GENERATING THE HOMOLOGY OF COVERS OF SURFACES

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Abstract. Putman and Wieland conjectured that if \( \tilde{\Sigma} \to \Sigma \) is a finite branched cover between closed oriented surfaces of sufficiently high genus, then the orbits of all nonzero elements of \( H_1(\tilde{\Sigma}; \mathbb{Q}) \) under the action of lifts to \( \tilde{\Sigma} \) of mapping classes on \( \Sigma \) are infinite. We prove that this holds if \( H_1(\tilde{\Sigma}; \mathbb{Q}) \) is generated by the homology classes of lifts of simple closed curves on \( \Sigma \). We also prove that the subspace of \( H_1(\tilde{\Sigma}; \mathbb{Q}) \) spanned by such lifts is a symplectic subspace. Finally, simple closed curves lie on subsurfaces homeomorphic to 2-holed spheres, and we prove that \( H_1(\tilde{\Sigma}; \mathbb{Q}) \) is generated by the homology classes of lifts of loops on \( \Sigma \) lying on subsurfaces homeomorphic to 3-holed spheres.

1. Introduction

Let \( \pi: \tilde{\Sigma} \to \Sigma \) be a finite branched cover between closed oriented surfaces. The homology of \( \tilde{\Sigma} \) encodes subtle information about the mapping class group of \( \Sigma \), and over the last decade has been intensely studied \([4, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18]\). Much of this is motivated by a conjecture of Putman–Wieland \([18]\) we discuss below. In this note, we prove this conjecture for covers \( \tilde{\Sigma} \) such that \( H_1(\tilde{\Sigma}; \mathbb{Q}) \) is generated by certain simple elements, and also prove that in general \( H_1(\tilde{\Sigma}; \mathbb{Q}) \) is generated by slightly more complicated elements.

1.1. Putman–Wieland conjecture. Mark \( \Sigma \) at each branch point of the branched cover \( \pi: \tilde{\Sigma} \to \Sigma \). Let \( \text{Mod}(\Sigma) \) be the pure mapping class group of \( \Sigma \), i.e., the group of isotopy classes of orientation-preserving homeomorphisms of \( \Sigma \) that fix each marked point. There is a finite-index subgroup \( \text{Mod}(\Sigma, \tilde{\Sigma}) \) of \( \text{Mod}(\Sigma) \) that can be lifted to \( \tilde{\Sigma} \) to give a well-defined action of \( \text{Mod}(\Sigma, \tilde{\Sigma}) \) on \( H_1(\tilde{\Sigma}; \mathbb{Q}) \). Putman–Wieland \([18]\) made the following conjecture.

Conjecture 1.1 ([18]). Let the notation be as above, and assume that the genus of \( \Sigma \) is sufficiently large. Consider some nonzero \( \vec{v} \in H_1(\tilde{\Sigma}; \mathbb{Q}) \). Then the \( \text{Mod}(\Sigma, \tilde{\Sigma}) \)-orbit of \( \vec{v} \) is infinite.

The main theorem of \([18]\) says that this holds if and only if the virtual first Betti number of the mapping class group is 0 when the genus is sufficiently large, which is a well-known conjecture of Ivanov \([6]\).

1.2. Simple closed curve homology. To prove Conjecture 1.1, it is natural to try to find generators for \( H_1(\tilde{\Sigma}; \mathbb{Q}) \). A first idea is that \( H_1(\tilde{\Sigma}; \mathbb{Q}) \) might be generated by lifts of simple closed curves. Define the simple closed curve homology of \( \tilde{\Sigma} \), denoted \( H_{\text{sc}}^1(\tilde{\Sigma}; \mathbb{Q}) \), to be the subspace of \( H_1(\tilde{\Sigma}; \mathbb{Q}) \) spanned by the homology classes of loops \( \tilde{\gamma} \) on \( \tilde{\Sigma} \) that avoid the branch points and project to simple closed curves \( \gamma \) on \( \Sigma \). The restriction of the branched covering map \( \pi: \tilde{\Sigma} \to \Sigma \) to \( \tilde{\gamma} \) is thus a possibly nontrivial cover \( \tilde{\gamma} \to \gamma \).

Unfortunately, this need not be all of \( H_1(\tilde{\Sigma}; \mathbb{Q}) \). For all closed surfaces \( \Sigma \) with \( \pi_1(\Sigma) \) nonabelian, Malestein–Putman \([14, \text{Theorem B}]\) constructed finite branched covers \( \tilde{\Sigma} \to \Sigma \) such that \( H_{\text{sc}}^1(\tilde{\Sigma}; \mathbb{Q}) \neq H_1(\tilde{\Sigma}; \mathbb{Q}) \). More recently, Klukowski \([8]\) constructed unbranched
covers with this property. Our first theorem is that Conjecture 1.1 does hold if if $H_1^{sc}(\tilde{\Sigma}; Q) = H_1(\Sigma; Q)$.

**Theorem A.** Let $\pi: \tilde{\Sigma} \to \Sigma$ be a finite branched cover between closed oriented surfaces such that $H_1^{sc}(\tilde{\Sigma}; Q) = H_1(\Sigma; Q)$. Then Conjecture 1.1 holds for $\pi: \Sigma \to \Sigma$.

This suggests that the examples from [14, Theorem B] and [8] might be good places to look for counterexamples to Conjecture 1.1.

### 1.3. Symplectic subspace.

Our next theorem clarifies the nature of the subspace $H_1^{sc}(\Sigma; Q)$ of $H_1(\Sigma; Q)$. By Poincaré duality, the algebraic intersection form $\omega$ on $H_1(\Sigma; Q)$ is a symplectic form, i.e., an alternating bilinear form that induces an isomorphism between $H_1(\Sigma; Q)$ and its dual. A subspace $V$ of $H_1(\Sigma; Q)$ is a symplectic subspace if the restriction of $\omega$ to $V$ is a symplectic form. We will prove the following:

**Theorem B.** Let $\pi: \tilde{\Sigma} \to \Sigma$ be a finite branched cover between closed oriented surfaces. Then $H_1^{sc}(\tilde{\Sigma}; Q)$ is a symplectic subspace of $H_1(\tilde{\Sigma}; Q)$.

In fact, we will prove something more general. A nontrivial simple closed curve on $\Sigma$ is a simple closed curve $\gamma$ that avoids the marked points and does not bound a disk containing at most one marked point. We will always consider such curves up to isotopy. The group $\text{Mod}(\Sigma)$ acts on the set of nontrivial simple closed curves on $\Sigma$, and the orbits of this action are the topological types of nontrivial simple closed curves.

