VIC-modules over noncommutative rings

Andrew Putman* Steven V Sam†

Abstract

For a finite ring R, not necessarily commutative, we prove that the category of $\mathtt{VIC}(R)$-modules over a left Noetherian ring $\mathbf{k}$ is locally Noetherian, generalizing a theorem of the authors that dealt with commutative R. As an application, we prove a very general twisted homology stability for $\mathrm{GL}_n(R)$ with R a finite noncommutative ring.

1 Introduction

The program of representation stability was introduced by Church and Farb [3, 6]. The idea is that many of the representations that occur in nature depend on a parameter n, and it is useful to study algebraic structures that encode all of these representations simultaneously. For instance, the cohomology groups of the space $\mathrm{Conf}_n(\mathbb{R}^2)$ of configurations of n labeled points in $\mathbb{R}^2$ are representations of the symmetric group S_n, which acts by permuting the n points. Individually, these are very hard to understand; however, taken together they have a lot of global structure, especially as $n \mapsto \infty$.

Representations of categories. This can be encoded in many ways. One of the most fruitful is Church–Ellenberg–Farb's [1] theory of FI-modules. Here FI is the category whose objects are the finite sets $[n] = \{1, \ldots, n\}$ and whose morphisms are injections. For a category C like FI and a ring $\mathbf{k}$, a C-*module* over $\mathbf{k}$ is a functor M from C to $\mathbf{k}$-Mod. Thus M consists of a $\mathbf{k}$-module M_c for every object $c \in \mathtt{C}$ and a $\mathbf{k}$-module map $f\colon M_c \to M_d$ for every C-morphism $f\colon c \to d$. For an FI-module M, we will write M_n for $M_{[n]}$. The FI-endomorphisms of $[n]$ are S_n, and these act on M_n, making each M_n a representation of S_n.

Example 1.1. For a fixed p, we can define an FI-module M over $\mathbb{Z}$ with $M_n = \mathrm{H}^p(\mathrm{Conf}_n(\mathbb{R}^2); \mathbb{Z})$. The induced $S_n = \mathrm{End}_{\mathtt{FI}}([n])$-action on M_n is precisely the S_n-action on $\mathrm{H}^p(\mathrm{Conf}_n(\mathbb{R}^2); \mathbb{Z})$ from the previous paragraph. We therefore get a single object encoding all these representations together with the various ways that they are related as $n \mapsto \infty$. □

Homological algebra. For a category C, the collection of C-modules over $\mathbf{k}$ forms an abelian category whose morphisms are natural transformations between functors $\mathtt{C} \to \mathbf{k}$-Mod. An important insight of Church–Ellenberg–Farb [1] is that one can do commutative and homological algebra in this category in a way that is very similar to the category $\mathbf{k}$-Mod. For instance, one can construct projective resolutions, take derived functors, etc.

*Supported in part by NSF grant DMS-1811322
†Supported in part by NSF grant DMS-1849173

Local Noetherianity. Perhaps the most important technical result for this is a version of the Hilbert basis theorem. A $\mathtt{C}$-module M over a ring $\mathbf{k}$ is *finitely generated* if there exist objects $c_1, \ldots, c_k \in \mathtt{C}$ and elements $x_i \in M_{c_i}$ such that the smallest $\mathtt{C}$-submodule of M containing all the x_i is M. In other words, for each $c \in \mathtt{C}$ the set

$$\bigoplus_{i=1}^{k} \left(\bigoplus_{f\colon c_i \to c} f\left(x_i\right) \right) \subset M_c$$

spans M_c. We say that the category of $\mathtt{C}$-modules over $\mathbf{k}$ is *locally Noetherian* if for all finitely generated $\mathtt{C}$-modules M over $\mathbf{k}$, all $\mathtt{C}$-submodules of M are finitely generated. Generalizing previous work that dealt for instance with fields $\mathbf{k}$ of characteristic 0, Church–Ellenberg–Farb–Nagpal [2] proved that the category of $\mathtt{FI}$-modules over a left Noetherian ring $\mathbf{k}$ is locally Noetherian.

VIC-modules. The category of $\mathtt{FI}$-modules encodes representations of the symmetric groups, and there has been a huge amount of work developing analogues for other families of groups (see, e.g., [8, 14, 15, 17, 20]). One particularly important family of groups are the general linear groups $\mathrm{GL}_n(R)$ over a ring R. Here it is natural to look at categories whose objects are the finite-rank free right R-modules R^n with $n \geq 0$. As for the morphisms, there are several potential choices. To help keep the notation for our morphisms straight, we will write $[R^n]$ when we mean to regard R^n as an object of one of our categories and R^n when we mean to regard it as an R-module.

- The category $\mathtt{V}(R)$, whose morphisms $[R^n] \to [R^m]$ are R-linear maps $R^n \to R^m$. Versions of this go back to work of Lannes and Schwartz and are the focus of the *Artinian conjecture* (see [11, Conjecture 3.12]), which was resolved independently by the authors [14] and by Sam–Snowden [17].
- The category $\mathtt{VI}(R)$, whose morphisms $[R^n] \to [R^m]$ are injective R-linear maps $f\colon R^n \to R^m$ that are splittable in the sense that there exists some $g\colon R^m \to R^n$ with $g \circ f = \mathrm{id}$. Equivalently, the image of f is a summand of R^m. This was introduced by Scorichenko in his thesis ([18]; see [7] for a published account).
- The category $\mathtt{VIC}(R)$, whose morphisms $[R^n] \to [R^m]$ are pairs (f_1, f_2), where $f_1\colon R^n \to R^m$ is an injective R-linear map and $f_2\colon R^m \to R^n$ is a splitting of f_1, so $f_2 \circ f_1 = \mathrm{id}$. This was introduced by the authors in [14].

Remark 1.2. One motivation for studying $\mathtt{VIC}(R)$ is that it is the only one of these categories where there is a functor $\mathtt{VIC}(R) \to \text{Groups}$ taking $R^n \in \mathtt{VIC}(R)$ to $\mathrm{GL}_n(R)$. For a morphism $(f_1, f_2)\colon [R^n] \to [R^m]$, the induced group homomorphism $\mathrm{GL}_n(R) \to \mathrm{GL}_m(R)$ is as follows. Set $C = \ker(f_2)$, so $R^m = \mathrm{im}(f_1) \oplus C$. Our homomorphism then takes $\phi \in \mathrm{GL}_n(R)$ to the map $R^m \to R^m$ obtained from $f_1 \circ \phi \circ f_1^{-1}\colon \mathrm{im}(f_1) \to \mathrm{im}(f_1)$ by extending over C by the identity. □

Remark 1.3. Our definition of $\mathtt{VIC}(R)$ is slightly different from the one in [14], which requires that a $\mathtt{VIC}(R)$-morphism (f_1, f_2) also have $\ker(f_2)$ free. For finite (and, more generally, Artinian) rings, this added condition is superfluous: $\ker(f_2)$ is in any case stably free, and for Artinian rings finitely generated stably free modules are free (see [13, Example I.4.7.3]; rings with this property are called *Hermite rings*). □

