GENERATING THE HOMOLOGY OF COVERS OF SURFACES

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Abstract. Putman and Wieland conjectured that if $\widetilde{\Sigma} \to \Sigma$ is a finite branched cover between closed oriented surfaces of sufficiently high genus, then the orbits of all nonzero elements of $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ under the action of lifts to $\widetilde{\Sigma}$ of mapping classes on Σ are infinite. We prove that this holds if $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ is generated by the homology classes of lifts of simple closed curves on Σ. We also prove that the subspace of $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ spanned by such lifts is a symplectic subspace. Finally, simple closed curves lie on subsurfaces homeomorphic to 2-holed spheres, and we prove that $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ is generated by the homology classes of lifts of loops on Σ lying on subsurfaces homeomorphic to 3-holed spheres.

1. Introduction

Let $\pi\colon \widetilde{\Sigma} \to \Sigma$ be a finite branched cover between closed oriented surfaces. The homology of $\widetilde{\Sigma}$ encodes subtle information about the mapping class group of Σ, and over the last decade has been intensely studied [5, 9, 10, 11, 12, 13, 14, 15, 17, 18, 20]. Much of this is motivated by a conjecture of Putman–Wieland [20] we discuss below. In this note, we prove this conjecture for covers $\widetilde{\Sigma}$ such that $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ is generated by certain simple elements, and also prove that in general $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ is generated by slightly more complicated elements.

1.1. **Putman–Wieland conjecture.** Mark Σ at each branch point of the branched cover $\pi\colon \widetilde{\Sigma} \to \Sigma$. Let $\mathrm{Mod}(\Sigma)$ be the pure mapping class group of Σ, i.e., the group of isotopy classes of orientation-preserving homeomorphisms of Σ that fix each marked point. There is a finite-index subgroup $\mathrm{Mod}(\Sigma,\widetilde{\Sigma})$ of $\mathrm{Mod}(\Sigma)$ that can be lifted to $\widetilde{\Sigma}$ to give a well-defined action of $\mathrm{Mod}(\Sigma,\widetilde{\Sigma})$ on $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$. Putman–Wieland [20] made the following conjecture.

Conjecture 1.1 ([20]). *Let the notation be as above, and assume that the genus of Σ is sufficiently large.*[1] *Consider some nonzero $\vec{v} \in \mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$. Then the $\mathrm{Mod}(\Sigma,\widetilde{\Sigma})$-orbit of $\vec{v}$ is infinite.*

The main theorem of [20] says that this holds if and only if the virtual first Betti number of the mapping class group is 0 when the genus is sufficiently large, which is a well-known conjecture of Ivanov [7].

1.2. **Simple closed curve homology.** To prove Conjecture 1.1, it is natural to try to find generators for $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$. A first idea is that $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ might be generated by lifts of simple closed curves. Define the *simple closed curve homology* of $\widetilde{\Sigma}$, denoted $\mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Q})$, to be the subspace of $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ spanned by the homology classes of loops $\widetilde{\gamma}$ on $\widetilde{\Sigma}$ that avoid the branch points and project to simple closed curves γ on Σ. The restriction of the branched covering map $\pi\colon \widetilde{\Sigma} \to \Sigma$ to $\widetilde{\gamma}$ is thus a possibly nontrivial cover $\widetilde{\gamma} \to \gamma$.

Unfortunately, this need not be all of $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$. For all closed surfaces Σ with $\pi_1(\Sigma)$ nonabelian, Malestein–Putman [15, Theorem B] constructed finite branched covers $\widetilde{\Sigma} \to \Sigma$

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[1] In [20] they further conjecture that this holds if the genus is at least 2, but counterexamples in genus 2 were found by Marković [17].

such that $\mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Q}) \neq \mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$. More recently, Klukowski [9] constructed unbranched covers with this property.[2] Our first theorem is that Conjecture 1.1 does hold if if $\mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Q}) = \mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$.

Theorem A. *Let $\pi\colon \widetilde{\Sigma} \to \Sigma$ be a finite branched cover between closed oriented surfaces. Consider some nonzero $\vec{v} \in \mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Q})$. Then the $\mathrm{Mod}(\Sigma,\widetilde{\Sigma})$-orbit of $\vec{v}$ is infinite. In particular, if $\mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Q}) = \mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ then Conjecture 1.1 holds for $\pi\colon \widetilde{\Sigma} \to \Sigma$.*

This suggests that the examples from [15] and [9] might be good places to look for counterexamples to Conjecture 1.1.

1.3. **Relationship to previous work.** Conjecture 1.1 has been proved in a variety of cases; see,[3] e.g., [5, 11, 12, 13, 14]. However, we know very little about when $\mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Q}) = \mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$. The only general result we are aware of is that this holds when $\widetilde{\Sigma} \to \Sigma$ is a finite unbranched abelian cover. This is implicit in work of Looijenga [14], and we provide a self-contained proof in Proposition 5.3 below.[4] Beyond this, it is unclear which known cases of Conjecture 1.1 follow from Theorem A.

Remark 1.2. It would be interesting to extend this to prove that $\mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Q}) = \mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ for finite branched abelian covers. □

Next, let $\pi\colon \widetilde{\Sigma} \to \Sigma$ be one of the examples from [15] or [9] where $\mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Q}) \neq \mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$. Below in Theorem B we will prove that $\mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Q})$ is a symplectic subspace of $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$, so

$$\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q}) = \mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Q}) \oplus \mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Q})^{\perp}.$$

Theorem A says that the $\mathrm{Mod}(\Sigma,\widetilde{\Sigma})$-orbit of all nonzero $\vec{v} \in \mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Q})$ is infinite. It turns out that there are also nonzero $\vec{v} \in \mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Q})^{\perp}$ whose $\mathrm{Mod}(\Sigma,\widetilde{\Sigma})$-orbits are infinite. Indeed, letting D be the deck group of $\pi\colon \widetilde{\Sigma} \to \Sigma$, it follows from the constructions in [15] and [9] that some D-isotypic subspace V of $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ lies in $\mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Q})^{\perp}$, and Landesman–Litt [12] proved that some nonzero $\vec{v} \in V$ has an infinite $\mathrm{Mod}(\Sigma,\widetilde{\Sigma})$-orbit.

1.4. **Symplectic subspace.** We next clarify the nature of the subspace $\mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Q})$ of $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$. By Poincaré duality, the algebraic intersection form ω on $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ is a *symplectic form*, i.e., an alternating bilinear form that induces an isomorphism between $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ and its dual. A subspace V of $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ is a *symplectic subspace* if the restriction of ω to V is a symplectic form. We will prove the following:

Theorem B. *Let $\pi\colon \widetilde{\Sigma} \to \Sigma$ be a finite branched cover between closed oriented surfaces. Then $\mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Q})$ is a symplectic subspace of $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$.*

In fact, we will prove something more general. A *nontrivial simple closed curve* on Σ is a simple closed curve γ that avoids the marked points and does not bound a disk containing at most one marked point. We will always consider such curves up to isotopy.[5] The group

[2]Earlier Koberda–Santharoubane [10] constructed unbranched covers of closed surfaces with $\mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Z}) \neq \mathrm{H}_1(\widetilde{\Sigma};\mathbb{Z})$. This is weaker since it is possible that in their examples $\mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Z})$ is a finite-index subgroup of $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Z})$.

[3]Not all of these papers explicitly prove cases of Conjecture 1.1, but it can be deduced from their results in the cases they cover.

[4]Proposition 5.3 is stronger than this: it shows that $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ is spanned by lifts of *nonseparating* simple closed curves.

[5]These are isotopies through nontrivial simple closed curves, so during the isotopies the curves cannot pass through the marked points.