If $\sigma$ is a set of topological types of nontrivial simple closed curves on $\Sigma$, then denote by $H_1^\sigma(\Sigma; Q)$ the subspace of $H_1(\Sigma; Q)$ spanned by the homology classes of loops $\tilde{\gamma}$ on $\tilde{\Sigma}$ that avoid the branch points and project to simple closed curves $\gamma$ on $\Sigma$ such that the topological type of $\gamma$ lies in $\sigma$. For instance, if $\sigma$ is the set of all topological types of nontrivial simple closed curves on $\Sigma$, then

$$H_1^\sigma(\Sigma; Q) = H_1^{sc}(\tilde{\Sigma}; Q).$$

The following therefore generalizes Theorem B:

**Theorem B’.** Let $\pi: \tilde{\Sigma} \to \Sigma$ be a finite branched cover between closed oriented surfaces and $\sigma$ be a set of topological types of nontrivial simple closed curves on $\Sigma$. Then $H_1^\sigma(\tilde{\Sigma}; Q)$ is a symplectic subspace of $H_1(\tilde{\Sigma}; Q)$.

We can also define $H_1^\sigma(\Sigma; Z)$ and $H_1^{sc}(\Sigma; Z)$, and it is natural to wonder whether Theorems B and B’ hold integrally. For Theorem B’, the answer is no in general:

**Theorem C.** Let $\Sigma$ be a closed oriented surface of genus at least 2 and let $\sigma$ be the set of nonseparating simple closed curves on $\Sigma$. Then there exists a finite unbranched cover $\pi: \tilde{\Sigma} \to \Sigma$ such that $H_1^\sigma(\tilde{\Sigma}; Z)$ is not a symplectic subspace of $H_1(\tilde{\Sigma}; Z)$.

Here $H_1^\sigma(\Sigma; Z)$ is a free abelian group, and a symplectic form on a free abelian group $A$ is an alternating $\mathbb{Z}$-valued bilinear form on $A$ that identifies $A$ with its dual $A^* = \text{Hom}(A, \mathbb{Z})$. Unfortunately, our proof of Theorem C breaks down if we allow separating curves, so we cannot answer the following question:

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2 Earlier Koberda–Santharoubane [9] constructed unbranched covers of closed surfaces with $H_1^{sc}(\tilde{\Sigma}; Z) \neq H_1(\Sigma; Z)$. This is weaker since it is possible that in their examples $H_1^{sc}(\tilde{\Sigma}; Z)$ is a finite-index subgroup of $H_1(\Sigma; Z)$.

3 These are isotopies through nontrivial simple closed curves, so during the isotopies the curves cannot pass through the marked points.

4 By the change of coordinates principle from [3, §1.3.2], the topological types are determined by the marked surface with boundary one gets by cutting $\Sigma$ open along $\gamma$. For instance, one topological type is the set of all nonseparating $\gamma$. 

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**Question 1.2.** Let $\pi: \tilde{\Sigma} \to \Sigma$ be a finite branched cover between closed oriented surfaces. Is $H^\text{sc}(\tilde{\Sigma}; \mathbb{Z})$ a symplectic subspace of $H_1(\tilde{\Sigma}; \mathbb{Z})$?

However, Theorem C suggests that the answer to this should be “no”.

**Remark 1.3.** An important ingredient in our proof of Theorem C is a theorem of Irmer [5] giving certain finite abelian covers $\pi: \tilde{\Sigma} \to \Sigma$ for which $H_1(\tilde{\Sigma}; \mathbb{Z})$ is a proper subgroup of $H_1(\tilde{\Sigma}; \mathbb{Z})$ (see Theorem 5.2.(ii) below). To make this paper more self-contained, we also include a simplified proof of this theorem.

**1.4. Pants homology.** Regular neighborhoods of simple closed curves on $\Sigma$ are homeomorphic to annuli, i.e., spheres with two boundary components. This suggests weakening the definition of simple closed curve homology as follows.

Recall that a pair of pants is a sphere with three holes. Define the pants homology of $\tilde{\Sigma}$, denoted $H^\text{pant}(\tilde{\Sigma}; \mathbb{Q})$, to be the subspace of $H_1(\tilde{\Sigma}; \mathbb{Q})$ spanned by the homology classes of loops $\tilde{\gamma}$ on $\tilde{\Sigma}$ such that there exists a subsurface $P \subset \Sigma$ homeomorphic to a pair of pants with $\pi(\tilde{\gamma}) \subset P$. Since every simple closed curve on $\Sigma$ is contained in some such $P$, we have

$$H^\text{sc}(\tilde{\Sigma}; \mathbb{Q}) \subset H^\text{pant}(\tilde{\Sigma}; \mathbb{Q}).$$

Our final main theorem is as follows. It answers positively a question of Kent [7].

**Theorem D.** Let $\pi: \tilde{\Sigma} \to \Sigma$ be a finite branched cover between closed oriented surfaces. Then $H^\text{pant}(\tilde{\Sigma}; \mathbb{Q}) = H_1(\tilde{\Sigma}; \mathbb{Q})$.

**Remark 1.4.** This also holds for punctured surfaces of finite type, which can be reduced to Theorem D as follows. Let $\pi: \tilde{\Sigma} \to \Sigma$ be a finite branched cover between punctured surfaces of finite type. Filling in the punctures yields a finite branched cover between closed surfaces to which one can apply Theorem D. To conclude, note that filling in the punctures has the effect of killing the homology classes of loops around the punctures, which lie in $H^\text{sc}(\tilde{\Sigma}; \mathbb{Q}) \subset H^\text{pant}(\tilde{\Sigma}; \mathbb{Q})$.

**Remark 1.5.** Theorem D might appear to contradict [14, Theorem C] and [8, Corollary 1.1.3], which give examples of finite covers $\pi: \tilde{\Sigma} \to \Sigma$ such that $H^\text{pant}(\tilde{\Sigma}; \mathbb{Q})$ is not spanned by the homology classes of loops $\tilde{\gamma}$ such that $\pi(\tilde{\gamma})$ is not in any given finite set of mapping class group orbits of curves. However, in the definition of $H^\text{pant}(\tilde{\Sigma}; \mathbb{Q})$ there is no restriction on the number of self-intersections of the projections of the curves to the $P$, so they do not fall into finitely many mapping class group orbits.  

The proof of Theorem D actually shows something stronger. A pants decomposition of $\Sigma$ is a collection $P = \{\delta_1, \ldots, \delta_n\}$ of disjoint simple closed curves on $\Sigma$ that avoid the branch points such that each component of $\Sigma \setminus \bigcup_{j=1}^n \delta_j$ is either a disk containing a single branch point or a pair of pants containing no branch points:

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5We do not require the boundary components of $P$ to be non-nullhomotopic curves on $\Sigma$, so this even holds if $\Sigma$ is a surface like a sphere that does not contain pairs of pants $P$ whose boundary components are non-nullhomotopic.

6Kent actually asked whether $H_1(\tilde{\Sigma}; \mathbb{Q})$ is generated by lifts of elements that do not fill $\Sigma$, which is much weaker than lying in a pair of pants.