Main theorem. Fix a left Noetherian ring $\mathbf{k}$. In [14], it is proven that for a finite *commutative* ring R, the categories of $\mathtt{V}(R)$- and $\mathtt{VI}(R)$- and $\mathtt{VIC}(R)$-modules over $\mathbf{k}$ are all locally Noetherian (see [17] for alternate proofs for $\mathtt{V}(R)$ and $\mathtt{VI}(R)$, but not for $\mathtt{VIC}(R)$). However, in many situations (e.g. in algebraic K-theory), it is important to study $\mathrm{GL}_n(R)$ where R is a noncommutative ring. For instance, R might be a group ring $\mathbb{F}_p[G]$ for a finite group G. Our main theorem addresses this more general situation:

Theorem A. *Let R be a finite ring, not necessarily commutative, and let $\mathbf{k}$ be a left Noetherian ring. Then the categories of $\mathtt{V}(R)$-modules and $\mathtt{VI}(R)$-modules and $\mathtt{VIC}(R)$-modules over $\mathbf{k}$ are locally Noetherian.*

Remark 1.4. For a category $\mathtt{C}$ like $\mathtt{VIC}(R)$, a $\mathtt{C}$-module is a functor $\mathtt{C} \to \mathbf{k}$-Mod for a ring $\mathbf{k}$. We allow $\mathbf{k}$ to also be noncommutative. In fact, for R commutative the proof of Theorem A in [14] works in that level of generality. □

Remark 1.5. For infinite commutative R, the authors proved in [14] that the categories of $\mathtt{V}(R)$- and $\mathtt{VI}(R)$- and $\mathtt{VIC}(R)$-modules over a ring $\mathbf{k}$ are *not* locally Noetherian. The same argument works for infinite noncommutative R. See [9] for one way to get around this for $R = \mathbb{Z}$. □

Application: twisted homological stability. A basic theorem of van der Kallen [19] says that for rings R satisfying mild hypotheses (for instance, all finite rings), the groups $\mathrm{GL}_n(R)$ satisfy *homological stability*, i.e. for all p, we have

$$\mathrm{H}_p(\mathrm{GL}_n(R); \mathbb{Z}) \cong \mathrm{H}_p(\mathrm{GL}_{n+1}(R); \mathbb{Z}) \quad \text{for } n \gg p.$$

In fact, building on ideas of Dwyer [5], van der Kallen is even able to prove this for certain twisted coefficient systems (those that are "polynomial" in an appropriate sense). For example, he is able to show for all $m \geq 0$ that we have

$$\mathrm{H}_p(\mathrm{GL}_n(R); (R^n)^{\otimes m}) \cong \mathrm{H}_p(\mathrm{GL}_{n+1}(R); (R^{n+1})^{\otimes m}) \quad \text{for } n \gg p.$$

In [14, §4], the authors showed how to deduce a much more general version of this for finite commutative rings from the local Noetherianity of $\mathtt{VIC}(R)$. Given our new Theorem A, the exact same argument gives the following result for finite noncommutative rings. For a $\mathtt{VIC}(R)$-module M, write M_n for the value of M on $[R^n] \in \mathtt{VIC}(R)$. The $\mathtt{VIC}(R)$-endomorphisms of $[R^n]$ are $\mathrm{GL}_n(R)$, so M_n is a representation of $\mathrm{GL}_n(R)$.

Theorem B. *Let R be a finite ring, not necessarily commutative, and let M be a finitely generated $\mathtt{VIC}(R)$-module over a left Noetherian ring $\mathbf{k}$. Then for all $p \geq 0$, we have*

$$\mathrm{H}_p(\mathrm{GL}_n(R); M_n) \cong \mathrm{H}_p(\mathrm{GL}_{n+1}(R); M_{n+1})$$

for $n \gg p$.

Remark 1.6. The proof of Theorem B for commutative rings in [14, §4] uses the more stringent definition of $\mathtt{VIC}(R)$ discussed in Remark 1.3, which as we discussed there is equivalent to ours for finite rings. □

Remark 1.7. In [19], van der Kallen also gives an explicit estimate of when this stability occurs. Since we apply our non-effective Noetherianity theorem, we are not able to give such an estimate. □

Ideas from proof. We will derive Theorem A for $\mathtt{V}(R)$ and $\mathtt{VI}(R)$ from the case of $\mathtt{VIC}(R)$, so we will focus on that category. In [14], this is dealt with for finite commutative R by a sort of Gröbner basis argument that was introduced to the theory of representation stability in [17] (though the general theorems of [17] do not apply to $\mathtt{VIC}(R)$; also, we remark that a similar kind of argument appeared much earlier in work of Richter [16]). We do the same thing, but the details are far harder. The main issue is that finite noncommutative rings are much more complicated than finite commutative rings. Indeed, the starting point of the proof in [14] is the fact that finite commutative rings are Artinian, and thus are the product of finitely many local rings. Local rings are not that different from fields, so in the end we can mostly focus on the case of finite fields. Unfortunately, noncommutative Artinian rings are not nearly as well-behaved, which greatly complicates the proof.

Convention: left vs right modules. Throughout this paper, we emphasize that column vectors R^n are considered as right R-modules. With this convention, the group $\mathrm{GL}_n(R)$ acts on R^n on the left by right R-module homomorphisms. If we wanted to deal with left R-modules, then we would have to use row vectors and have $\mathrm{GL}_n(R)$ act on the right.

Outline. We start in §2 by reducing to proving local Noetherianity for an "ordered" version of $\mathtt{VIC}(R)$ called $\mathtt{OVIC}(R)$. The rest of the paper is devoted to this: in §3, we discuss the structure of finite noncommutative rings, in §4 we define $\mathtt{OVIC}(R)$ and give its basic properties, and finally in §5 we prove that the category of $\mathtt{OVIC}(R)$-modules is locally Noetherian.

Remark 1.8. Some parts of our argument are the same as in [14], but we tried to make this paper mostly self-contained at least for $\mathtt{VIC}(R)$. The fact that we will focus on this single category will allow us to write in a much less abstract way, so one side benefit is that we think some of the details of the proof here will be a little easier to parse. □

Acknowledgments. We would like to thank Benson Farb and Andrew Snowden for helpful comments, and Peter Patzt for pointing out a small mistake in an earlier version of this paper.

2 Reduction to ordered VIC

Instead of working with $\mathtt{VIC}(R)$ directly, our proof will focus on a subcategory $\mathtt{OVIC}(R)$. The "O" stands for "ordered". Its main properties are as follows:

Theorem 2.1. *Let R be a finite ring. There exists a subcategory $\mathtt{OVIC}(R)$ of $\mathtt{VIC}(R)$ with the following properties:*

(a) The objects of $\mathtt{OVIC}(R)$ are the same as $\mathtt{VIC}(R)$: the finite-rank free R-modules R^n for $n \geq 0$.