$\mathrm{Mod}(\Sigma)$ acts on the set of nontrivial simple closed curves on Σ, and the orbits of this action are the *topological types* of nontrivial simple closed curves.[6]

If σ is a set of topological types of nontrivial simple closed curves on Σ, then denote by $\mathrm{H}_1^{\sigma}(\widetilde{\Sigma};\mathbb{Q})$ the subspace of $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ spanned by the homology classes of loops $\widetilde{\gamma}$ on $\widetilde{\Sigma}$ that avoid the branch points and project to simple closed curves γ on Σ such that the topological type of γ lies in σ. For instance, if σ is the set of all topological types of nontrivial simple closed curves on Σ, then

$$\mathrm{H}_1^{\sigma}(\Sigma;\mathbb{Q}) = \mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Q}).$$

The following therefore generalizes Theorem B:

Theorem B′. *Let $\pi\colon \widetilde{\Sigma} \to \Sigma$ be a finite branched cover between closed oriented surfaces and σ be a set of topological types of nontrivial simple closed curves on Σ. Then $\mathrm{H}_1^{\sigma}(\widetilde{\Sigma};\mathbb{Q})$ is a symplectic subspace of $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$.*

We can also define $\mathrm{H}_1^{\sigma}(\widetilde{\Sigma};\mathbb{Z})$ and $\mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Z})$, and it is natural to wonder whether Theorems B and B′ hold integrally. For Theorem B′, the answer is no in general:

Theorem C. *Let Σ be a closed oriented surface of genus at least 2 and let σ be the set of nonseparating simple closed curves on Σ. Then there exists a finite unbranched cover $\pi\colon \widetilde{\Sigma} \to \Sigma$ such that $\mathrm{H}_1^{\sigma}(\widetilde{\Sigma};\mathbb{Z})$ is not a symplectic subspace of $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Z})$.*

Here $\mathrm{H}_1^{\sigma}(\widetilde{\Sigma};\mathbb{Z})$ is a free abelian group, and a symplectic form on a free abelian group A is an alternating $\mathbb{Z}$-valued bilinear form on A that identifies A with its dual $A^* = \mathrm{Hom}(A,\mathbb{Z})$. Unfortunately, our proof of Theorem C breaks down if we allow separating curves, so we cannot answer the following question:

Question 1.3. *Let $\pi\colon \widetilde{\Sigma} \to \Sigma$ be a finite branched cover between closed oriented surfaces. Is $\mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Z})$ a symplectic subspace of $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Z})$?*

However, Theorem C suggests that the answer to this should be "no".

Remark 1.4. An important ingredient in our proof of Theorem C is a theorem of Irmer [6] giving certain finite abelian covers $\pi\colon \widetilde{\Sigma} \to \Sigma$ for which $\mathrm{H}_1^{\sigma}(\widetilde{\Sigma};\mathbb{Z})$ is a proper subgroup of $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Z})$ (see Theorem 5.2.(ii) below). To make this paper more self-contained, we also include a simplified proof of this theorem. □

1.5. **Pants homology.** Regular neighborhoods of simple closed curves on Σ are homeomorphic to annuli, i.e., spheres with two boundary components. This suggests weakening the definition of simple closed curve homology as follows.

Recall that a *pair of pants* is a sphere with three holes. Define the *pants homology* of $\widetilde{\Sigma}$, denoted $\mathrm{H}_1^{\mathrm{pant}}(\widetilde{\Sigma};\mathbb{Q})$, to be the subspace of $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ spanned by the homology classes of loops $\widetilde{\gamma}$ on $\widetilde{\Sigma}$ such that there exists a subsurface $P \subset \Sigma$ homeomorphic to a pair of pants with $\pi(\widetilde{\gamma}) \subset P$. Since every simple closed curve on Σ is contained in some[7] such P, we have

$$\mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Q}) \subset \mathrm{H}_1^{\mathrm{pant}}(\widetilde{\Sigma};\mathbb{Q}).$$

Our final main theorem is as follows. It answers positively a question[8] of Kent [8].

[6] By the change of coordinates principle from [4, §1.3.2], the topological types are determined by the marked surface with boundary one gets by cutting Σ open along γ. For instance, one topological type is the set of all nonseparating γ.

[7] We do not require the boundary components of P to be non-nullhomotopic curves on Σ, so this even holds if Σ is a surface like a sphere that does not contain pairs of pants P whose boundary components are non-nullhomotopic.

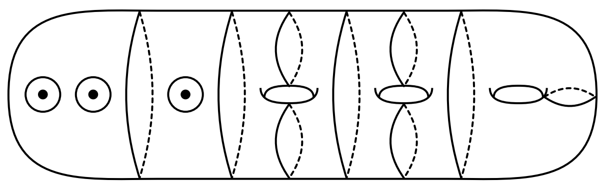
[8] Kent actually asked whether $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ is generated by lifts of elements that do not fill Σ, which is much weaker than lying in a pair of pants.

Theorem D. *Let $\pi\colon \widetilde{\Sigma} \to \Sigma$ be a finite branched cover between closed oriented surfaces. Then $\mathrm{H}_1^{\mathrm{pant}}(\widetilde{\Sigma};\mathbb{Q}) = \mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$.*

Remark 1.5. This also holds for punctured surfaces of finite type, which can be reduced to Theorem D as follows. Let $\pi\colon \widetilde{\Sigma} \to \Sigma$ be a finite branched cover between punctured surfaces of finite type. Filling in the punctures yields a finite branched cover between closed surfaces to which one can apply Theorem D. To conclude, note that filling in the punctures has the effect of killing the homology classes in $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ of loops around the punctures, which lie in $\mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma};\mathbb{Q}) \subset \mathrm{H}_1^{\mathrm{pant}}(\widetilde{\Sigma};\mathbb{Q})$. □

Remark 1.6. Theorem D might appear to contradict [15, Theorem C] and [9, Corollary 1.1.3], which give examples of finite covers $\pi\colon \widetilde{\Sigma} \to \Sigma$ such that $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ is not spanned by the homology classes of loops $\widetilde{\gamma}$ such that $\pi(\widetilde{\gamma})$ is not in any given finite set of mapping class group[9] orbits of curves. However, in the definition of $\mathrm{H}_1^{\mathrm{pant}}(\widetilde{\Sigma};\mathbb{Q})$ there is no restriction on the number of self-intersections of the projections of the curves to the P, so they do not fall into finitely many mapping class group orbits. □

The proof of Theorem D actually shows something stronger. A *pants decomposition* of Σ is a collection $\mathcal{P} = \{\delta_1, \ldots, \delta_n\}$ of disjoint simple closed curves on Σ that avoid the branch points such that each component of $\Sigma \setminus \cup_{j=1}^{n} \delta_j$ is either a disk containing a single branch point or a pair of pants containing no branch points:



We will prove the following, which implies Theorem D:

Theorem D′. *Let $\pi\colon \widetilde{\Sigma} \to \Sigma$ be a finite branched cover between closed oriented surfaces and $\mathcal{P}$ be a pants decomposition of Σ. Let σ be the set of topological types of nontrivial curves appearing in $\mathcal{P}$. Then $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ is spanned by $\mathrm{H}_1^{\sigma}(\widetilde{\Sigma};\mathbb{Q})$ and the set of homology classes of cycles $\widetilde{\gamma}$ on $\widetilde{\Sigma}$ such that $\pi(\widetilde{\gamma})$ is disjoint from all curves in $\mathcal{P}$.*

We can also define $\mathrm{H}_1^{\mathrm{pant}}(\widetilde{\Sigma};\mathbb{Z})$, and pose the following question:

Question 1.7. *Let $\pi\colon \widetilde{\Sigma} \to \Sigma$ be a finite branched cover between closed oriented surfaces. Is $\mathrm{H}_1^{\mathrm{pant}}(\widetilde{\Sigma};\mathbb{Z}) = \mathrm{H}_1(\widetilde{\Sigma};\mathbb{Z})$?*

Our proof of Theorems D and D′ shows that Question 1.7 has a positive answer if Question 1.3 does. Though we expect that Question 1.3 has a negative answer, we do not know what answer to expect for Question 1.7.

1.6. **Outline.** We prove Theorem A in §2, Theorems B and B′ in §3, Theorems D and D′ in §4, and Theorem C in §5.

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2. Simple closed curve homology and Dehn twists

This section contains the proof of Theorem A.

[9] or even automorphism group of free group

2.1. **Notation.** Fix a closed oriented surface Σ and a finite branched cover $\pi\colon \widetilde{\Sigma} \to \Sigma$. The surface $\widetilde{\Sigma}$ is thus also a closed oriented surface. Let $g \geq 0$ be its genus, let

$$H = \mathrm{H}_1(\widetilde{\Sigma}; \mathbb{Q}) \cong \mathbb{Q}^{2g},$$

and let ω be the algebraic intersection form on H. By Poincaré duality, ω is a symplectic form, i.e., an alternating form that induces an isomorphism between H and its dual. The symplectic group $\mathrm{Sp}(H, \omega) \cong \mathrm{Sp}_{2g}(\mathbb{Q})$ acts on H.