7or even automorphism group of free group
We will prove the following, which implies Theorem D:

**Theorem D'.** Let \( \pi: \tilde{\Sigma} \to \Sigma \) be a finite branched cover between closed oriented surfaces and \( \mathcal{P} \) be a pants decomposition of \( \Sigma \). Let \( \sigma \) be the set of topological types of nontrivial curves appearing in \( \mathcal{P} \). Then \( H_1(\tilde{\Sigma}; \mathbb{Q}) \) is spanned by \( H_1^\sigma(\tilde{\Sigma}; \mathbb{Q}) \) and the set of homology classes of cycles \( \tilde{\gamma} \) on \( \tilde{\Sigma} \) such that \( \pi(\tilde{\gamma}) \) is disjoint from all curves in \( \mathcal{P} \).

We can also define \( H^\text{pant}_1(\tilde{\Sigma}; \mathbb{Z}) \), and pose the following question:

**Question 1.6.** Let \( \pi: \tilde{\Sigma} \to \Sigma \) be a finite branched cover between closed oriented surfaces. Is \( H^\text{pant}_1(\tilde{\Sigma}; \mathbb{Z}) = H_1(\tilde{\Sigma}; \mathbb{Z}) ? \)

Our proof of Theorems D and D' shows that Question 1.6 has a positive answer if Question 1.2 does. Though we expect that Question 1.2 has a negative answer, we do not know what answer to expect for Question 1.6.

1.5. **Outline.** We prove Theorem A in §2, Theorems B and B' in §3, Theorems D and D' in §4, and Theorem C in §5.

1.6. **Acknowledgements.** We would like to thank Eduard Looijenga for his help, especially with Theorem B.

2. Simple closed curve homology and Dehn twists

This section contains the proof of Theorem A.

2.1. **Notation.** Fix a closed oriented surface \( \Sigma \) and a finite branched cover \( \pi: \tilde{\Sigma} \to \Sigma \). The surface \( \tilde{\Sigma} \) is thus also a closed oriented surface. Let \( g \geq 0 \) be its genus, let

\[
H = H_1(\tilde{\Sigma}; \mathbb{Q}) \cong \mathbb{Q}^{2g},
\]

and let \( \omega \) be the algebraic intersection form on \( H \). By Poincaré duality, \( \omega \) is a symplectic form, i.e., an alternating form that induces an isomorphism between \( H \) and its dual. The symplectic group \( \text{Sp}(H, \omega) \cong \text{Sp}_{2g}(\mathbb{Q}) \) acts on \( H \).

2.2. **Lifting Dehn twists.** Recall from §1.3 that a nontrivial simple closed curve on \( \Sigma \) is a simple closed curve that avoids the marked points and does not bound a disk containing at most one marked point. Consider a nontrivial simple closed curve \( \gamma \) on \( \Sigma \). The preimage \( \pi^{-1}(\gamma) \) is a disjoint union of simple closed curves. Enumerate them as

\[
\pi^{-1}(\gamma) = \tilde{\gamma}_1 \sqcup \cdots \sqcup \tilde{\gamma}_k.
\]

For each \( 1 \leq j \leq k \), the map

\[
\pi|_{\tilde{\gamma}_j}: \tilde{\gamma}_j \to \gamma
\]

is a finite unbranched cover. Let \( d_j \) be its degree. Set

\[
de(\gamma) = \text{lcm}(d_1, \ldots, d_k) \quad \text{and} \quad e_j = d(\gamma)/d_j \quad \text{for } 1 \leq j \leq k.
\]

If \( T_\gamma \) and \( T_{\tilde{\gamma}_j} \) denote the Dehn twists about \( \gamma \) and \( \tilde{\gamma}_j \), respectively, then \( T_\gamma^{e_j} \) lifts to the product

\[
T^{e_1}_{\tilde{\gamma}_1} \cdots T^{e_k}_{\tilde{\gamma}_k}.
\]
Let \( \tilde{\tau}_{\gamma} \) be the image of this product of powers of Dehn twists in 
\[ \text{Sp}(H, \omega) \cong \text{Sp}_{2g}(\mathbb{Q}). \]

The element \( \tilde{\tau}_{\gamma} \) acts on \( H \) as follows:
\[ \tilde{\tau}_{\gamma}(h) = h + \sum_{j=1}^{k} e_{j} \omega(h, [\tilde{\gamma}_{j}]) \cdot [\tilde{\gamma}_{j}] \quad \text{for } h \in H. \]

For a set \( \sigma \) of topological types of nontrivial simple closed curves on \( \Sigma \), define \( D_{\sigma} \) to be the subgroup of \( \text{Sp}(H, \omega) \) generated by the set of all \( \tilde{\tau}_{\gamma} \) as \( \gamma \) ranges over nontrivial simple closed curves on \( \Sigma \) whose topological type lies in \( \sigma \).

2.3. 

**Fixed set of lifted twists.** As in §1.3, let \( \sigma \) be a set of topological types of nontrivial simple closed curves on \( \Sigma \) and define
\[ H^{\sigma} = H^{\sigma}_{1}(\tilde{\Sigma}; \mathbb{Q}) \subset H_{1}(\tilde{\Sigma}; \mathbb{Q}) = H. \]

The following lemma will be fundamental to our paper:

**Lemma 2.1.** Let the notation be as above. Then\(^8\) \( H^{D_{\sigma}} \) equals the orthogonal complement \((H^{\sigma})^\perp\) of \( H^{\sigma} \) with respect to \( \omega \).

**Proof.** Let \( \gamma \) be a nontrivial simple closed curve on \( \Sigma \) whose topological type lies in \( \sigma \). As we did in §2.2 above, write \( \pi^{-1}(\gamma) \) as a disjoint union of simple closed curves on \( \tilde{\Sigma} \):
\[ \pi^{-1}(\gamma) = \tilde{\gamma}_{1} \sqcup \cdots \sqcup \tilde{\gamma}_{k}. \]

As in that section, there are positive integers \( e_{1}, \ldots, e_{k} \) such that the generator \( \tilde{\tau}_{\gamma} \in D_{\sigma} \) acts on \( H \) as follows:
\[ \tilde{\tau}_{\gamma}(h) = h + \sum_{j=1}^{k} e_{j} \omega(h, [\tilde{\gamma}_{j}]) \cdot [\tilde{\gamma}_{j}] \quad \text{for } h \in H. \]

Each \( [\tilde{\gamma}_{j}] \) lies in \( H^{\sigma} \), so it is immediate from this formula that \( (H^{\sigma})^\perp \subset H^{D_{\sigma}} \). For the other inclusion, consider some \( h_{0} \in H^{D_{\sigma}} \). We then know that \( \tilde{\tau}_{\gamma}(h_{0}) = h_{0} \), so from the above
\[ \sum_{j=1}^{k} e_{j} \omega(h_{0}, [\tilde{\gamma}_{j}]) \cdot [\tilde{\gamma}_{j}] = 0. \]