(b) *Every* $\texttt{VIC}(R)$*-morphism* $f\colon [R^d] \to [R^n]$ *can be factored as*

$$[R^d] \xrightarrow{f_1} [R^d] \xrightarrow{f_2} [R^n],$$

where $f_1\colon [R^d] \to [R^d]$ *is a* $\texttt{VIC}(R)$*-morphism and* $f_2\colon [R^d] \to [R^n]$ *is an* $\texttt{OVIC}(R)$*-morphism.*

(c) *The category of* $\texttt{OVIC}(R)$*-modules over a left Noetherian ring* $\mathbf{k}$ *is locally Noetherian.*

The proof of Theorem 2.1 is spread throughout the rest of the paper: in §3, we discuss some ring-theoretic preliminaries, in §4 we construct $\texttt{OVIC}(R)$ and prove part (b) of Theorem 2.1 (see Proposition 4.5), and finally in §5 we prove part (c) of Theorem 2.1 (see Proposition 5.4). Here we will show how to use Theorem 2.1 to prove Theorem A.

Proof of Theorem A, assuming Theorem 2.1. Let R be a finite ring, not necessarily commutative, and let $\mathbf{k}$ be a left Noetherian ring. In [14, §2.4], the local Noetherianity of the categories of $\texttt{V}(R)$- and $\texttt{VI}(R)$-modules over $\mathbf{k}$ for finite commutative rings R are derived from the local Noetherianity of the category of $\texttt{VIC}(R)$-modules over $\mathbf{k}$. This derivation does not make use of the commutativity of R, so we must just prove that the category of $\texttt{VIC}(R)$-modules over $\mathbf{k}$ is locally Noetherian.

Let M be a finitely generated $\texttt{VIC}(R)$-module over $\mathbf{k}$. Our goal is to prove that every $\texttt{VIC}(R)$-submodule of M is finitely generated. Theorem 2.1 says that for the subcategory $\texttt{OVIC}(R)$ of $\texttt{VIC}(R)$, the category of $\texttt{OVIC}(R)$-modules over $\mathbf{k}$ is locally Noetherian. Via restriction, we can regard M as an $\texttt{OVIC}(R)$-module, so it is enough to prove that M is finitely generated as an $\texttt{OVIC}(R)$-module.

We will do this by studying representable $\texttt{VIC}(R)$-modules, which function similarly to free modules. For $d \geq 0$, let $P(d)$ be the $\texttt{VIC}(R)$-module defined via the formula

$$P(d)_n = \mathbf{k}[\mathrm{Hom}_{\texttt{VIC}(R)}(R^d, R^n)] \qquad (n \geq 0).$$

By Theorem 2.1, every $\texttt{VIC}(R)$-module morphism $f\colon [R^d] \to [R^n]$ can be factored as

$$[R^d] \xrightarrow{f_1} [R^d] \xrightarrow{f_2} [R^n],$$

where $f_1\colon [R^d] \to [R^d]$ is a $\texttt{VIC}(R)$-morphism and $f_2\colon [R^d] \to [R^n]$ is an $\texttt{OVIC}(R)$-morphism. This implies that as an $\texttt{OVIC}(R)$-module, $P(d)$ is generated by the set $\mathrm{Hom}_{\texttt{VIC}(R)}(R^d, R^d) \subset P(d)_d$, which is finite since R is a finite ring.

For all $x \in M_d$ there exists a $\texttt{VIC}(R)$-morphism $P(d) \to M$ taking the element $\mathrm{id}\colon [R^d] \to [R^d]$ of $P(d)_d = \mathbf{k}[\mathrm{Hom}_{\texttt{VIC}(R)}(R^d, R^d)]$ to x. The image of this $\texttt{VIC}(R)$-morphism is the $\texttt{VIC}(R)$-submodule spanned by x. Since M is finitely generated, for some $d_1, \ldots, d_k \geq 1$ we can find elements $x_i \in M_{d_i}$ such that $\{x_1, \ldots, x_k\}$ generates M. Associated to these x_i is a surjective $\texttt{VIC}(R)$-morphism

$$\bigoplus_{i=1}^{k} P(d_i) \longrightarrow M.$$

Since each $P(d_i)$ is finitely generated as an $\texttt{OVIC}(R)$-module, so is M. $\square$

3 The structure of Artinian rings

To discuss $\texttt{OVIC}(R)$, we will need some basic facts about finite rings. In fact, the results we need hold more generally for Artinian rings, so we will state them in this level of generality. A suitable textbook reference is [12]. Throughout this section, R is an Artinian ring.

Peirce decomposition, I. We begin with some generalities (see [12, §21]). Assume that $\{e_1, \ldots, e_\mu\}$ are idempotent elements of R that are orthogonal (i.e. $e_i e_j = 0$ for distinct $1 \le i, j \le \mu$) and satisfy

$$1 = e_1 + \cdots + e_\mu.$$

Each $e_i R e_j$ is an additive subgroup of R, and we have the Peirce decomposition

$$R = \bigoplus_{i,j=1}^{\mu} e_i R e_j. \tag{3.1}$$

To make this a ring isomorphism, view elements of the right hand side as $\mu \times \mu$ matrices whose (i, j)-entries lie in $e_i R e_j$. Using the fact that

$$(e_i R e_k)(e_k R e_j) \subset e_i R e_j,$$

we can multiply these matrices as usual, turning the right hand side of (3.1) into a ring and (3.1) into a ring isomorphism. Since $e_i R e_j \subset R$, we can view (3.1) as an embedding $\Phi\colon R \hookrightarrow \mathrm{Mat}_\mu(R)$ that we will call the *Peirce embedding*.

Peirce decomposition, II. Continue with the notation of the previous paragraph. A more conceptual way to think about the Peirce embedding is as follows. Each $e_i R$ is a right R-module, and letting R_R denote R considered as a right R-module we have

$$R_R = \bigoplus_{i=1}^{\mu} e_i R.$$

The ring R acts on the left on R_R by right R-module endomorphisms, and in fact $R \cong \mathrm{End}(R_R)$. We thus have

$$R = \mathrm{End}(R_R) = \bigoplus_{i,j=1}^{\mu} \mathrm{Hom}(e_j R, e_i R). \tag{3.2}$$

