})$.

Proof. Theorem 17.2 says that $\mathfrak{K}(H)$ is a finite-dimensional algebraic representation of $\mathrm{Sp}(H_{\mathbb{Z}})$, so it is enough to construct an $\mathrm{Sp}(H_{\mathbb{Z}})$-equivariant surjection $\rho \colon \mathfrak{K}(H) \to \Lambda_g$.

We define ρ on generators, and then check that the corresponding map takes relations to relations. Let V be a genus-1 symplectic summand of $H_{\mathbb{Z}}$ and let $\kappa \in \wedge^2 V_{\mathbb{Q}}^{\perp}$. We then have generators $\llbracket V, \kappa \rrbracket$ and $\llbracket \kappa, V \rrbracket$ for $\mathfrak{K}(H)$. Define

$$\rho(\llbracket V, \kappa \rrbracket) = \langle\!\langle V, \kappa \rangle\!\rangle_{\beta} \quad \text{and} \quad \rho(\llbracket \kappa, V \rrbracket) = -\langle\!\langle \kappa, V \rangle\!\rangle_{\alpha}.$$

The minus sign is there to ensure:

Claim. *The map* ρ *takes relations to relations, and thus gives a well-defined map.*

Proof of claim. It follows from Lemma 14.3 that ρ respects the linearity relations, so we must check the other two:

- Let V and W be genus-1 symplectic summand of $H_{\mathbb{Z}}$ with $W \subset V^{\perp}$. We can then apply Lemma 16.1 to see that
$$\rho(\llbracket V, \omega_W \rrbracket) = \langle\!\langle V, \omega_W \rangle\!\rangle_{\beta} = -\langle\!\langle \omega_V, W \rangle\!\rangle_{\alpha} = \rho(\llbracket \omega_V, W \rrbracket).$$
- Let V be a genus-1 symplectic summand of $H_{\mathbb{Z}}$. We can apply Lemma 16.2 to see that
$$\rho(\llbracket V, \omega_{V^{\perp}} \rrbracket) = \langle\!\langle V, \omega_{V^{\perp}} \rangle\!\rangle_{\beta} = -\langle\!\langle \omega_{V^{\perp}}, V \rangle\!\rangle_{\alpha} = \rho_2(\llbracket \omega_{V^{\perp}}, V \rrbracket). \qquad \square$$

The map ρ is $\mathrm{Sp}(H_{\mathbb{Z}})$-equivariant by construction, and is surjective since its image contains all the generators for Λ_g identified by Lemma 15.1. The theorem follows. □

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