Taking the algebraic intersection with \( h_{0} \), we deduce that
\[ \sum_{j=1}^{k} e_{j} \omega(h_{0}, [\tilde{\gamma}_{j}])^{2} = 0. \]

Since \( e_{j} \geq 1 \) for all \( 1 \leq j \leq k \), this implies that \( \omega(h_{0}, [\tilde{\gamma}_{j}]) = 0 \) for all \( 1 \leq j \leq k \). This holds for all choices of \( \gamma \) and all components of the preimage \( \pi^{-1}(\gamma) \). These generate \( H^{\sigma} \), so we conclude that \( h_{0} \in (H^{\sigma})^\perp \), as desired. \( \square \)

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\(^8\)Here the superscript indicates that we are taking invariants: \( H^{D_{\sigma}} = \{ h \in H \mid d \cdot h = h \text{ for all } d \in D_{\sigma} \} \).
2.4. **Putman–Wieland conjecture.** We now prove Theorem A.

*Proof of Theorem A.* We start by recalling the statement. As is the case throughout this paper, $\Sigma$ is a closed oriented surface and $\pi: \tilde{\Sigma} \to \Sigma$ is a finite branched cover. We make the assumption that $H^{sc}_{1}(\tilde{\Sigma}; \mathbb{Q}) = H_{1}(\tilde{\Sigma}; \mathbb{Q})$. Our goal is to prove that Conjecture 1.1 holds for $\pi: \tilde{\Sigma} \to \Sigma$. This conjecture asserts that if $\text{Mod}(\Sigma, \tilde{\Sigma})$ is the liftable subgroup of the mapping class group $\text{Mod}(\Sigma)$ and $\vec{v} \in H_{1}(\tilde{\Sigma}; \mathbb{Q})$ is nonzero, then the $\text{Mod}(\Sigma, \tilde{\Sigma})$-orbit of $\vec{v}$ is infinite.

Let $\sigma$ be the set of all topological types of nontrivial simple closed curves on $\Sigma$. Since $H^{sc}_{1}(\tilde{\Sigma}; \mathbb{Q}) = H_{1}(\tilde{\Sigma}; \mathbb{Q})$, Lemma 2.1 implies that $\vec{v}$ is not fixed by the group $D_{\sigma}$, so there exists some nontrivial simple closed curve $\gamma$ on $\Sigma$ such that $\bar{\tau}_{\gamma}(\vec{v}) \neq \vec{v}$. Some power of $\bar{\tau}_{\gamma}$ is the image in the symplectic group of an element of $\text{Mod}(\Sigma, \tilde{\Sigma})$, so it is enough to prove that the elements $\bar{\tau}_{\gamma}^{n}(\vec{v})$ as $n$ ranges over $\mathbb{Z}$ are all distinct.

Write $\pi^{-1}(\gamma)$ as a disjoint union of simple closed curves on $\tilde{\Sigma}$:

$$\pi^{-1}(\gamma) = \tilde{\gamma}_1 \sqcup \cdots \sqcup \tilde{\gamma}_k.$$  

There are then positive integers $e_1, \ldots, e_k$ such that

$$\bar{\tau}_{\gamma}(\vec{v}) = \vec{v} + \sum_{j=1}^{k} e_j \omega(\vec{v}, [\tilde{\gamma}_j]) \cdot [\tilde{\gamma}_j].$$

Setting

$$\vec{w} = \sum_{j=1}^{k} e_j \omega(\vec{v}, [\tilde{\gamma}_j]) \cdot [\tilde{\gamma}_j],$$

the fact that $\bar{\tau}_{\gamma}(\vec{v}) \neq \vec{v}$ implies that $\vec{w} \neq 0$. For $n \in \mathbb{Z}$, we have

$$\bar{\tau}_{\gamma}^{n}(\vec{v}) = \vec{v} + n\vec{w}.$$  

Since $\vec{w} \neq 0$, the elements $\vec{v} + n\vec{w}$ as $n$ ranges over $\mathbb{Z}$ are all distinct, as desired. □

3. **The symplectic nature of simple closed curves**

This section contains the proof of Theorem B′ (which generalizes Theorem B).

### 3.1. **Notation.** The notation is similar to that of §2:

- $\pi: \tilde{\Sigma} \to \Sigma$ is a finite branched cover between closed oriented surfaces, and $g$ is the genus of $\tilde{\Sigma}$.
- $H = H_{1}(\tilde{\Sigma}; \mathbb{Q}) \cong \mathbb{Q}^{2g}$, and $\omega$ is the algebraic intersection form on $H$.
- $\sigma$ is a set of topological types of nontrivial simple closed curves on $\Sigma$.
- $D_{\sigma}$ is the subgroup of $\text{Sp}(H, \omega) \cong \text{Sp}_{2g}(\mathbb{Q})$ generated by the elements $\bar{\tau}_{\gamma}$ as $\gamma$ ranges over nontrivial simple closed curves on $\Sigma$ whose topological type lies in $\sigma$.

### 3.2. **Symplectic criterion.** We will need the following criterion for a subspace of $H$ to be a symplectic subspace:

**Lemma 3.1.** Let the notation be as above, and let $D$ be a subgroup of $\text{Sp}(H, \omega)$. Assume that the action of $D$ on $H$ is semisimple.\(^9\) Then\(^10\) $H^{D}$ is a symplectic subspace of $H$.

This is a well-known result; see, for instance, [2, Lemme 4.14]. For completeness, we include a proof.

\(^9\)That is, the $D$-representation $H$ decomposes as a direct sum of irreducible representations.

\(^10\)Here just like in Lemma 2.1 the superscript indicates we are taking invariants.
Proof of Lemma 3.1. The symplectic form \( \omega \) induces a \( D \)-equivariant isomorphism
\[
\phi: H \xrightarrow{\cong} H^*.
\]
We want to prove \( \omega \) also induces an isomorphism between \( H^D \) and \( (H^D)^* \). Letting \( \iota: H^D \hookrightarrow H \) be the inclusion and letting \( \iota^*: H^* \to (H^D)^* \) be its dual,\(^{11}\) our goal is equivalent to proving that the composition
\[
(3.1) \quad H^D \xrightarrow{\iota} H \xrightarrow{\phi} H^* \xrightarrow{\iota^*} (H^D)^*
\]
is an isomorphism.
Since \( \phi \) is a \( D \)-equivariant isomorphism, it restricts to an isomorphism on \( D \)-invariants, i.e., an isomorphism
\[
\phi^D: H^D \xrightarrow{\cong} (H^*)^D.
\]
A linear map \( \lambda: H \to \mathbb{Q} \) in \( H^* \) is \( D \)-invariant if and only if it factors through the \( D \)-coinvariants
\[
H_D = H/\langle d \cdot h - h \mid d \in D \text{ and } h \in H \rangle.
\]
We thus get an isomorphism
\[
\mu: (H^*)^D \xrightarrow{\cong} (H_D)^*.
\]
The projection \( H \to H_D \) restricts to a map \( \eta: H^D \to H_D \). Since the action of \( D \) is semisimple, the map \( \eta \) is an isomorphism.\(^{12}\) Taking its dual, we get an isomorphism
\[
\eta^*: (H_D)^* \xrightarrow{\cong} (H^D)^*.
\]
The composition
\[
(3.2) \quad H^D \xrightarrow{\phi^D} (H^*)^D \xrightarrow{\mu} (H_D)^* \xrightarrow{\eta^*} (H^D)^*
\]
of isomorphisms is an isomorphism, and reflecting on the maps we see that the compositions (3.1) and (3.2) are the same. We conclude that (3.1) is an isomorphism, as desired. \( \square \)