For all $1 \le i, j \le \mu$, we have $\mathrm{Hom}(e_j R, e_i R) = e_i R e_j$, where $\phi \in \mathrm{Hom}(e_j R, e_i R)$ corresponds to the element $\phi(e_j) \in e_i R e_j$. Making these identifications turns (3.2) into (3.1). This makes it clear that the Peirce embedding reflects the left action of R on R_R; indeed, using

$$R_R = \bigoplus_{i=1}^{\mu} e_i R,$$

we can embed R_R into the set of length-μ column vectors R^μ, which is itself a right R-module. The matrices $\mathrm{Mat}_\mu(R)$ act on R^μ, and we have a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\cong} & \mathrm{End}(R_R) \\ {\scriptstyle \Phi}\downarrow & & \downarrow \\ \mathrm{Mat}_\mu(R) & \xrightarrow{\cong} & \mathrm{End}(R^\mu). \end{array}$$

Jacobson radical. Let $J(R)$ be the Jacobson radical of R. By definition, $J(R)$ consists of all $y \in R$ such that for all $x, z \in R$, the element $1 - xyz$ is a unit. Since R is Artinian, $J(R)$ can also be characterized as the largest ideal of R that is nilpotent, i.e. such that $J(R)^k = 0$ for $k \gg 0$ (see [12, Theorem 4.12]). Let $\overline{R} = R/J(R)$. For $x \in R$, let $\overline{x} \in \overline{R}$ be its image. Also, for a matrix $M \in \mathrm{Mat}_{n,m}(R)$, let $\overline{M} \in \mathrm{Mat}_{n,m}(\overline{R})$ be its image. The following simple fact will be very important for us.

Lemma 3.1. *Let R be a ring and let $M \in \mathrm{Mat}_n(R)$ for some $n \geq 1$. Then M is invertible if and only if $\overline{M}$ is invertible.*

Proof. We have $J(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(J(R))$ (see [12, p. 61]), so $\overline{\mathrm{Mat}_n(R)} = \mathrm{Mat}_n(\overline{R})$. The result now follows from the fact that for *any* ring R, an element $x \in R$ is invertible if and only if $\overline{x} \in \overline{R}$ is invertible. □

Artin–Wedderburn. The fact that R is Artinian implies that $\overline{R}$ is semisimple (see [12, Theorem 4.14]), which by the Artin–Wedderburn Theorem [12, Theorem 3.5] means that

$$\overline{R} \cong \mathrm{Mat}_{\mu_1}(\mathbb{D}_1) \times \cdots \times \mathrm{Mat}_{\mu_q}(\mathbb{D}_q) \tag{3.3}$$

for division rings $\mathbb{D}_1, \ldots, \mathbb{D}_q$. We remark that when R is finite as it is in most of this paper, Wedderburn's Little Theorem [12, Theorem 13.1] implies that the $\mathbb{D}_k$ are actually (commutative) fields. The decomposition (3.3) arises from orthogonal idempotents $\overline{e}_i^k \in \overline{R}$ for $1 \leq k \leq q$ and $1 \leq i \leq \mu_k$ satisfying

$$1 = (\overline{e}_1^1 + \cdots + \overline{e}_{\mu_1}^1) + \cdots + (\overline{e}_1^q + \cdots + \overline{e}_{\mu_q}^q) \quad \text{and} \quad \overline{e}_i^k \overline{R} \cong \mathbb{D}_k^{\mu_k}. \tag{3.4}$$

Here $\mathbb{D}_k^{\mu_k}$ denotes the right $\overline{R}$-module consisting of length-μ_k column vectors with entries in $\mathbb{D}_k$. Setting $\mu = \mu_1 + \cdots + \mu_q$, the Peirce embedding associated to (3.4) is precisely the embedding $\overline{\Phi} \colon \overline{R} \hookrightarrow \mathrm{Mat}_\mu(\overline{R})$ taking an element of $\overline{R}$ to the matrices in (3.3), arranged as diagonal blocks in $\mathrm{Mat}_\mu(\overline{R})$.

Lifting idempotents. Since $J(R)$ is nilpotent, idempotents in $\overline{R}$ can be lifted to R (see [12, Theorem 21.28]; we remark that a ring R such that $\overline{R}$ is semisimple and all idempotents in $\overline{R}$ can be lifted to R is called *semiperfect*). Combined with [12, Proposition 21.25] and the proof of [12, Theorem 23.6], this implies we can find orthogonal idempotents $e_i^k \in R$ for $1 \leq k \leq q$ and $1 \leq i \leq \mu_k$ lifting the $\overline{e}_i^k$ such that

$$1 = (e_1^1 + \cdots + e_{\mu_1}^1) + \cdots + (e_1^q + \cdots + e_{\mu_q}^q). \tag{3.5}$$

What is more, by [12, Proposition 21.21], we have

$$e_i^k R \cong e_{i'}^{k'} R \quad \Leftrightarrow \quad \overline{e}_i^k \overline{R} \cong \overline{e}_{i'}^{k'} \overline{R} \quad \Leftrightarrow \quad k = k'. \tag{3.6}$$

For $1 \leq h, k \leq q$, let

$$\mathbb{L}_{hk} = e_1^h R e_1^k \cong \mathrm{Hom}(e_1^k R, e_1^h R).$$

We thus have

$$\overline{\mathbb{L}}_{kk} = \mathbb{D}_k \quad \text{and} \quad \overline{\mathbb{L}}_{hk} = 0 \quad \text{for } h \neq k.$$

The rings $\mathbb{L}_{kk}$ are thus local rings, and the $\mathbb{L}_{hk}$ are additive subgroups of $J(R)$.

Summary. Recall that $\mu = \mu_1 + \cdots + \mu_q$. Using the isomorphisms (3.6), the Peirce embedding $\Phi\colon R \hookrightarrow \mathrm{Mat}_\mu(R)$ associated to (3.5) can be identified with a ring homomorphism that takes $x \in R$ to a $q \times q$ block matrix of the form

$$\Phi(x) = (\Phi_{hk}(x))_{h,k=1}^{q} \quad \text{with} \quad \Phi_{hk}(x) \in \mathrm{Mat}_{\mu_h,\mu_k}(\mathbb{L}_{hk}).$$

Moreover,

$$\overline{\Phi(x)} = \overline{\Phi}(\overline{x}) = \left(\overline{\Phi_{hk}(x)}\right)_{h,k=1}^{q} \quad \text{with} \quad \overline{\Phi_{hh}(x)} \in \mathrm{Mat}_{\mu_h}(\mathbb{D}_h) \text{ and } \overline{\Phi_{hk}(x)} = 0 \text{ for } h \neq k.$$

We will call this the *Artin–Wedderburn embedding* of R.

4 Ordered VIC: definition and basic properties

This section defines the subcategory $\mathtt{OVIC}(R)$ of $\mathtt{VIC}(R)$ and proves some basic facts about it. We do this in two steps: in §4.1, we deal with semisimple rings, and in §4.2 we deal with Artinian rings (and thus general finite rings).