2.2. **Lifting Dehn twists.** Recall from §1.4 that a nontrivial simple closed curve on Σ is a simple closed curve that avoids the marked points and does not bound a disk containing at most one marked point. Consider a nontrivial simple closed curve γ on Σ. The preimage $\pi^{-1}(\gamma)$ is a disjoint union of simple closed curves. Enumerate them as

$$\pi^{-1}(\gamma) = \widetilde{\gamma}_1 \sqcup \cdots \sqcup \widetilde{\gamma}_k.$$

For each $1 \leq j \leq k$, the map

$$\pi|_{\widetilde{\gamma}_j} \colon \widetilde{\gamma}_j \to \gamma$$

is a finite unbranched cover. Let d_j be its degree. Set

$$d(\gamma) = \mathrm{lcm}(d_1, \ldots, d_k) \quad \text{and} \quad e_j = d(\gamma)/d_j \quad \text{for } 1 \leq j \leq k. \tag{2.1}$$

If T_γ and $T_{\widetilde{\gamma}_j}$ denote the Dehn twists about γ and $\widetilde{\gamma}_j$, then $T_\gamma^{d(\gamma)}$ lifts to the product

$$T_{\widetilde{\gamma}_1}^{e_1} \cdots T_{\widetilde{\gamma}_k}^{e_k}.$$

Let $\widetilde{\tau}_\gamma$ be the image of this product of powers of Dehn twists in $\mathrm{Sp}(H, \omega) \cong \mathrm{Sp}_{2g}(\mathbb{Q})$. The element $\widetilde{\tau}_\gamma$ acts on H as follows:

$$\widetilde{\tau}_\gamma(h) = h + \sum_{j=1}^{k} e_j \omega(h, [\widetilde{\gamma}_j]) \cdot [\widetilde{\gamma}_j] \quad \text{for } h \in H.$$

For a set σ of topological types of nontrivial simple closed curves on Σ, define D_σ to be the subgroup of $\mathrm{Sp}(H, \omega)$ generated by the set of all $\widetilde{\tau}_\gamma$ as γ ranges over nontrivial simple closed curves on Σ whose topological type lies in σ.

2.3. **Fixed set of lifted twists.** As in §1.4, let σ be a set of topological types of nontrivial simple closed curves on Σ and define

$$H^\sigma = \mathrm{H}_1^\sigma(\widetilde{\Sigma}; \mathbb{Q}) \subset \mathrm{H}_1(\widetilde{\Sigma}; \mathbb{Q}) = H.$$

The following lemma will be fundamental to our paper:

Lemma 2.1. *Let the notation be as above. Then*[10] H^{D_σ} *equals the orthogonal complement* $(H^\sigma)^\perp$ *of* H^σ *with respect to* ω.

Proof. Let γ be a nontrivial simple closed curve on Σ whose topological type lies in σ. As we did in §2.2 above, write $\pi^{-1}(\gamma)$ as a disjoint union of simple closed curves on $\widetilde{\Sigma}$:

$$\pi^{-1}(\gamma) = \widetilde{\gamma}_1 \sqcup \cdots \sqcup \widetilde{\gamma}_k.$$

As in that section, there are positive integers $e_1, \ldots, e_k$ such that the generator $\widetilde{\tau}_\gamma \in D_\sigma$ acts on H as follows:

$$\widetilde{\tau}_\gamma(h) = h + \sum_{j=1}^{k} e_j \omega(h, [\widetilde{\gamma}_j]) \cdot [\widetilde{\gamma}_j] \quad \text{for } h \in H.$$

[10] Here the superscript indicates that we are taking invariants: $H^{D_\sigma} = \{h \in H \mid d \cdot h = h \text{ for all } d \in D_\sigma\}$.

Each $[\widetilde{\gamma}_j]$ lies in H^σ, so it is immediate from this formula that $(H^\sigma)^\perp \subset H^{D_\sigma}$. For the other inclusion, consider some $h_0 \in H^{D_\sigma}$. We then know that $\widetilde{\tau}_\gamma(h_0) = h_0$, so from the above

$$\sum_{j=1}^{k} e_j \omega(h_0, [\widetilde{\gamma}_j]) \cdot [\widetilde{\gamma}_j] = 0.$$

Taking the algebraic intersection with h_0, we deduce that

$$\sum_{j=1}^{k} e_j \omega(h_0, [\widetilde{\gamma}_j])^2 = 0.$$

Since $e_j \geq 1$ for all $1 \leq j \leq k$, this implies that $\omega(h_0, [\widetilde{\gamma}_j]) = 0$ for all $1 \leq j \leq k$. This holds for all choices of γ and all components of the preimage $\pi^{-1}(\gamma)$. These generate H^σ, so we conclude that $h_0 \in (H^\sigma)^\perp$, as desired. □

2.4. **Putman–Wieland conjecture.** We now prove Theorem A.

Proof of Theorem A. We start by recalling the statement. Let $\pi\colon \widetilde{\Sigma} \to \Sigma$ be a finite branched cover between closed oriented surfaces. Let $\mathrm{Mod}(\Sigma, \widetilde{\Sigma})$ be the subgroup of the mapping class group $\mathrm{Mod}(\Sigma)$ that lifts to $\widetilde{\Sigma}$. Consider some nonzero $\vec{v} \in \mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma}; \mathbb{Q})$. Our goal is to prove the $\mathrm{Mod}(\Sigma, \widetilde{\Sigma})$-orbit of $\vec{v}$ is infinite.

Let σ be the set of all topological types of nontrivial simple closed curves on Σ. Since $\mathrm{H}_1^{\mathrm{scc}}(\widetilde{\Sigma}; \mathbb{Q}) = \mathrm{H}_1(\widetilde{\Sigma}; \mathbb{Q})$, Lemma 2.1 implies that $\vec{v}$ is not fixed by the group D_σ, so there exists some nontrivial simple closed curve γ on Σ such that $\widetilde{\tau}_\gamma(\vec{v}) \neq \vec{v}$. Some power of $\widetilde{\tau}_\gamma$ is the image in the symplectic group of an element of $\mathrm{Mod}(\Sigma, \widetilde{\Sigma})$, so it is enough to prove that the elements $\widetilde{\tau}_\gamma^n(\vec{v})$ as n ranges over $\mathbb{Z}$ are all distinct.

Write $\pi^{-1}(\gamma)$ as a disjoint union of simple closed curves on $\widetilde{\Sigma}$:

$$\pi^{-1}(\gamma) = \widetilde{\gamma}_1 \sqcup \cdots \sqcup \widetilde{\gamma}_k.$$

There are then positive integers $e_1, \ldots, e_k$ such that

$$\widetilde{\tau}_\gamma(\vec{v}) = \vec{v} + \sum_{j=1}^{k} e_j \omega(\vec{v}, [\widetilde{\gamma}_j]) \cdot [\widetilde{\gamma}_j].$$

Setting

$$\vec{w} = \sum_{j=1}^{k} e_j \omega(\vec{v}, [\widetilde{\gamma}_j]) \cdot [\widetilde{\gamma}_j],$$

the fact that $\widetilde{\tau}_\gamma(\vec{v}) \neq \vec{v}$ implies that $\vec{w} \neq 0$. For $n \in \mathbb{Z}$, we have $\widetilde{\tau}_\gamma^n(\vec{v}) = \vec{v} + n\vec{w}$. Since $\vec{w} \neq 0$, the elements $\vec{v} + n\vec{w}$ as n ranges over $\mathbb{Z}$ are all distinct, as desired. □

3. The symplectic nature of simple closed curves

This section contains the proof of Theorem B′ (which generalizes Theorem B).

3.1. **Notation.** The notation is similar to that of §2:

- $\pi\colon \widetilde{\Sigma} \to \Sigma$ is a finite branched cover between closed oriented surfaces, and g is the genus of $\widetilde{\Sigma}$.
- $H = \mathrm{H}_1(\widetilde{\Sigma}; \mathbb{Q}) \cong \mathbb{Q}^{2g}$, and ω is the algebraic intersection form on H.
- σ is a set of topological types of nontrivial simple closed curves on Σ.
- D_σ is the subgroup of $\mathrm{Sp}(H, \omega) \cong \mathrm{Sp}_{2g}(\mathbb{Q})$ generated by the elements $\widetilde{\tau}_\gamma$ as γ ranges over nontrivial simple closed curves on Σ whose topological type lies in σ.

3.2. **Symplectic criterion.** We will need the following criterion for a subspace of H to be a symplectic subspace:

Lemma 3.1. *Let the notation be as above, and let D be a subgroup of $\mathrm{Sp}(H, \omega)$. Assume that the action of D on H is semisimple.*[11] *Then*[12] *H^D is a symplectic subspace of H.*

This is well-known; see, e.g., [3, Lemme 4.14]. For completeness, we include a proof.