3.3. Semisimplicity. Our goal is to apply Lemma 3.1 to the group \( D_\sigma \) from §3.1, which requires verifying the following:

Lemma 3.2. Let the notation be as above. Then the group \( D_\sigma \) acts semisimply on \( H \).

Proof. Let \( D_\sigma \) be the Zariski closure of \( D_\sigma \) in \( \text{Sp}(H, \omega) \). It is enough to prove that \( D_\sigma \) is a semisimple algebraic group.

Regard \( \Sigma \) as a closed surface with marked points at the branch points of \( \pi: \tilde{\Sigma} \to \Sigma \). To simplify things, if there are no branch points introduce a single additional marked point on \( \Sigma \), and regard its preimage in \( \tilde{\Sigma} \) as a collection of branch points of order 1. Let \( \mathcal{M}(\Sigma) \) be the moduli space of Riemann surfaces \( S \) with marked points such that \( S \cong \Sigma \) as surfaces with marked points. The (orbifold) fundamental group of \( \mathcal{M}(\Sigma) \) is thus the mapping class group \( \text{Mod}(\Sigma) \).

We can find a finite-index subgroup \( \Gamma \) of \( \text{Mod}(\Sigma) \) such that each element of \( \Gamma \) can be lifted to a homeomorphism of \( \tilde{\Sigma} \) fixing all the marked points. Since there is at least one marked point, these lifts are unique up to homotopy, so \( \Gamma \) acts on \( H_1(\tilde{\Sigma}; \mathbb{Q}) \) in a well-defined way. Shrinking \( \Gamma \) if necessary, we can also assume that \( \Gamma \) is torsion-free. Let \( \mathcal{M}_\Gamma(\Sigma) \) be the cover of \( \mathcal{M}(\Sigma) \) corresponding to \( \Gamma \).

\(^{11}\)This dual restricts a linear map \( \lambda: H \to \mathbb{Q} \) to \( H^D \).

\(^{12}\)Indeed, if \( H = H^D \oplus V_1 \oplus \cdots \oplus V_n \) with the \( V_i \) nontrivial irreducible representations of \( D \), then
\[
H_D = (H^D)_D \oplus (V_1)_D \oplus \cdots \oplus (V_n)_D = H^D \oplus 0 \oplus \cdots \oplus 0 = H^D.
\]
Since $\Gamma$ is torsion-free, $M_{\Gamma}(\Sigma)$ is a fine moduli space. It thus has a universal curve $U \to M_{\Gamma}(\Sigma)$ whose fiber over $S \in M_{\Gamma}(\Sigma)$ is $S$. Replacing $\Gamma$ by a deeper finite-index subgroup if necessary, we can find a fiberwise branched cover $\tilde{U} \to M_{\Gamma}(\Sigma)$ of $U \to M_{\Gamma}(\Sigma)$ whose fibers are the branched cover $\tilde{\Sigma}$ of $\Sigma$.

The monodromy representation of $\pi_1(M_{\Gamma}(\Sigma)) \cong \Gamma$ on $H_1$ of the fibers is thus exactly the action of $\Gamma$ on $H = H_1(\tilde{\Sigma}; \mathbb{Q})$ obtained by lifting mapping classes through the branched cover $\tilde{\Sigma} \to \Sigma$. The image of this representation lies in $\text{Sp}(H, \omega)$. Let $G$ be the Zariski closure in $\text{Sp}(H, \omega)$ of the image of $\Gamma$. Deligne’s semisimplicity theorem [1] implies that $G$ is a semisimple algebraic group.

From the definition (2.1) of $d(\gamma)$ for nontrivial simple closed curves $\gamma$ on $\Sigma$, it only achieves finitely many values (depending on the degree of the cover $\tilde{\Sigma} \to \Sigma$). Pick some $m \geq 1$ such that the following two properties hold for each nontrivial simple closed curve $\gamma$ whose topological type lies in $\sigma$:

- $d(\gamma)$ divides $m$.
- $T^m_\gamma \in \Gamma$.

Let $E$ be the subgroup of $\text{Mod}(\Sigma)$ generated by all the $T^m_\gamma$ as $\gamma$ ranges over nontrivial simple closed curves on $\Sigma$ whose topological type lies in $\sigma$. For such a $\gamma$, we have

$$fT^m_\gamma f^{-1} = T^m_{f(\gamma)}$$

for all $f \in \text{Mod}(\Sigma)$. It follows that $E$ is a normal subgroup of $\text{Mod}(\Sigma)$. By construction, $E \subset \Gamma$. For each nontrivial simple closed curve $\gamma$ on $\Sigma$ whose topological type lies in $\sigma$, recall that $\tilde{\tau}_\gamma$ is the image of $T^{d(\gamma)}_\gamma$ in $\text{Sp}(H, \omega)$. The Zariski closure in $\text{Sp}(H, \omega)$ of the subgroup generated by $\tilde{\tau}_\gamma$ is the one-parameter subgroup $\tilde{\tau}_{\gamma,t}$ defined by

$$\tilde{\tau}_{\gamma,t}(h) = h + \sum_{j=1}^k t e_j \omega(h, [\tilde{\gamma}_j]) \cdot [\tilde{\gamma}_j]$$

for $h \in H$ and $t \in \mathbb{Q}$.

The group $D_\sigma$ is generated by these one-parameter subgroups. The Zariski closure in $\text{Sp}(H, \omega)$ of the subgroup generated by

$$T^m_\gamma = \left(T^{d(\gamma)}_\gamma\right)^{m/d(\gamma)}$$

is the same one-parameter subgroup $\tilde{\tau}_{\gamma,t}$. It follows that the Zariski closure of the image of $E$ in $\text{Sp}(H, \omega)$ is also $D_\sigma$. Since $E$ is a normal subgroup of $\text{Mod}(\Sigma)$, it follows that $D_\sigma$ is a normal subgroup of $G$. Since $G$ is semisimple, so is $D_\sigma$, as desired.