4.1 Ordered VIC for semisimple rings

We start by introducing the notation we will use in this section. Let R be a semisimple ring, so

$$R \cong \mathrm{Mat}_{\mu_1}(\mathbb{D}_1) \times \cdots \times \mathrm{Mat}_{\mu_q}(\mathbb{D}_q) \tag{4.1}$$

for $\mu_1, \ldots, \mu_q \geq 1$ and division rings $\mathbb{D}_1, \ldots, \mathbb{D}_q$. Set $\mu = \mu_1 + \cdots + \mu_q$ and let $\Phi\colon R \hookrightarrow \mathrm{Mat}_\mu(R)$ be the Artin–Wedderburn embedding of R. For $x \in R$, the matrix $\Phi(x)$ thus consists of the matrices in (4.1), arranged as diagonal blocks in $\mathrm{Mat}_\mu(\overline{R})$.

Decomposing maps. Consider an R-linear map $h\colon R^m \to R^n$. Via (4.1), we can identify h with a collection of $\mathbb{D}_k$-linear maps $h_k\colon \mathbb{D}_k^{\mu_k m} \to \mathbb{D}_k^{\mu_k n}$ for $1 \leq k \leq q$. The matrix of h_k is a submatrix of the matrix corresponding to the R-linear map $\Phi(h)\colon R^{\mu m} \to R^{\mu n}$ obtained by applying Φ to each entry of the matrix representing h.

Distinguished bases. We will need notation for the collections of basis elements of $R^{\mu m}$ and $R^{\mu n}$ corresponding to these submatrices. The *distinguished basis of* $R^{\mu m}$ is defined as follows. For each $1 \leq k \leq q$, let $\{\vec{v}(k)_1, \ldots, \vec{v}(k)_{\mu_k m}\}$ be the portion of the standard basis of $R^{\mu m}$ corresponding to the columns of $\Phi(h)$ whose nonzero entries are required to be in $\mathbb{D}_k$, arranged in their natural increasing order. In its natural ordering, the standard basis for $R^{\mu m}$ is thus

$$\vec{v}(1)_1, \ldots, \vec{v}(1)_{\mu_1}, \vec{v}(2)_1, \ldots, \vec{v}(2)_{\mu_2}, \ldots, \vec{v}(q)_1, \ldots, \vec{v}(q)_{\mu_q}$$

followed by

$$\vec{v}(1)_{\mu_1+1}, \ldots, \vec{v}(1)_{\mu_1+\mu_1}, \vec{v}(2)_{\mu_2+1}, \ldots, \vec{v}(2)_{\mu_2+\mu_2}, \ldots, \vec{v}(q)_{\mu_q+1}, \ldots, \vec{v}(q)_{\mu_q+\mu_q},$$

etc., finally ending with

$$\vec{v}(1)_{(m-1)\mu_1+1}, \ldots, \vec{v}(1)_{(m-1)\mu_1+\mu_1}, \ldots, \vec{v}(q)_{(m-1)\mu_q+1}, \ldots, \vec{v}(q)_{(m-1)\mu_q+\mu_q}.$$

Similarly, the *distinguished basis of* $R^{\mu n}$ is defined by letting $\{\vec{w}(k)_1, \ldots, \vec{w}(k)_{\mu_k n}\}$ for $1 \leq k \leq q$ be the portion of the standard basis of $R^{\mu n}$ corresponding to the rows of $\Phi(h)$ whose nonzero entries are required to be in $\mathbb{D}_k$, arranged in their natural increasing order. For all $1 \leq k \leq q$ and $0 \leq j \leq \mu_k m$, we thus have

$$\Phi(h)(\vec{v}(k)_j) \subset \bigoplus_{i=1}^{\mu_k n} \vec{w}(k)_i \cdot \mathbb{D}_k.$$

Surjective maps. Now assume that $h\colon R^m \to R^n$ is a surjective R-linear map. The maps $h_k\colon \mathbb{D}_k^{\mu_k m} \to \mathbb{D}_k^{\mu_k n}$ discussed above are thus also surjective. Recall that linear algebra over division rings is very similar to linear algebra over fields. In particular, notions of basis, dimension, etc. make sense in this noncommutative context. Considerations of dimension show that there exists some subset $S \subset \{1, \ldots, \mu_k m\}$ such that $\{\Phi(h)(\vec{v}(k)_i) \mid i \in S\}$ is a basis for the $\mathbb{D}_k$-submodule of $R^{\mu_k n}$ spanned by $\{\vec{w}(k)_1, \ldots, \vec{w}(k)_{\mu_k n}\}$. Order $\mu_k n$-element subsets of $\{1, \ldots, \mu_k m\}$ with the lexicographic order, and define $\mathfrak{S}(h,k)$ to be the smallest such S. The following lemma gives an alternate characterization of $\mathfrak{S}(h,k)$:

Lemma 4.1. *Let $h\colon R^m \to R^n$ be a surjective R-linear map. For $1 \leq k \leq q$, write $\mathfrak{S}(h,k) = \{j_1 < j_2 < \cdots < j_{\mu_k n}\}$. Then the j_i are the unique elements of $\{1, \ldots, \mu_k m\}$ satisfying the following two conditions:*

- *$\{\Phi(h)(\vec{v}(k)_{j_1}), \ldots, \Phi(h)(\vec{v}(k)_{j_{\mu_k n}})\}$ is a basis for the $\mathbb{D}_k$-module*

$$\bigoplus_{i=1}^{\mu_k n} \vec{w}(k)_i \cdot \mathbb{D}_k.$$

- *Consider $1 \leq j \leq \mu_k m$, and let $1 \leq i_0 \leq \mu_k n$ be the largest index such that $j_{i_0} \leq j$. Then*

$$\Phi(h)(\vec{v}(k)_j) \in \bigoplus_{i=1}^{i_0} \Phi(h)(\vec{v}(k)_{j_i}) \cdot \mathbb{D}_k.$$

Proof. Immediate. □

Column-adapted maps. This allows us to make the following definition. A surjective R-linear map $h\colon R^m \to R^n$ is *column-adapted* if it satisfies the following condition for each $1 \leq k \leq q$. Write $\mathfrak{S}(h,k) = \{j_1 < j_2 < \cdots < j_{\mu_k n}\}$. We then require that $\Phi(h)(\vec{v}(k)_{j_i}) = \vec{w}(k)_i$ for all $1 \leq i \leq \mu_k n$. One should regard these matrices as being generalizations of upper triangular matrices. This class of maps is closed under composition:

Lemma 4.2. *Let $h_1\colon R^m \to R^n$ and $h_2\colon R^n \to R^\ell$ be column-adapted maps. Then $h_2 \circ h_1\colon R^m \to R^\ell$ is column-adapted.*

Proof. Let $\vec{v}(k)_i$ and $\vec{w}(k)_i$ and $\vec{u}(k)_i$ be the distinguished bases for $R^{\mu m}$ and $R^{\mu n}$ and $R^{\mu \ell}$, respectively. Fix some $1 \leq k \leq q$, and write

$$\mathfrak{S}(h_1, k) = \{j_1 < j_2 < \cdots < j_{\mu_k n}\},$$
$$\mathfrak{S}(h_2, k) = \{j'_1 < j'_2 < \cdots < j'_{\mu_k \ell}\}.$$