Proof of Lemma 3.1. The symplectic form ω induces a D-equivariant isomorphism

$$\phi\colon H \xrightarrow{\cong} H^*.$$

We want to prove ω also induces an isomorphism between H^D and $(H^D)^*$. Letting $\iota\colon H^D \hookrightarrow H$ be the inclusion and letting $\iota^*\colon H^* \to (H^D)^*$ be its dual,[13] our goal is equivalent to proving that the composition

$$H^D \xrightarrow{\iota} H \xrightarrow{\phi} H^* \xrightarrow{\iota^*} (H^D)^* \tag{3.1}$$

is an isomorphism.

Since ϕ is a D-equivariant isomorphism, it restricts to an isomorphism on D-invariants, i.e., an isomorphism

$$\phi^D\colon H^D \xrightarrow{\cong} (H^*)^D.$$

A linear map $\lambda\colon H \to \mathbb{Q}$ in H^* is D-invariant if and only if it factors through the D-coinvariants

$$H_D = H/\langle d \cdot h - h \mid d \in D \text{ and } h \in H\rangle.$$

We thus get an isomorphism

$$\mu\colon\ (H^*)^D \xrightarrow{\cong} (H_D)^*.$$

The projection $H \to H_D$ restricts to a map $\eta\colon H^D \to H_D$. Since the action of D is semisimple, the map η is an isomorphism.[14] Taking its dual, we get an isomorphism

$$\eta^*\colon\ (H_D)^* \xrightarrow{\cong} \left(H^D\right)^*.$$

The composition

$$H^D \xrightarrow{\phi^D} (H^*)^D \xrightarrow{\mu} (H_D)^* \xrightarrow{\eta^*} \left(H^D\right)^* \tag{3.2}$$

of isomorphisms is an isomorphism, and reflecting on the maps we see that the compositions (3.1) and (3.2) are the same. We conclude that (3.1) is an isomorphism, as desired. □

3.3. **Semisimplicity.** Our goal is to apply Lemma 3.1 to the group D_σ from §3.1, which requires verifying the following:

Lemma 3.2. *Let the notation be as above. Then the group D_σ acts semisimply on H.*

Proof. Let $\mathbf{D}_\sigma$ be the Zariski closure of D_σ in $\mathrm{Sp}(H, \omega)$. It is enough to prove that $\mathbf{D}_\sigma$ acts semisimply on H. For this, it is enough to prove that $\mathbf{D}_\sigma$ is a semisimple algebraic group.[15]

Regard Σ as a closed surface with marked points at the branch points of $\pi\colon \widetilde{\Sigma} \to \Sigma$. To simplify things, if there are no branch points introduce a single additional marked point on

[11] That is, the D-representation H decomposes as a direct sum of irreducible representations.

[12] Here just like in Lemma 2.1 the superscript indicates we are taking invariants.

[13] This dual restricts a linear map $\lambda\colon H \to \mathbb{Q}$ to H^D.

[14] Indeed, if $H = H^D \oplus V_1 \oplus \cdots \oplus V_n$ with the V_i nontrivial irreducible representations of D, then $H_D = (H^D)_D \oplus (V_1)_D \oplus \cdots \oplus (V_n)_D = H^D \oplus 0 \oplus \cdots \oplus 0 = H^D$.

[15] See [1, 19] for textbook references on algebraic groups. One key property of semisimple algebraic groups over $\mathbb{Q}$ is that all of their finite-dimensional representations are semisimple [19, Proposition 22.41].

Σ, and regard its preimage in $\widetilde{\Sigma}$ as a collection of branch points of order 1. Let $\mathcal{M}(\Sigma)$ be the moduli space of Riemann surfaces S with marked points such that $S \cong \Sigma$ as surfaces with marked points. The (orbifold) fundamental group of $\mathcal{M}(\Sigma)$ is thus the mapping class group $\mathrm{Mod}(\Sigma)$.

We can find a finite-index subgroup Γ of $\mathrm{Mod}(\Sigma)$ such that each element of Γ can be lifted to a homeomorphism of $\widetilde{\Sigma}$ fixing all the marked points. Since there is at least one marked point, these lifts are unique up to homotopy, so Γ acts on $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ in a well-defined way. Shrinking Γ if necessary, we can also assume that Γ is torsion-free. Let $\mathcal{M}_\Gamma(\Sigma)$ be the cover of $\mathcal{M}(\Sigma)$ corresponding to Γ.

Since Γ is torsion-free, $\mathcal{M}_\Gamma(\Sigma)$ is a fine moduli space. It thus has a universal curve $\mathcal{U} \to \mathcal{M}_\Gamma(\Sigma)$ whose fiber over $S \in \mathcal{M}_\Gamma(\Sigma)$ is S. Replacing Γ by a deeper finite-index subgroup if necessary, we can find a fiberwise branched cover $\widetilde{\mathcal{U}} \to \mathcal{M}_\Gamma(\Sigma)$ of $\mathcal{U} \to \mathcal{M}_\Gamma(\Sigma)$ whose fibers are the branched cover $\widetilde{\Sigma}$ of Σ.

The monodromy representation of $\pi_1(\mathcal{M}_\Gamma(\Sigma)) \cong \Gamma$ on H_1 of the fibers is thus exactly the action of Γ on $H = \mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ obtained by lifting mapping classes through the branched cover $\widetilde{\Sigma} \to \Sigma$. The image of this representation lies in $\mathrm{Sp}(H,\omega)$. Let $\mathbf{G}$ be the Zariski closure in $\mathrm{Sp}(H,\omega)$ of the image of Γ. Deligne's semisimplicity theorem [2, Corollaire 4.2.9] implies that $\mathbf{G}$ is a semisimple algebraic group.

From the definition (2.1) of $d(\gamma)$ for nontrivial simple closed curves γ on Σ, it only achieves finitely many values (depending on the degree of the cover $\widetilde{\Sigma} \to \Sigma$). Pick some $m \geq 1$ such that the following two properties hold for each nontrivial simple closed curve γ whose topological type lies in σ:

- $d(\gamma)$ divides m.
- $T_\gamma^m \in \Gamma$.

Let E be the subgroup of $\mathrm{Mod}(\Sigma)$ generated by all the T_γ^m as γ ranges over nontrivial simple closed curves on Σ whose topological type lies in σ. For such a γ, we have

$$f T_\gamma^m f^{-1} = T_{f(\gamma)}^m \quad \text{for all } f \in \mathrm{Mod}(\Sigma).$$

It follows that E is a normal subgroup of $\mathrm{Mod}(\Sigma)$. By construction, $E \subset \Gamma$. For each nontrivial simple closed curve γ on Σ whose topological type lies in σ, recall that $\widetilde{\tau}_\gamma$ is the image of $T_\gamma^{d(\gamma)}$ in $\mathrm{Sp}(H,\omega)$. The Zariski closure in $\mathrm{Sp}(H,\omega)$ of the subgroup generated by $\widetilde{\tau}_\gamma$ is[16] the one-parameter subgroup $\widetilde{\tau}_{\gamma,t}$ defined by

$$\widetilde{\tau}_{\gamma,t}(h) = h + \sum_{j=1}^{k} t e_j \omega(h, [\widetilde{\gamma}_j]) \cdot [\widetilde{\gamma}_j] \quad \text{for } h \in H \text{ and } t \in \mathbb{Q}.$$

The group $\mathbf{D}_\sigma$ is generated by these one-parameter subgroups.[17] Since one-parameter subgroups are connected, we deduce that $\mathbf{D}_\sigma$ is connected.

The Zariski closure in $\mathrm{Sp}(H,\omega)$ of the subgroup generated by

$$T_\gamma^m = \left(T_\gamma^{d(\gamma)}\right)^{m/d(\gamma)}$$

is the same one-parameter subgroup $\widetilde{\tau}_{\gamma,t}$. It follows that the Zariski closure of the image of E in $\mathrm{Sp}(H,\omega)$ is also $\mathbf{D}_\sigma$. Since E is a normal subgroup of $\mathrm{Mod}(\Sigma)$, it follows that $\mathbf{D}_\sigma$ is a normal subgroup of $\mathbf{G}$. Since $\mathbf{G}$ is semisimple, so[18] is $\mathbf{D}_\sigma$, as desired. □

[16] The point here is that the subgroup generated by $\widetilde{\tau}_\gamma$ is the integer points in the one-parameter subgroup $\widetilde{\tau}_{\gamma,t}$, and the Zariski closure of $\mathbb{Z}$ in $\mathbb{Q}$ is $\mathbb{Q}$.

[17] This uses the fact that the subgroup of an algebraic group generated by a set of algebraic subgroups is algebraic, i.e., Zariski closed [1, Proposition 2.2].

[18] Every connected normal subgroup of a semisimple algebraic group is semisimple [19, Theorem 21.51].