3.4. Symplectic subspace. We now prove Theorem $B'$.

**Proof of Theorem $B'$.** The statement we must prove is as follows. Let $\pi: \tilde{\Sigma} \to \Sigma$ be a finite branched cover between closed oriented surfaces and $\sigma$ be a set of topological types of nontrivial simple closed curves on $\Sigma$. We must show that $H^\sigma = H_1(\tilde{\Sigma}; \mathbb{Q})$ is a symplectic subspace of $H = H_1(\Sigma; \mathbb{Q})$, or equivalently that $(H^\sigma)^\perp$ is a symplectic subspace. Lemma 2.1 implies that

$$(H^\sigma)^\perp = H^{D_\sigma},$$

and Lemma 3.2 says that the group $D_\sigma$ acts semisimply on $H$. The result thus follows from Lemma 3.1. \qed
4. Pants homology

In this section, we prove Theorem D′ (which implies Theorem D).

Proof of Theorem D′. We first recall the statement. Let \( \pi: \tilde{\Sigma} \to \Sigma \) be a finite branched cover between closed oriented surfaces and \( \mathcal{P} \) be a pants decomposition of \( \Sigma \). Let \( \sigma \) be the set of topological types of nontrivial curves appearing in \( \mathcal{P} \). We must prove that \( H = H_1(\tilde{\Sigma}; \mathbb{Q}) \) is spanned by \( H_1(\tilde{\Sigma}; \mathbb{Q}) \) and the set of homology classes of cycles \( \tilde{\gamma} \) on \( \tilde{\Sigma} \) such that \( \pi(\tilde{\gamma}) \) is disjoint from all curves in \( \mathcal{P} \).

Theorem B′ says that \( H_1(\tilde{\Sigma}; \mathbb{Q}) \) is a symplectic subspace of \( H_1(\Sigma; \mathbb{Q}) \), so \( H = H_1(\tilde{\Sigma}; \mathbb{Q}) \oplus (H_1(\tilde{\Sigma}; \mathbb{Q}))^\perp \).

It is thus enough to prove that \( (H_1(\tilde{\Sigma}; \mathbb{Q}))^\perp \) is spanned by the homology classes of cycles \( \tilde{\gamma} \) on \( \tilde{\Sigma} \) such that \( \pi(\tilde{\gamma}) \) is disjoint from all the curves in \( \mathcal{P} \).

Recall that we are working with homology with rational coefficients. Every element of \( H_1(\tilde{\Sigma}; \mathbb{Q}) \) is a multiple of an integral class, and every integral class can be represented by an oriented multicurve. Therefore, consider an oriented multicurve \( \tilde{\gamma} \) on \( \tilde{\Sigma} \) such that \( \pi(\tilde{\gamma}) \) is disjoint from all the curves in \( \mathcal{P} \).

Our pants decomposition \( \mathcal{P} \) looks like the following:

Write \( \mathcal{P} = \{\delta_1, \ldots, \delta_n\} \). Call the \( \delta_j \) that bound disks containing marked points the boundary loops (red in the above figure) and the other \( \delta_j \) the interior loops (blue in the above figure). Enumerate the components of \( \pi^{-1}(\delta_j) \) as \( j \) ranges over \( 1 \leq j \leq n \) as \( \{\tilde{\delta}_1, \ldots, \tilde{\delta}_m\} \). Call the \( \tilde{\delta}_j \) that project to boundary loops the lifted boundary loops and the \( \tilde{\delta}_j \) that project to interior loops the lifted interior loops.

Put the oriented multicurve \( \tilde{\gamma} \) in general position with respect to the \( \tilde{\delta}_j \). The lifted boundary loops bound disks in \( \tilde{\Sigma} \) containing a single branch point. Isotope \( \tilde{\gamma} \) such that it is disjoint from all these disks, and in particular is disjoint from all the lifted boundary loops.

Let \( \omega \) be the algebraic intersection form. Consider a lifted interior loop \( \tilde{\delta}_j \). Since \( [\tilde{\delta}_j] \in H_1(\Sigma; \mathbb{Q}) \) and \( [\tilde{\gamma}] \in (H_1(\Sigma; \mathbb{Q}))^\perp \), we have \( \omega([\tilde{\gamma}], [\tilde{\delta}_j]) = 0 \). This implies that the number of positively oriented intersection points of \( \tilde{\gamma} \) with \( \tilde{\delta}_j \) is the same as the number of negatively oriented intersection points. We can then modify \( \tilde{\gamma} \) as follows to make it disjoint from \( \tilde{\delta}_j \):

The result is an oriented multicurve that is homologous to \( \tilde{\gamma} \). Doing this for each lifted interior loop, we obtain an oriented multicurve \( \tilde{\gamma}' \) such that \( \tilde{\gamma}' = [\tilde{\gamma}] \) and such that \( \pi(\tilde{\gamma}') \) is disjoint from all the \( \tilde{\delta}_j \) and does not lie in any of the disks bounded by lifted boundary loops. This implies \( \pi(\tilde{\gamma}) \) is disjoint from all the curves in \( \mathcal{P} \), as desired. \( \square \)
5. A non-symplectic example

This section contains the proof of Theorem C.

5.1. Reduction. Theorem C asserts that for all closed oriented surfaces Σ of genus at least 2, there exists a finite unbranched cover \( \pi: \tilde{\Sigma} \to \Sigma \) such that for σ the set of nonseparating simple closed curves on Σ, the subspace \( H^1_\sigma(\tilde{\Sigma}; \mathbb{Z}) \) is not a symplectic subspace of \( H_1(\tilde{\Sigma}; \mathbb{Z}) \).

We start with the following.

**Lemma 5.1.** Let \( V \) be a finitely generated free abelian group equipped with a symplectic form and let \( W \) be a subgroup of \( V \). Assume that \( W \) is a symplectic subspace of \( V \) and that \( W \otimes \mathbb{Q} = V \otimes \mathbb{Q} \). Then \( W = V \).

**Proof.** Since \( W \) is a symplectic subspace of \( V \), we have \( V = W \oplus W^\perp \). Since \( W \otimes \mathbb{Q} = V \otimes \mathbb{Q} = (W \otimes \mathbb{Q}) \oplus (W^\perp \otimes \mathbb{Q}) \), it follows that \( W^\perp \otimes \mathbb{Q} = 0 \). We conclude that \( W^\perp = 0 \) and thus that \( W = V \). □

It is therefore enough to construct a finite unbranched cover \( \pi: \tilde{\Sigma} \to \Sigma \) such that \( H^1_\sigma(\tilde{\Sigma}; \mathbb{Q}) = H_1(\tilde{\Sigma}; \mathbb{Q}) \) but \( H^1_\sigma(\tilde{\Sigma}; \mathbb{Z}) \neq H_1(\tilde{\Sigma}; \mathbb{Z}) \).