For $1 \leq i \leq \mu_k \ell$, define $j''_i = j_{j'_i}$. We thus have

$$\{j''_1 < j''_2 < \cdots < j''_{\mu_k \ell}\} \tag{4.2}$$

and

$$h_2 \circ h_1(\vec{v}(k)_{j''_i}) = h_2 \circ h_1(\vec{v}(k)_{j_{j'_i}}) = h_2(\vec{w}(k)_{j'_i}) = \vec{u}(k)_i.$$

From this, it is easy to see that (4.2) satisfies the criterion of Lemma 4.1, so $\mathfrak{S}(h_2 \circ h_1, k)$ equals (4.2) and $h_2 \circ h_1$ is column-adapted. □

Ordered VIC, semisimple case. From the above, it makes sense to define $\mathtt{OVIC}(R)$ to be the subcategory of $\mathtt{VIC}(R)$ whose objects are all the R^n with $n \geq 1$ and whose morphisms $f\colon [R^n] \to [R^m]$ are all the $\mathtt{VIC}(R)$-morphisms $f = (f', f'')$ such that f'' is column-adapted. Since the only column-adapted maps $R^n \to R^n$ are the identity, it follows that the identity is the only $\mathtt{OVIC}(R)$-endomorphism of $[R^n]$. In the next section, we will show how to generalize all of this to the case of Artinian R, and thus in particular to all finite R.

4.2 Ordered VIC for general Artinian rings

Let R be an Artinian ring. The structure of R was discussed in §3. The quotient ring $\overline{R} = R/J(R)$ is semisimple, so

$$\overline{R} \cong \mathrm{Mat}_{\mu_1}(\mathbb{D}_1) \times \cdots \times \mathrm{Mat}_{\mu_q}(\mathbb{D}_q)$$

for $\mu_1, \ldots, \mu_q \geq 1$ and division rings $\mathbb{D}_1, \ldots, \mathbb{D}_q$. Set $\mu = \mu_1 + \cdots + \mu_q$. Let $\Phi\colon R \hookrightarrow \mathrm{Mat}_\mu(R)$ and $\overline{\Phi}\colon \overline{R} \hookrightarrow \mathrm{Mat}_\mu(\overline{R})$ be the Artin–Wedderburn embeddings of R and $\overline{R}$, so $\overline{\Phi(x)} = \overline{\Phi}(\overline{x})$ for all $x \in R$. Also, for $1 \leq h, k \leq q$ let $\mathbb{L}_{hk} \subset R$ be as defined in §3, so the $\mathbb{L}_{kk}$ are local rings and

$$\overline{\mathbb{L}}_{kk} = \mathbb{D}_k \quad \text{and} \quad \overline{\mathbb{L}}_{hk} = 0 \quad \text{for } h \neq k.$$

The Artin–Wedderburn embedding $\Phi\colon R \hookrightarrow \mathrm{Mat}_\mu(R)$ can then be decomposed into a $q \times q$ block matrix of the form

$$\Phi(x) = (\Phi_{hk}(x))_{h,k=1}^q \quad \text{with} \quad \Phi_{hk}(x) \in \mathrm{Mat}_{\mu_h, \mu_k}(\mathbb{L}_{hk}),$$

and

$$\overline{\Phi(x)} = \overline{\Phi}(\overline{x}) = \left(\overline{\Phi_{hk}(x)}\right)_{h,k=1}^q \quad \text{with} \quad \overline{\Phi_{hh}(x)} \in \mathrm{Mat}_{\mu_h}(\mathbb{D}_h) \text{ and } \overline{\Phi_{hk}(x)} = 0 \text{ for } h \neq k.$$

Distinguished bases. Consider an R-linear map $h\colon R^m \to R^n$. Let $\overline{h}\colon \overline{R}^m \to \overline{R}^n$ be the induced map, and let $\Phi(h)\colon R^{\mu m} \to R^{\mu n}$ and $\overline{\Phi}(\overline{h})\colon \overline{R}^{\mu m} \to \overline{R}^{\mu n}$ be the maps obtained by applying Φ and $\overline{\Phi}$ to the entries of matrices representing h and $\overline{h}$, respectively. For $0 \le k \le q$, let

$$\{\overline{\vec{v}(k)_1}, \ldots, \overline{\vec{v}(k)_{\mu_k m}}\} \quad \text{and} \quad \{\overline{\vec{w}(k)_1}, \ldots, \overline{\vec{w}(k)_{\mu_k n}}\} \tag{4.3}$$

be the distinguished bases for $\overline{R}^{\mu m}$ and $\overline{R}^{\mu n}$ discussed in §4.1. These were introduced to make sense of $\overline{\Phi}(\overline{h})$. We will need the exact same bases for $R^{\mu m}$ and $R^{\mu n}$, so let

$$\{\vec{v}(k)_1, \ldots, \vec{v}(k)_{\mu_k m}\} \quad \text{and} \quad \{\vec{w}(k)_1, \ldots, \vec{w}(k)_{\mu_k n}\}$$

be the subsets of the standard bases for $R^{\mu m}$ and $R^{\mu n}$ that map to (4.3) under the maps $R^{\mu m} \to \overline{R}^{\mu m}$ and $R^{\mu n} \to \overline{R}^{\mu n}$. For all $1 \le k \le q$ and $1 \le j \le \mu_k m$, we thus have

$$\Phi(h)(\vec{v}(k)_j) \in \bigoplus_{h=1}^{q} \left(\bigoplus_{i=1}^{\mu_h n} \vec{w}(h)_i \cdot \mathbb{L}_{hk} \right). \tag{4.4}$$

S-function. Given a surjective map $h\colon R^m \to R^n$, the induced map $\overline{h}\colon \overline{R}^m \to \overline{R}^n$ is also surjective. For $1 \le k \le q$, we define

$$\mathfrak{S}(h,k) = \mathfrak{S}(\overline{h},k) \subset \{1, \ldots, \mu_k m\},$$

so $|\mathfrak{S}(h,k)| = \mu_k n$.

Column-adapted maps. A surjective map $h\colon R^m \to R^n$ is said to be *column-adapted* if it satisfies the following two conditions:

(i) The map $\overline{h}\colon \overline{R}^m \to \overline{R}^n$ is column-adapted in the sense of §4.1.

(ii) For each $1 \le k \le q$, write $\mathfrak{S}(h,k) = \{j_1 < j_2 < \cdots < j_{\mu_k n}\}$. We then require that $\Phi(h)(\vec{v}(k)_{j_i}) = \vec{w}(k)_i$ for all $1 \le i \le \mu_k n$.