3.4. **Symplectic subspace.** We now prove Theorem B$'$.

Proof of Theorem B$'$. The statement we must prove is as follows. Let $\pi\colon \widetilde{\Sigma} \to \Sigma$ be a finite branched cover between closed oriented surfaces and σ be a set of topological types of nontrivial simple closed curves on Σ. We must show that $H^\sigma = \mathrm{H}_1^\sigma(\widetilde{\Sigma};\mathbb{Q})$ is a symplectic subspace of $H = \mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$, or equivalently that $(H^\sigma)^\perp$ is a symplectic subspace. Lemma 2.1 implies that

$$(H^\sigma)^\perp = H^{D_\sigma},$$

and Lemma 3.2 says that the group D_σ acts semisimply on H. The result thus follows from Lemma 3.1. □

4. Pants homology

In this section, we prove Theorem D$'$ (which implies Theorem D).

Proof of Theorem D$'$. We first recall the statement. Let $\pi\colon \widetilde{\Sigma} \to \Sigma$ be a finite branched cover between closed oriented surfaces and $\mathcal{P}$ be a pants decomposition of Σ. Let σ be the set of topological types of nontrivial curves appearing in $\mathcal{P}$. We must prove that $H = \mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ is spanned by $H^\sigma = \mathrm{H}_1^\sigma(\widetilde{\Sigma};\mathbb{Q})$ and the set of homology classes of cycles $\widetilde{\gamma}$ on $\widetilde{\Sigma}$ such that $\pi(\widetilde{\gamma})$ is disjoint from all curves in $\mathcal{P}$.

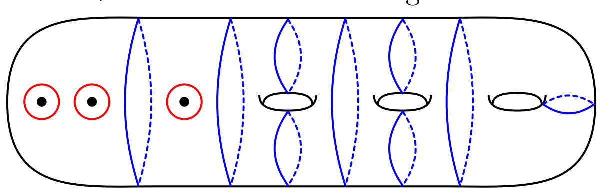
Theorem B$'$ says that H^σ is a symplectic subspace of H, so

$$H = H^\sigma \oplus (H^\sigma)^\perp.$$

It is thus enough to prove that $(H^\sigma)^\perp$ is spanned by the homology classes of cycles $\widetilde{\gamma}$ on $\widetilde{\Sigma}$ such that $\pi(\widetilde{\gamma})$ is disjoint from all the curves in $\mathcal{P}$.

Recall that we are working with homology with rational coefficients. Every element of H (and hence $(H^\sigma)^\perp$) is a multiple of an integral class, and every integral class can be represented by an oriented multicurve. Therefore, consider an oriented multicurve $\widetilde{\gamma}$ on $\widetilde{\Sigma}$ such that $[\widetilde{\gamma}] \in (H^\sigma)^\perp$. It is enough to prove that $\widetilde{\gamma}$ is homologous to an oriented multicurve $\widetilde{\gamma}'$ such that $\pi(\widetilde{\gamma})$ is disjoint from all the curves in $\mathcal{P}$.

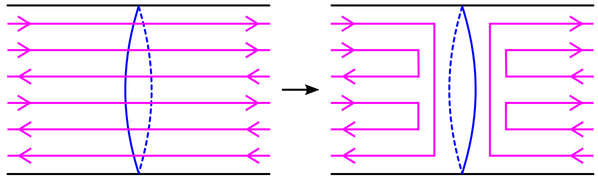
Our pants decomposition $\mathcal{P}$ looks like the following:



Write $\mathcal{P} = \{\delta_1, \ldots, \delta_n\}$. Call the δ_j that bound disks containing marked points the *boundary loops* (red in the above figure) and the other δ_j the *interior loops* (blue in the above figure). Enumerate the components of $\pi^{-1}(\delta_j)$ as j ranges over $1 \leq j \leq n$ as $\{\widetilde{\delta}_1, \ldots, \widetilde{\delta}_m\}$. Call the $\widetilde{\delta}_j$ that project to boundary loops the *lifted boundary loops* and the $\widetilde{\delta}_j$ that project to interior loops the *lifted interior loops*.

Put the oriented multicurve $\widetilde{\gamma}$ in general position with respect to the $\widetilde{\delta}_j$. The lifted boundary loops bound disks in $\widetilde{\Sigma}$ containing a single branch point. Isotope $\widetilde{\gamma}$ such that it is disjoint from all these disks, and in particular is disjoint from all the lifted boundary loops.

Let ω be the algebraic intersection form. Consider a lifted interior loop $\widetilde{\delta}_j$. Since $[\widetilde{\delta}_j] \in H^\sigma$ and $[\widetilde{\gamma}] \in (H^\sigma)^\perp$, we have $\omega([\widetilde{\gamma}], [\widetilde{\delta}_j]) = 0$. This implies that the number of positively oriented intersection points of $\widetilde{\gamma}$ with $\widetilde{\delta}_j$ is the same as the number of negatively oriented intersection points. We can then modify $\widetilde{\gamma}$ as follows to make it disjoint from $\widetilde{\delta}_j$:



The result is an oriented multicurve that is homologous to $\widetilde{\gamma}$. Doing this for each lifted interior loop, we obtain an oriented multicurve $\widetilde{\gamma}'$ such that $[\widetilde{\gamma}'] = [\widetilde{\gamma}]$ and such that $\widetilde{\gamma}'$ is disjoint from all the $\widetilde{\delta}_j$ and does not lie in any of the disks bounded by lifted boundary loops. This implies $\pi(\widetilde{\gamma})$ is disjoint from all the curves in $\mathcal{P}$, as desired. □

5. A non-symplectic example

This section contains the proof of Theorem C, which asserts that for all closed oriented surfaces Σ of genus at least 2, there exists a finite unbranched cover $\pi\colon \widetilde{\Sigma} \to \Sigma$ such that for σ the set of nonseparating simple closed curves on Σ, the subspace $\mathrm{H}_1^{\sigma}(\widetilde{\Sigma};\mathbb{Z})$ is not a symplectic subspace of $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Z})$..

5.1. **Reduction.** We start with the following.

Lemma 5.1. *Let V be a finitely generated free abelian group equipped with a symplectic form and let W be a subgroup of V. Assume that W is a symplectic subspace of V and that $W \otimes_{\mathbb{Z}} \mathbb{Q} = V \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $W = V$.*

Proof. Since W is a symplectic subspace of V, we have $V = W \oplus W^{\perp}$. Since

$$W \otimes_{\mathbb{Z}} \mathbb{Q} = V \otimes_{\mathbb{Z}} \mathbb{Q} = (W \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus \left(W^{\perp} \otimes_{\mathbb{Z}} \mathbb{Q}\right),$$

it follows that $W^{\perp} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. We conclude that $W^{\perp} = 0$ and thus that $W = V$. □

It is therefore enough to construct a finite unbranched cover $\pi\colon \widetilde{\Sigma} \to \Sigma$ such that

$$\mathrm{H}_1^{\sigma}(\widetilde{\Sigma};\mathbb{Q}) = \mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q}) \quad \text{but} \quad \mathrm{H}_1^{\sigma}(\widetilde{\Sigma};\mathbb{Z}) \neq \mathrm{H}_1(\widetilde{\Sigma};\mathbb{Z}).$$

For $\ell \geq 2$, let $\pi\colon \Sigma[\ell] \to \Sigma$ be the cover corresponding to the homomorphism

$$\pi_1(\Sigma) \longrightarrow \mathrm{H}_1(\Sigma;\mathbb{Z}/\ell).$$

By the above, it is enough to prove the following theorem, which we will do in the remainder of this section:

Theorem 5.2. *Let Σ be a closed oriented surface of genus at least 2 and σ be the set of nonseparating simple closed curves on Σ. Fix some $\ell \geq 2$. The following then hold:*

(i) We have $\mathrm{H}_1^{\sigma}(\Sigma[\ell];\mathbb{Q}) = \mathrm{H}_1(\Sigma[\ell];\mathbb{Q})$.
(ii) If $\ell \geq 3$, then $\mathrm{H}_1^{\sigma}(\Sigma[\ell];\mathbb{Z}) \neq \mathrm{H}_1(\Sigma[\ell];\mathbb{Z})$.

Part (ii) is a theorem of Irmer [6, Lemma 6]. We will give a simplified version of her argument below that avoids most of its complicated combinatorial group theory.