For \( \ell \geq 2 \), let \( \pi: \Sigma[\ell] \to \Sigma \) be the cover corresponding to the homomorphism \( \pi_1(\Sigma) \to H_1(\Sigma; \mathbb{Z}/\ell) \).

By the above, it is enough to prove the following theorem, which we will do in the remainder of this section:

**Theorem 5.2.** Let Σ be a closed oriented surface of genus at least 2 and σ be the set of nonseparating simple closed curves on Σ. Fix some \( \ell \geq 2 \). The following then hold:

(i) We have \( H^1_\sigma(\Sigma[\ell]; \mathbb{Q}) = H_1(\Sigma[\ell]; \mathbb{Q}) \).

(ii) If \( \ell \geq 3 \), then \( H^1_\sigma(\Sigma[\ell]; \mathbb{Z}) \neq H_1(\Sigma[\ell]; \mathbb{Z}) \).

Part (ii) is a theorem of Irmer [5, Lemma 6]. We will give a simplified version of her argument below that avoids most of its complicated combinatorial group theory.

5.2. Rational equality. We start by proving part (i) of Theorem 5.2

**Proof of Theorem 5.2, part (i).** It is enough to prove that \( H^1_\sigma(\Sigma[\ell]; \mathbb{Q})^\perp = 0 \). During the proof of Theorem B', we showed that

\[ H^1_\sigma(\Sigma[\ell]; \mathbb{Q})^\perp = H_1(\Sigma[\ell]; \mathbb{Q})^{D_\sigma}. \]

It is thus enough to show that the group \( D_\sigma \) fixes no nonzero vectors in \( H_1(\Sigma[\ell]; \mathbb{Q}) \).

Let \( G_\ell \cong H_1(\Sigma; \mathbb{Z}/\ell) \) be the deck group of the cover \( \Sigma[\ell] \to \Sigma \). The actions of \( G_\ell \) and \( D_\sigma \) on \( H_1(\Sigma[\ell]; \mathbb{Q}) \) commute, so the action of \( D_\sigma \) preserves the decomposition of \( H_1(\Sigma[\ell]; \mathbb{Q}) \) into \( G_\ell \)-isotypic components. The action of the generators for \( D_\sigma \) on these isotypic components was studied in detail in [13], and it follows from these results that indeed it fixes no nonzero vectors in any isotypic component. □

13In fact, the orbits of all nonzero vectors in any isotypic component are infinite, as predicted by Conjecture 1.1.
5.3. Nilpotent preliminaries. Before we can prove part (ii) of Theorem 5.2, we need some preliminary results. Let $F_n$ be the free group on $\{x_1, \ldots, x_n\}$. Fix some $\ell \geq 3$. Define\(^{14}\)

\[
\hat{\ell} = \begin{cases} 
\ell & \text{if } \ell \text{ is odd,} \\
\ell/2 & \text{if } \ell \text{ is even.}
\end{cases}
\]

Since $\ell \geq 3$, we have $\hat{\ell} \geq 2$. Define $N_n[\ell]$ to be the quotient of $F_n$ by the normal subgroup generated by the following elements:\(^{15}\)

- The third term $[F_n, [F_n, F_n]]$ of the lower central series.
- The subgroup $[F_n, F_n^{\times \ell}]$, i.e., the subgroup generated by commutators $[u, v^{\hat{\ell}}]$ as $u$ and $v$ range over elements of $F_n$.

We will use boldface letters to denote elements of $N_n[\ell]$, and in particular will let $\{x_1, \ldots, x_n\}$ be the generators of $N_n[\ell]$ coming from the generators $\{x_1, \ldots, x_n\}$ for $F_n$. The abelianization of $N_n[\ell]$ is $\mathbb{Z}^n$, and for $u \in N_n[\ell]$ we will write $\overline{u} \in \mathbb{Z}^n$ for its image in the abelianization and $\widehat{\overline{u}} \in (\mathbb{Z}/\hat{\ell})^n$ for the image of $\overline{u}$ under the mod-$\hat{\ell}$ reduction map.

The following lemma clarifies the nature of $N_n[\ell]$:

**Lemma 5.3.** For $n \geq 2$ and $\ell \geq 3$, we have a central extension

\[ 1 \longrightarrow \Lambda^2(\mathbb{Z}/\hat{\ell})^n \longrightarrow N_n[\ell] \longrightarrow \mathbb{Z}^n \longrightarrow 1. \]

Here the map $N_n[\ell] \to \mathbb{Z}^n$ is the abelianization map taking $u \in N_n[\ell]$ to $\overline{u} \in \mathbb{Z}^n$, and for $u, v \in N_n[\ell]$ the commutator $[u, v] \in N_n[\ell]$ is the central element $\overline{u} \wedge \overline{v} \in \Lambda^2(\mathbb{Z}/\hat{\ell})^n$.

**Proof.** It is immediate from Magnus–Witt’s work on the lower central series of a free group ([15, 20]; see [19] for a textbook account) that

\[
\frac{[F_n, F_n]}{[F_n, [F_n, F_n]]} \cong \Lambda^2\mathbb{Z}^n,
\]

with $[u, v] \in [F_n, F_n]$ mapping to $\overline{u} \wedge \overline{v} \in \Lambda^2\mathbb{Z}^n$. Here $\overline{u}, \overline{v} \in \mathbb{Z}^n$ are the images of $u, v \in F_n$ in its abelianization. This fits into a central extension

\[ 1 \longrightarrow \Lambda^2\mathbb{Z}^n \longrightarrow \frac{F_n}{[F_n, [F_n, F_n]]} \longrightarrow \mathbb{Z}^n \longrightarrow 1. \]

To get $N_n[\ell]$ from the middle group in this extension, one quotients out the image of $[F_n, F_n^{\times \hat{\ell}}]$, which maps to the kernel of the map

\[ \Lambda^2\mathbb{Z}^n \longrightarrow \Lambda^2(\mathbb{Z}/\hat{\ell})^n. \]

The lemma follows. \(\square\)

In the rest of this section, we will identify $\Lambda^2(\mathbb{Z}/\hat{\ell})^n$ with the corresponding central subgroup of $N_n[\ell]$. The following calculation lies at the heart of our arguments:

**Lemma 5.4.** For $n \geq 2$ and $\ell \geq 3$, we have

\[ (uv)^\ell = u^\ell v^\ell \quad \text{for all } u, v \in N_n[\ell]. \]

**Proof.** To transform $(uv)^\ell$ into $u^\ell v^\ell$, we must commute each $u$ past all the $v$ terms to its left. Each time we commute a $u$ past a $v$, we must introduce a commutator $[v, u] = \overline{v} \wedge \overline{u}$. This commutator is central, so it can moved all the way to the right. The first $u$ must be

\(^{14}\)When reading this for the first time, it might be easier to assume that $\ell$ is odd, so $\hat{\ell} = \ell$.