This class of maps is closed under composition:

Lemma 4.3. *Let $h_1\colon R^m \to R^n$ and $h_2\colon R^n \to R^\ell$ be column-adapted maps. Then $h_2 \circ h_1\colon R^m \to R^\ell$ is column-adapted.*

Proof. By Lemma 4.2, the map $\overline{h_2 \circ h_1} = \overline{h}_2 \circ \overline{h}_1$ is column-adapted, so condition (i) is satisfied for $h_2 \circ h_1$. The same argument used in the proof of Lemma 4.2 then shows that condition (ii) is satisfied for $h_2 \circ h_1$. The lemma follows. □

Canonical splittings. One of the key features of column-adapted maps is the following lemma. We will call the map g constructed in it the *canonical splitting* of h; as the lemma says, it only depends on $\mathfrak{S}(h,k)$ for $1 \le k \le q$.

Lemma 4.4. *For each $1 \le k \le q$, let $S(k) \subset \{1, \ldots, \mu_k m\}$ be an $\mu_k n$-element set. There then exists an R-linear map $g\colon R^n \to R^m$ such that if $h\colon R^m \to R^n$ is a column-adapted map with $\mathfrak{S}(h,k) = S(k)$ for all $1 \le k \le q$, then $h \circ g = id$.*

Proof. Let $\vec{v}(k)_i$ and $\vec{w}(k)_i$ be the distinguished bases for $R^{\mu m}$ and $R^{\mu n}$, respectively. For $1 \leq k \leq q$, write

$$S(k) = \{j(k)_1, \ldots, j(k)_{\mu_k n}\}.$$

Define $G\colon R^{\mu n} \to R^{\mu m}$ via the formula

$$G(\vec{w}(k)_i) = \vec{v}(k)_{j(k)_i} \qquad (1 \leq k \leq q, 1 \leq i \leq \mu_k n).$$

Since for all $1 \leq k \leq q$ and $1 \leq i \leq \mu_k n$ we trivially have

$$G(\vec{w}(k)_i) \in \bigoplus_{h=1}^{q} \left(\bigoplus_{j=1}^{\mu_h m} \vec{v}(h)_j \cdot \mathbb{L}_{hk} \right),$$

it follows that there exists some $g\colon R^n \to R^m$ with $\Phi(g) = G$. If $h\colon R^m \to R^n$ is a column-adapted map with $\mathfrak{S}(h,k) = S(k)$ for all $1 \leq k \leq q$, then for all $1 \leq k \leq q$ and $1 \leq i \leq \mu_k n$ we have

$$\Phi(h) \circ \Phi(g)(\vec{w}(k)_i) = \Phi(h)(\vec{v}_{j(k)_i}) = \vec{w}(k)_i,$$

so $\Phi(h) \circ \Phi(g) = \mathrm{id}$ and thus $h \circ g = \mathrm{id}$. □

Ordered VIC, Artinian case. From the above, it makes sense to define $\mathtt{OVIC}(R)$ to be the subcategory of $\mathtt{VIC}(R)$ whose objects are all the R^n with $n \geq 0$ and whose morphisms $f\colon [R^n] \to [R^m]$ are all the $\mathtt{VIC}(R)$-morphisms $f = (f', f'')$ such that f'' is column-adapted. Since the only column-adapted maps $R^n \to R^n$ are the identity, it follows that the identity is the only $\mathtt{OVIC}(R)$-endomorphism of $[R^n]$.

Factoring VIC-morphisms. The following proposition says that $\mathtt{OVIC}(R)$ satisfies conclusion (b) of Theorem 2.1:

Proposition 4.5. *Let R be an Artinian ring. Every $\mathtt{VIC}(R)$-morphism $f\colon [R^d] \to [R^n]$ can be factored as*

$$[R^d] \xrightarrow{f_1} [R^d] \xrightarrow{f_2} [R^n],$$

where $f_1\colon [R^d] \to [R^d]$ is a $\mathtt{VIC}(R)$-morphism and $f_2\colon [R^d] \to [R^n]$ is an $\mathtt{OVIC}(R)$-morphism.

Proof. Write $f = (f', f'')$, where $f'\colon R^d \to R^n$ is an injection and $f''\colon R^n \to R^d$ is a splitting of f', so $f'' \circ f' = \mathrm{id}$.

Let $\vec{v}(k)_i$ and $\vec{w}(k)_i$ be the distinguished bases of $R^{\mu n}$ and $R^{\mu d}$, respectively. Also, write

$$\mathfrak{S}(f'', k) = \{j(k)_1 < \cdots < j(k)_{\mu_k d}\} \subset \{1, \ldots, \mu_k n\}.$$

Define $G\colon R^{\mu d} \to R^{\mu d}$ via the formula

$$G(\vec{w}(k)_i) = \Phi(f'')(\vec{v}(k)_{j(k)_i}) \qquad (1 \leq k \leq q, 1 \leq i \leq \mu_k d).$$

Using (4.4) for $h = f''$, for $1 \leq k \leq q$ and $1 \leq i \leq \mu_k d$ we have

$$G(\vec{w}(k)_i) \in \bigoplus_{h=1}^{q} \left(\bigoplus_{j=1}^{\mu_h d} \vec{w}(h)_j \cdot \mathbb{L}_{hk} \right).$$

From this, we see that there exists some $g\colon R^d \to R^d$ such that $G = \Phi(g)$.

Since the columns of $\Phi(\overline{g})$ are a basis for $\overline{R}^{\mu d}$, it follows that $\overline{g}$ is an isomorphism, so by Lemma 3.1 it follows that g is an isomorphism. By construction, the map $g^{-1} \circ f''$ is column-adapted, so $f_2 = (f' \circ g, g^{-1} \circ f'')$ is an $\mathtt{OVIC}(R)$-morphism. Setting $f_1 = (g^{-1}, g)$, the map f_1 is a $\mathtt{VIC}(R)$-morphism and $f = f_2 \circ f_1$, as desired. □

Free and dependent rows. Consider an $\mathtt{OVIC}(R)$-morphism $f\colon [R^n] \to [R^m]$ with $f = (f', f'')$. The condition that f'' is column-adapted is a condition on the columns of $\Phi(f'') \in \mathrm{Mat}_{\mu n, \mu m}(R)$. We now discuss the rows of $\Phi(f') \in \mathrm{Mat}_{\mu m, \mu n}(R)$. We will call the rows of $\Phi(f')$ that lie in $\mathfrak{S}(f'', k) \subset \{1, \ldots, \mu_k m\}$ for some $1 \leq k \leq q$ the *dependent rows*, and all the other rows will be called the *free rows*. The reason for this terminology is the following lemma:

Lemma 4.6. *Let R be an Artinian ring. Consider $\mathtt{OVIC}(R)$-morphisms $f_1, f_2\colon [R^n] \to [R^m]$ with $f_i = (f'_i, f''_i)$. Assume that $f''_1 = f''_2$ and that the free rows of $\Phi(f'_1)$ and $\Phi(f'_2)$ are equal. Then $f_1 = f_2$.*