5.2. **Rational equality.** We start by proving part (i) of Theorem 5.2. In fact, we prove a more general result:

Proposition 5.3. *Let Σ be a closed surface and σ be the set of nonseparating simple closed curves on Σ. Let $\widetilde{\Sigma} \to \Sigma$ be a finite unbranched abelian cover. Then $\mathrm{H}_1^{\sigma}(\widetilde{\Sigma};\mathbb{Q}) = \mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$.*

Proof. It is enough to prove that $\mathrm{H}_1^\sigma(\widetilde{\Sigma};\mathbb{Q})^\perp = 0$. During the proof of Theorem B′, we showed that

$$\mathrm{H}_1^\sigma(\widetilde{\Sigma};\mathbb{Q})^\perp = \mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})^{D_\sigma}.$$

It is thus enough to show that the group D_σ fixes no nonzero vectors in $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$. In fact, the D_σ-orbits of all nonzero vectors in $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ are infinite. This follows from work of Looijenga [14], but since it is only implicit in [14] we give a complete proof.

Let G be the deck group of the finite abelian cover $\widetilde{\Sigma} \to \Sigma$. The actions of G and D_σ on $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ commute, so the action of D_σ preserves the decomposition of $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$ into G-isotypic components. Let V be an irreducible representation of G over $\mathbb{Q}$ and let W be the V-isotypic component of $\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})$. We must prove that the D_σ-orbits of all nonzero vectors in W are infinite.

Since V is an irreducible representation of the finite abelian group G, there is a finite cyclic quotient $\phi\colon G \to \mathbb{Z}/d$ such that[19] action of G on V factors through ϕ. Let

$$\Sigma[\phi] = \widetilde{\Sigma}/\ker(\phi),$$

and let $\pi\colon \Sigma[\phi] \to \Sigma$ be the projection, so $\pi\colon \Sigma[\phi] \to \Sigma$ is the degree-d cyclic cover corresponding to $\ker(\phi)$. The group $\ker(\phi)$ acts trivially on V and hence on W, so W is a subrepresentation of[20]

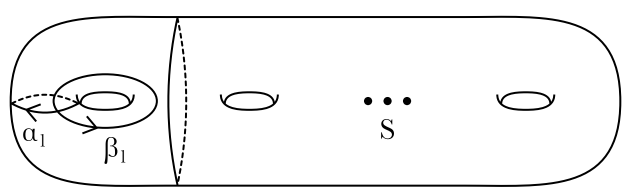
$$\mathrm{H}_1(\widetilde{\Sigma};\mathbb{Q})_{\ker(\phi)} \cong \mathrm{H}_1(\widetilde{\Sigma}/\ker(\phi);\mathbb{Q}) \cong \mathrm{H}_1(\Sigma[\phi];\mathbb{Q}).$$

Letting $v \in \mathrm{H}_1(\Sigma[\phi];\mathbb{Q})$ be nonzero, it is thus enough to prove that the D_σ-orbit of v is infinite.

Let ω be the algebraic intersection form on $\mathrm{H}_1(\Sigma;\mathbb{Q})$. There is a surjection $\mathrm{H}_1(\Sigma;\mathbb{Z}) \to G$. Pick a surjection $\widetilde{\phi}\colon \mathrm{H}_1(\Sigma;\mathbb{Z}) \to \mathbb{Z}$ making the diagram

$$\begin{tikzcd} \mathrm{H}_1(\Sigma;\mathbb{Z}) \arrow[r, two heads, "\widetilde{\phi}"] \arrow[d, two heads] & \mathbb{Z} \arrow[d, two heads] \\ G \arrow[r, two heads, "\phi"] & \mathbb{Z}/d \end{tikzcd}$$

commute. Since ω is a symplectic form on $\mathrm{H}_1(\Sigma;\mathbb{Z})$, there exists some $a_1 \in \mathrm{H}_1(\Sigma;\mathbb{Z})$ such that $\widetilde{\phi}(x) = \omega(a_1, x)$ for all $x \in \mathrm{H}_1(\Sigma;\mathbb{Z})$. Since $\widetilde{\phi}$ is surjective, a_1 is primitive[21] and thus there exists some oriented nonseparating simple closed curve α_1 such that $[\alpha_1] = a_1$. Let β_1 and S be as follows:

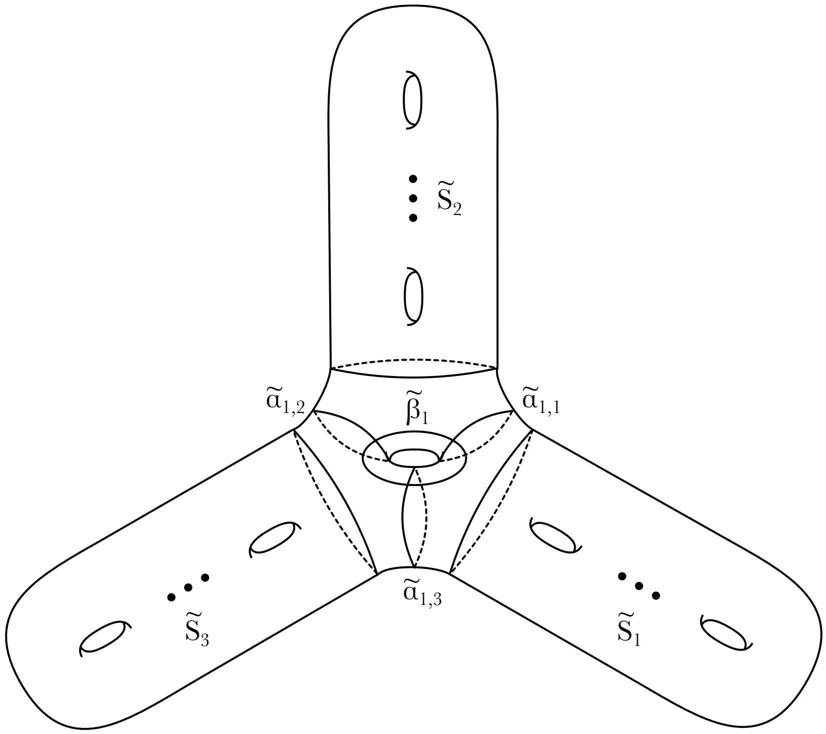


[19]Here is a sketch of this standard fact. Since G is abelian, the action of G on V comes from a homomorphism $\iota\colon G \to \mathrm{End}_G(V)$. Since V is irreducible, Schur's Lemma says that $\mathrm{End}_G(V)$ is a division algebra over $\mathbb{Q}$. Let F be the $\mathbb{Q}$-subalgebra of $\mathrm{End}_G(V)$ generated by $\mathrm{Im}(\iota)$. It is an easy exercise to show that for $f \in \mathrm{End}_G(V)$ nonzero, f^{-1} can be expressed as a polynomial in f. It follows that F is closed under taking multiplicative inverses. Since G is abelian, this implies that F is a commutative division ring, i.e., a field. The result now follows from the fact that a finite subgroup of $F^\times$ like the image of $\iota\colon G \to \mathrm{End}_G(V)$ must be cyclic.

[20]Here the subscript indicates that we are taking the $\ker(\phi)$-coinvariants and the first isomorphism follows from the transfer map.

[21]That is, not divisible by any integers greater than 1.

The case $d = 3$ of our cover $\pi\colon \Sigma[\phi] \to \Sigma$ is then as follows:



More generally, we have

- $\pi^{-1}(S) = \widetilde{S}_1 \sqcup \cdots \sqcup \widetilde{S}_d$ with each $\widetilde{S}_i$ projecting homeomorphically to S; and
- $\pi^{-1}(\beta_1) = \widetilde{\beta}_1$, where $\widetilde{\beta}_1$ is a simple closed curve that d-fold covers β_1; and
- $\pi^{-1}(\alpha_1) = \widetilde{\alpha}_{1,1} \sqcup \cdots \sqcup \widetilde{\alpha}_{1,d}$, where $\widetilde{\alpha}_{1,i}$ is a simple closed curve projecting homeomorphically to α_1.

The curves $\widetilde{\alpha}_{1,i}$ are all homologous, and we have

$$\mathrm{H}_1(\Sigma[\phi]; \mathbb{Q}) = \langle [\widetilde{\alpha}_{1,1}], [\widetilde{\beta}_1] \rangle \oplus \bigoplus_{i=1}^{d} \mathrm{H}_1(\widetilde{S}_i; \mathbb{Q}).$$

Recall that we are trying to prove that the nonzero $v \in \mathrm{H}_1(\Sigma[\phi]; \mathbb{Q})$ has an infinite D_σ-orbit. In fact, we will find some $\widetilde{\tau} \in D_\sigma$ such that the elements $\{\widetilde{\tau}^n(v) \mid n \geq 1\}$ are all distinct. Write

$$v = \lambda[\widetilde{\alpha}_{1,1}] + \nu[\widetilde{\beta}_1] + \sum_{i=1}^{d} v_i \quad \text{with } \lambda, \nu \in \mathbb{Q} \text{ and } v_i \in \mathrm{H}_1(\widetilde{S}_i; \mathbb{Q}).$$

There are three cases.