\(^{15}\)Here “N” stands for “nilpotent”.
We must prove that

Whether $\ell$ is even or odd, the integer $\ell(\ell - 1)/2$ is divisible by $\hat{\ell}$. Since $[v, u] \in \wedge^2(\mathbb{Z}/\hat{\ell})^n$, this implies that $[v, u]^{\ell(\ell - 1)/2} = 1$. The lemma follows. \hfill $\square$

Define $P_n[\ell]$ to be the subgroup of $N_n[\ell]$ generated by $\{u^\ell \mid u \in N_n[\ell]\}$ and define $A_n[\ell]$ to be the subgroup generated by $P_n[\ell]$ and $\wedge^2(\mathbb{Z}/\hat{\ell})^n$. We then have the following:

**Lemma 5.5.** For $n \geq 2$ and $\ell \geq 3$, the subgroup $P_n[\ell]$ is a central subgroup of $N_n[\ell]$ with $P_n[\ell] = Z^n$, and $A_n[\ell] = P_n[\ell] \times \wedge^2(\mathbb{Z}/\hat{\ell})^n$.

**Proof.** The fact that $P_n[\ell]$ is a central subgroup follows from the fact that

$$[u^\ell, v] = u^\ell \wedge v = \ell (u \wedge v) = 0 \quad \text{for all } u, v \in N_n[\ell].$$

Recall that $N_n[\ell]$ is generated by the elements $x_1, \ldots, x_n$, which map to a basis for the abelianization $\mathbb{Z}^n$. The elements $x_i^\ell \in N_n[\ell]$ are central and map to linearly independent elements in the abelianization, so

$$P_n'[\ell] = \left\{x_1^{k_1} \cdots x_n^{k_n} \mid k_1, \ldots, k_n \in \mathbb{Z}\right\}$$

is a central subgroup satisfying $P_n'[\ell] = Z^n$. Moreover, letting $A_n'[\ell]$ be the subgroup of $N_n[\ell]$ generated by $P_n'[\ell]$ and $\wedge^2(\mathbb{Z}/\hat{\ell})^n$, we clearly have $A_n'[\ell] = P_n'[\ell] \times \wedge^2(\mathbb{Z}/\hat{\ell})^n$.

To prove the lemma, it is therefore enough to prove that $P_n[\ell] = P_n'[\ell]$. Since $x_i^\ell \in P_n'[\ell]$ for all $1 \leq i \leq n$, we have $P_n'[\ell] \subset P_n[\ell]$. For the reverse inclusion, consider some $u \in N_n[\ell]$. We must prove that $u^\ell \in P_n'[\ell]$. We can find $k_1, \ldots, k_n \in \mathbb{Z}$ such that

$$u = x_1^{k_1} \cdots x_n^{k_n} c.$$

Applying Lemma 5.4 repeatedly, we deduce that

$$u^\ell = x_1^{\ell k_1} \cdots x_n^{\ell k_n} c^\ell = x_1^{\ell k_1} \cdots x_n^{\ell k_n} \in P_n'[\ell].$$

\hfill $\square$

5.4. **Integral inequality.** We now prove part (ii) of Theorem 5.2

**Proof of Theorem 5.2, part (ii).** We first recall the statement. Let $\Sigma$ be a closed oriented surface of genus $g \geq 2$ and $\sigma$ be the set of nonseparating simple closed curves on $\Sigma$. Fix some $\ell \geq 3$, and as above let

$$\hat{\ell} = \begin{cases} \ell & \text{if } \ell \text{ is odd,} \\ \ell/2 & \text{if } \ell \text{ is even.} \end{cases}$$

Since $\ell \geq 3$, we have $\hat{\ell} \geq 2$. We must prove that $H^2_\sigma(\Sigma[\ell]; \mathbb{Z}) \neq H_1(\Sigma[\ell]; \mathbb{Z})$.

Recall that $\Sigma[\ell]$ is the cover corresponding to the homomorphism

$$\pi_1(\Sigma) \to H_1(\Sigma[\ell]; \mathbb{Z}) \cong (\mathbb{Z}/\hat{\ell})^{2g}.$$

It follows that $\pi_1(\Sigma[\ell])$ is the kernel of this map, so $\pi_1(\Sigma[\ell])$ is the subgroup of $\pi_1(\Sigma)$ generated by the following two subgroups:

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16The purpose of using $\hat{\ell}$ is to ensure this.
17Here “P” stands for “power subgroup”.
18Here “A” stands for “abelian subgroup”; see Lemma 5.5.
• The commutator subgroup $[\pi_1(\Sigma), \pi_1(\Sigma)]$.
• The subgroup $P$ generated by $\{x^\ell \mid x \in \pi_1(\Sigma)\}$.

Each nonseparating simple closed curve $x \in \pi_1(\Sigma)$ maps to a primitive\(^{19}\) element of $H_1(\Sigma; \mathbb{Z})$, so the minimal power of $x$ that lies in $\pi_1(\Sigma[\ell])$ is $x^\ell$. It follows that the image $\overline{P}$ of $P$ in $H_1(\Sigma[\ell]; \mathbb{Z})$ contains $H_1(\Sigma; \mathbb{Z})$. It is enough therefore to prove that $\overline{P} \neq H_1(\Sigma[\ell]; \mathbb{Z})$.

Let $\{a_1, b_1, \ldots, a_g, b_g\}$ be the standard generating set for $\pi_1(\Sigma)$ satisfying the surface relation $[a_1, b_1] \cdots [a_g, b_g] = 1$. We can then define a homomorphism $\phi : \pi_1(\Sigma) \to N_g[\ell]$ via the formulas

$$\phi(a_i) = x_i \quad \text{and} \quad \phi(b_i) = 1 \quad \text{for } 1 \leq i \leq g.$$ 

The map $\phi$ takes $[\pi_1(\Sigma), \pi_1(\Sigma)]$ to the central subgroup $\wedge^2(\mathbb{Z}/\ell)^g$ and $P$ to the central subgroup $P_g[\ell]$ (see Lemma 5.5). It follows that $\phi$ takes $\pi_1(\Sigma[\ell])$ surjectively onto the abelian subgroup $A_g[\ell] = P_g[\ell] \times \wedge^2(\mathbb{Z}/\ell)^g$ identified by Lemma 5.5. The restriction of $\phi$ to $\pi_1(\Sigma[\ell])$ thus factors through $H_1(\Sigma[\ell]; \mathbb{Z})$, and takes $\overline{P} \subset H_1(\Sigma[\ell]; \mathbb{Z})$ to the proper subgroup $P_g[\ell]$ of $A_g[\ell]$. The theorem follows. \(\square\)

References


\(^{19}\)That is, not divisible by any integers except $\pm 1$. 


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