The first is $\lambda \neq 0$. In this case, $T_{\beta_1}^d \in \mathrm{Mod}(\Sigma)$ lifts to $T_{\widetilde{\beta}_1} \in \mathrm{Mod}(\Sigma[\phi])$. We have

$$T_{\widetilde{\beta}_1}^n(v) = \lambda[\widetilde{\alpha}_{1,1}] + (\nu - n\lambda)[\widetilde{\beta}_1] + \sum_{i=1}^{d} v_i \quad \text{for } n \geq 1.$$

These are all distinct elements. Since D_σ is defined via the lifts to the cover[22] $\widetilde{\Sigma} \to \Sigma$, there is some $\ell \geq 1$ (necessarily divisible by d) such that the element $\widetilde{\tau}_{\beta_1} \in D_\sigma$ is induced by the lift of $T_{\beta_1}^{\ell} = (T_{\beta_1}^d)^{\ell/d}$. We conclude that the elements $\left\{\widetilde{\tau}_{\beta_1}^n(v) \mid n \geq 1\right\}$ are all distinct.

[22]Remember that the degree-d cyclic cover $\Sigma[\phi] \to \Sigma$ is a subcover of $\widetilde{\Sigma} \to \Sigma$, i.e., the map $\widetilde{\Sigma} \to \Sigma$ factors as $\widetilde{\Sigma} \to \Sigma[\phi] \to \Sigma$.

The second is $\nu \neq 0$. In this case, $T_{\alpha_1} \in \mathrm{Mod}(\Sigma)$ lifts to $T_{\widetilde{\alpha}_{1,1}} \cdots T_{\widetilde{\alpha}_{1,d}} \in \mathrm{Mod}(\Sigma[\phi])$. Since the $\widetilde{\alpha}_{1,i}$ are all homologous, we have

$$(T_{\widetilde{\alpha}_{1,1}} \cdots T_{\widetilde{\alpha}_{1,d}})^n(v) = (\lambda + dn\nu)[\widetilde{\alpha}_{1,1}] + \nu[\widetilde{\beta}_1] + \sum_{i=1}^{d} v_i \quad \text{for } n \geq 1.$$

These are all distinct elements. Just like in the previous case, we conclude that the elements $\left\{\widetilde{\tau}^n_{\alpha_1}(v) \mid n \geq 1\right\}$ are all distinct.

The third is that some v_i is nonzero. Reordering, assume that $v_1 \neq 0$. Let $\overline{v}_1 \in \mathrm{H}_1(S;\mathbb{Q})$ be the image of $v_1 \in \mathrm{H}_1(\widetilde{S}_1;\mathbb{Q})$. Pick an oriented simple closed curve γ on S with $\omega([\gamma], \overline{v}_1)$ nonzero. Let $\widetilde{\gamma}_1 \sqcup \cdots \sqcup \widetilde{\gamma}_d$ be the preimage of γ in $\Sigma[\phi]$, ordered such that $\widetilde{\gamma}_i \in \widetilde{S}_i$. By construction, $T_\gamma \in \mathrm{Mod}(\Sigma)$ lifts to $T_{\widetilde{\gamma}_1} \cdots T_{\widetilde{\gamma}_d} \in \mathrm{Mod}(\Sigma[\phi])$. We have

$$(T_{\widetilde{\gamma}_1} \cdots T_{\widetilde{\gamma}_d})^n(v) = \lambda[\widetilde{\alpha}_{1,1}] + \nu[\widetilde{\beta}_1] + \sum_{i=1}^{d}(v_i + n\omega([\widetilde{\gamma}_i], v_i)[\widetilde{\gamma}_i]) \quad \text{for } n \geq 1.$$

Since $\omega([\widetilde{\gamma}_1], v_1) = \omega([\gamma], \overline{v}_1) \neq 0$, these are all distinct. Just like before, we conclude that the elements $\left\{\widetilde{\tau}^n_{\gamma}(v) \mid n \geq 1\right\}$ are all distinct. □

5.3. **Nilpotent preliminaries.** Before we can prove part (ii) of Theorem 5.2, we need some preliminary results. Let F_n be the free group on $\{x_1, \ldots, x_n\}$. Fix some $\ell \geq 3$. Define[23]

$$\widehat{\ell} = \begin{cases} \ell & \text{if } \ell \text{ is odd,} \\ \ell/2 & \text{if } \ell \text{ is even.} \end{cases}$$

Since $\ell \geq 3$, we have $\widehat{\ell} \geq 2$. Define $N_n[\ell]$ to be the quotient of F_n by the normal subgroup generated by the following elements:[24]

- The third term $[F_n, [F_n, F_n]]$ of the lower central series.
- The subgroup $[F_n, F_n^{\times\widehat{\ell}}]$, i.e., the subgroup generated by commutators $[u, v^{\widehat{\ell}}]$ as u and v range over elements of F_n.

We will use boldface letters to denote elements of $N_n[\ell]$, and in particular will let $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be the generators of $N_n[\ell]$ coming from the generators $\{x_1, \ldots, x_n\}$ for F_n. The abelianization of $N_n[\ell]$ is $\mathbb{Z}^n$, and for $\mathbf{u} \in N_n[\ell]$ we will write $\overline{\mathbf{u}} \in \mathbb{Z}^n$ for its image in the abelianization and $\widehat{\overline{\mathbf{u}}} \in (\mathbb{Z}/\widehat{\ell})^n$ for the image of $\overline{\mathbf{u}}$ under the mod-$\widehat{\ell}$ reduction map.

The following lemma clarifies the nature of $N_n[\ell]$:

Lemma 5.4. *For $n \geq 2$ and $\ell \geq 3$, we have a central extension*

$$1 \longrightarrow \wedge^2(\mathbb{Z}/\widehat{\ell})^n \longrightarrow N_n[\ell] \longrightarrow \mathbb{Z}^n \longrightarrow 1.$$

Here the map $N_n[\ell] \to \mathbb{Z}^n$ is the abelianization map taking $\mathbf{u} \in N_n[\ell]$ to $\overline{\mathbf{u}} \in \mathbb{Z}^n$, and for $\mathbf{u}, \mathbf{v} \in N_n[\ell]$ the commutator $[\mathbf{u}, \mathbf{v}] \in N_n[\ell]$ is the central element $\widehat{\overline{\mathbf{u}}} \wedge \widehat{\overline{\mathbf{v}}} \in \wedge^2(\mathbb{Z}/\widehat{\ell})^n$.

Proof. It is immediate from Magnus–Witt's work on the lower central series of a free group ([16, 22]; see [21] for a textbook account) that

$$\frac{[F_n, F_n]}{[F_n, [F_n, F_n]]} \cong \wedge^2 \mathbb{Z}^n,$$

[23] When reading this for the first time, it might be easier to assume that ℓ is odd, so $\widehat{\ell} = \ell$.

[24] Here "N" stands for "nilpotent".

with $[u,v] \in [F_n, F_n]$ mapping to $\overline{u} \wedge \overline{v} \in \wedge^2 \mathbb{Z}^n$. Here $\overline{u}, \overline{v} \in \mathbb{Z}^n$ are the images of $u, v \in F_n$ in its abelianization. This fits into a central extension

$$1 \longrightarrow \wedge^2 \mathbb{Z}^n \longrightarrow \frac{F_n}{[F_n,[F_n,F_n]]} \longrightarrow \mathbb{Z}^n \longrightarrow 1.$$

To get $N_n[\ell]$ from the middle group in this extension, one quotients out the image of $[F_n, F_n^{\times \widehat{\ell}}]$, which maps to the kernel of the map

$$\wedge^2 \mathbb{Z}^n \longrightarrow \wedge^2 (\mathbb{Z}/\widehat{\ell})^n.$$

The lemma follows. □

In the rest of this section, we will identify $\wedge^2(\mathbb{Z}/\widehat{\ell})^n$ with the corresponding central subgroup of $N_n[\ell]$. The following calculation lies at the heart of our arguments:

Lemma 5.5. *For $n \geq 2$ and $\ell \geq 3$, we have $(\mathbf{u}\mathbf{v})^\ell = \mathbf{u}^\ell \mathbf{v}^\ell$ for all $\mathbf{u}, \mathbf{v} \in N_n[\ell]$.*

Proof. To transform $(\mathbf{u}\mathbf{v})^\ell$ into $\mathbf{u}^\ell\mathbf{v}^\ell$, we must commute each $\mathbf{u}$ past all the $\mathbf{v}$ terms to its left. Each time we commute a $\mathbf{u}$ past a $\mathbf{v}$, we must introduce a commutator $[\mathbf{v},\mathbf{u}] = \widehat{\overline{\mathbf{v}}} \wedge \widehat{\overline{\mathbf{u}}}$. This commutator is central, so it can moved all the way to the right. The first $\mathbf{u}$ must be commuted with 0 copies of $\mathbf{v}$, the second with 1 copy of $\mathbf{v}$, the third with 2 copies of $\mathbf{v}$, etc. In the end, we see that

$$(\mathbf{u}\mathbf{v})^\ell = \mathbf{u}^\ell \mathbf{v}^\ell [\mathbf{v},\mathbf{u}]^{0+1+2+\cdots+(\ell-1)} = \mathbf{u}^\ell \mathbf{v}^\ell [\mathbf{v},\mathbf{u}]^{\ell(\ell-1)/2}.$$

Whether ℓ is even or odd,[25] the integer $\ell(\ell-1)/2$ is divisible by $\widehat{\ell}$. Since $[\mathbf{v},\mathbf{u}] \in \wedge^2(\mathbb{Z}/\widehat{\ell})^n$, this implies that $[\mathbf{v},\mathbf{u}]^{\ell(\ell-1)/2} = 1$. The lemma follows. □

Define $P_n[\ell]$ to be[26] the subgroup of $N_n[\ell]$ generated by $\left\{\mathbf{u}^\ell \mid \mathbf{u} \in N_n[\ell]\right\}$ and define $A_n[\ell]$ to be the subgroup[27] of $N_n[\ell]$ generated by $P_n[\ell]$ and $\wedge^2(\mathbb{Z}/\widehat{\ell})^n$. We then have:

Lemma 5.6. *For $n \geq 2$ and $\ell \geq 3$, the subgroup $P_n[\ell]$ is a central subgroup of $N_n[\ell]$ with $P_n[\ell] \cong \mathbb{Z}^n$, and $A_n[\ell] = P_n[\ell] \times \wedge^2(\mathbb{Z}/\widehat{\ell})^n$.*

Proof. The fact that $P_n[\ell]$ is a central subgroup follows from the fact that

$$[\mathbf{u}^\ell, \mathbf{v}] = \widehat{\overline{\mathbf{u}}}^\ell \wedge \widehat{\overline{\mathbf{v}}} = \ell\left(\widehat{\overline{\mathbf{u}}} \wedge \widehat{\overline{\mathbf{v}}}\right) = 0 \quad \text{for all } \mathbf{u},\mathbf{v} \in N_n[\ell].$$

Recall that $N_n[\ell]$ is generated by the elements $\mathbf{x}_1, \ldots, \mathbf{x}_n$, which map to a basis for the abelianization $\mathbb{Z}^n$. The elements $\mathbf{x}_i^\ell \in N_n[\ell]$ are central and map to linearly independent elements in the abelianization, so

$$P_n'[\ell] = \left\{\mathbf{x}_1^{\ell k_1} \cdots \mathbf{x}_n^{\ell k_n} \mid k_1, \ldots, k_n \in \mathbb{Z}\right\}$$

is a central subgroup satisfying $P_n'[\ell] \cong \mathbb{Z}^n$. Moreover, letting $A_n'[\ell]$ be the subgroup of $N_n[\ell]$ generated by $P_n'[\ell]$ and $\wedge^2(\mathbb{Z}/\widehat{\ell})^n$, we clearly have $A_n'[\ell] = P_n'[\ell] \times \wedge^2(\mathbb{Z}/\widehat{\ell})^n$.

To prove the lemma, it is therefore enough to prove that $P_n[\ell] = P_n'[\ell]$. Since $\mathbf{x}_i^\ell \in P_n[\ell]$ for all $1 \leq i \leq n$, we have $P_n'[\ell] \subset P_n[\ell]$. For the reverse inclusion, consider some $\mathbf{u} \in N_n[\ell]$. We must prove that $\mathbf{u}^\ell \in P_n'[\ell]$. We can find $k_1, \ldots, k_n \in \mathbb{Z}$ and

$$\mathbf{c} \in [P_n[\ell], P_n[\ell]] = \wedge^2(\mathbb{Z}/\widehat{\ell})^n$$

such that $\mathbf{u} = \mathbf{x}_1^{k_1} \cdots \mathbf{x}_n^{k_n}\mathbf{c}$. Applying Lemma 5.5 repeatedly, we deduce that

$$\mathbf{u}^\ell = \mathbf{x}_1^{\ell k_1} \cdots \mathbf{x}_n^{\ell k_n} \mathbf{c}^\ell = \mathbf{x}_1^{\ell k_1} \cdots \mathbf{x}_n^{\ell k_n} \in P_n'[\ell].$$ □

[25] The purpose of using $\widehat{\ell}$ is to ensure this.

[26] Here "P" stands for "power subgroup".

[27] Here "A" stands for "abelian subgroup"; see Lemma 5.6.

5.4. **Integral inequality.** We now prove part (ii) of Theorem 5.2

Proof of Theorem 5.2, part (ii). We first recall the statement. Let Σ be a closed oriented surface of genus $g \geq 2$ and σ be the set of nonseparating simple closed curves on Σ. Fix some $\ell \geq 3$, and as above let

$$\widehat{\ell} = \begin{cases} \ell & \text{if } \ell \text{ is odd,} \\ \ell/2 & \text{if } \ell \text{ is even.} \end{cases}$$

Since $\ell \geq 3$, we have $\widehat{\ell} \geq 2$. We must prove that $\mathrm{H}_1^{\sigma}(\Sigma[\ell]; \mathbb{Z}) \neq \mathrm{H}_1(\Sigma[\ell]; \mathbb{Z})$.

Recall that $\Sigma[\ell]$ is the cover corresponding to the homomorphism

$$\pi_1(\Sigma) \to \mathrm{H}_1(\Sigma; \mathbb{Z}/\ell) \cong (\mathbb{Z}/\ell)^{2g}.$$

It follows that $\pi_1(\Sigma[\ell])$ is the kernel of this map, so $\pi_1(\Sigma[\ell])$ is the subgroup of $\pi_1(\Sigma)$ generated by the following two subgroups:

- The commutator subgroup $[\pi_1(\Sigma), \pi_1(\Sigma)]$.
- The subgroup P generated by $\{x^{\ell} \mid x \in \pi_1(\Sigma)\}$.

Each nonseparating simple closed curve $x \in \pi_1(\Sigma)$ maps to a primitive[28] element of $\mathrm{H}_1(\Sigma; \mathbb{Z})$, so the minimal power of x that lies in $\pi_1(\Sigma[\ell])$ is x^{ℓ}. It follows that the image $\overline{P}$ of P in $\mathrm{H}_1(\Sigma[\ell]; \mathbb{Z})$ contains $\mathrm{H}_1^{\sigma}(\Sigma[\ell]; \mathbb{Z})$. It is enough therefore to prove that $\overline{P} \neq \mathrm{H}_1(\Sigma[\ell]; \mathbb{Z})$.

Let $\{a_1, b_1, \ldots, a_g, b_g\}$ be the standard generating set for $\pi_1(\Sigma)$ satisfying the surface relation $[a_1, b_1] \cdots [a_g, b_g] = 1$. We can then define a homomorphism $\phi\colon \pi_1(\Sigma) \to N_g[\ell]$ via the formulas

$$\phi(a_i) = \mathbf{x}_i \quad \text{and} \quad \phi(b_i) = 1 \quad \text{for } 1 \leq i \leq g.$$

The map ϕ takes $[\pi_1(\Sigma), \pi_1(\Sigma)]$ to the central subgroup $\wedge^2(\mathbb{Z}/\widehat{\ell})^g$ and P to the central subgroup $P_g[\ell]$ (see Lemma 5.6). It follows that ϕ takes $\pi_1(\Sigma[\ell])$ surjectively onto the abelian subgroup $A_g[\ell] = P_g[\ell] \times \wedge^2(\mathbb{Z}/\widehat{\ell})^g$ identified by Lemma 5.6. The restriction of ϕ to $\pi_1(\Sigma[\ell])$ thus factors through $\mathrm{H}_1(\Sigma[\ell]; \mathbb{Z})$, and takes $\overline{P} \subset \mathrm{H}_1(\Sigma[\ell]; \mathbb{Z})$ to the proper subgroup $P_g[\ell]$ of $A_g[\ell]$. The theorem follows. □

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