Proof. What this lemma is saying is that the dependent rows of $\Phi(f'_i)$ are determined by the free rows together with the fact that $f''_i \circ f'_i = \mathrm{id}$. This is a simple fact about matrix multiplication that is easier to grasp from an example rather than a formal proof: if for instance we have

$$\Phi(f''_i) = \begin{pmatrix} * & 1 & 0 & * & 0 & * \\ * & 0 & 1 & * & 0 & * \\ * & 0 & 0 & * & 1 & * \end{pmatrix} \quad \text{and} \quad \Phi(f'_i) = \begin{pmatrix} * & * & * \\ \diamond & \diamond & \diamond \\ \diamond & \diamond & \diamond \\ * & * & * \\ \diamond & \diamond & \diamond \\ * & * & * \end{pmatrix},$$

then the $\diamond$ entries are the dependent rows, and are determined by the $*$ entries together with the fact that $\Phi(f''_i) \circ \Phi(f'_i) = \mathrm{id}$. □

5 Ordered VIC: local Noetherianity

The goal of this section is to prove that the category of $\mathtt{OVIC}(R)$-modules is locally Noetherian for a finite ring R. This is proved in §5.3, which is preceded by two preliminary sections: §5.1 discusses well partial orders, and §5.2 constructs a specific ordering that is needed for the proof.

5.1 Well partial orders

Let $(\mathfrak{P}, \preceq)$ be a poset. We say that $\mathfrak{P}$ is *well partially ordered* if for any infinite sequence

$$p_1, p_2, p_3, \ldots \qquad (p_i \in \mathfrak{P}),$$

we can find indices $i_1 < i_2 < i_3 < \cdots$ such that

$$p_{i_1} \preceq p_{i_2} \preceq p_{i_3} \preceq \cdots. \tag{5.1}$$

In fact, it is enough to just prove that

$$\text{there exist indices } i < j \text{ with } p_i \preceq p_j. \tag{5.2}$$

Here's a quick proof of this. Letting $I = \{i \mid \text{there does not exist } j > i \text{ with } p_j \succeq p_i\}$, if I is infinite then it provides a sequence of elements of $\mathfrak{P}$ violating (5.2), so I must be finite and we can find the sequence (5.1) starting with any index larger than all the indices in I.

We will need the following specific well partial ordering. Fix a finite set Σ, and let Σ^* be the set of words $s_1 \cdots s_n$ whose letters s_i are in Σ. Define a partial ordering on Σ^* by saying that $s_1 \cdots s_n \preceq t_1 \cdots t_m$ if there exists a strictly increasing function $f\colon \{1, \ldots, n\} \to \{1, \ldots, m\}$ with the following two properties:

- $s_i = t_{f(i)}$ for $1 \leq i \leq n$, and
- for all $1 \leq j \leq m$, there exists some $1 \leq i \leq n$ such that $f(i) \leq j$ and $t_{f(i)} = t_j$.

We then have the following theorem, which is a variant on Higman's Lemma [10].

Lemma 5.1 ([17, Proposition 8.2.1])**.** *For all finite sets Σ, the ordering $(\Sigma^*, \preceq)$ is a well partial ordering.*

Remark 5.2. An alternate proof of Lemma 5.1 can be found in [4, Proof of Prop. 7.5]. □

5.2 An ordering of the generators

The key to our proof that the category of $\texttt{OVIC}(R)$-modules is locally Noetherian is the following lemma.

Lemma 5.3. *Let R be a finite ring and let $d \geq 0$. Define*

$$\mathfrak{P}(d) = \bigsqcup_{n=0}^{\infty} \mathrm{Hom}_{\texttt{OVIC}(R)}(R^d, R^n).$$

There then exists a well partial ordering $\preceq$ on $\mathfrak{P}(d)$ along with an extension $\leq$ of $\preceq$ to a total ordering such that the following holds. Consider $\texttt{OVIC}(R)$-morphisms $f\colon [R^d] \to [R^n]$ and $g\colon [R^d] \to [R^m]$ with $f \preceq g$. There then exists an $\texttt{OVIC}(R)$-morphism $\phi\colon [R^n] \to [R^m]$ with the following two properties:

(i) $g = \phi \circ f$, and

(ii) if $h\colon [R^d] \to [R^n]$ is an $\texttt{OVIC}(R)$-morphism such that $h < f$, then

$$\phi \circ h < \phi \circ f = g.$$

Proof. The notation will be as in §4.2. Our finite ring R is Artinian, so $\overline{R} = R/J(R)$ is semisimple and

$$\overline{R} \cong \mathrm{Mat}_{\mu_1}(\mathbb{D}_1) \times \cdots \times \mathrm{Mat}_{\mu_q}(\mathbb{D}_q)$$

for $\mu_1, \ldots, \mu_q \geq 1$ and division rings $\mathbb{D}_1, \ldots, \mathbb{D}_q$. Set $\mu = \mu_1 + \cdots + \mu_q$. Let $\Phi\colon R \hookrightarrow \mathrm{Mat}_\mu(R)$ and $\overline{\Phi}\colon \overline{R} \hookrightarrow \mathrm{Mat}_\mu(\overline{R})$ be the Artin–Wedderburn embeddings of R and $\overline{R}$, so $\overline{\Phi(x)} = \overline{\Phi}(\overline{x})$ for all $x \in R$.

Step 1. *We construct the total order $\leq$ on $\mathfrak{P}(d)$.*

Fix an arbitrary total order on $R^{\mu d}$. Consider $\mathtt{OVIC}(R)$-morphisms $f\colon [R^d] \to [R^n]$ and $g\colon [R^d] \to [R^m]$ in $\mathfrak{P}(d)$. Write $f = (f', f'')$ and $g = (g', g'')$. We then determine if $f \leq g$ via the following procedure:

- If $n < m$, then $f < g$.
- Otherwise, assume that $n = m$. For each $1 \leq k \leq q$, we have the $\mu_k d$-element subsets $\mathfrak{S}(f'', k)$ and $\mathfrak{S}(g'', k)$ of $\{1, \ldots, \mu_k m\}$. Order $\mu_k d$-element subsets of $\{1, \ldots, \mu_k m\}$ using the lexicographic order, and then further order tuples $(I_1, \ldots, I_q)$ with I_k a $\mu_k d$-element subset of $\{1, \ldots, \mu_k m\}$ using the lexicographic ordering. If
$$(\mathfrak{S}(f'', 1), \ldots, \mathfrak{S}(f'', q)) < (\mathfrak{S}(g'', 1), \ldots, \mathfrak{S}(g'', q))$$
using this order, then $f < g$.
- Otherwise, assume that $n = m$ and that $\mathfrak{S}(f'', k) = \mathfrak{S}(g'', k)$ for all $1 \leq k \leq q$. Compare the columns of $\Phi(f'') \in \mathrm{Mat}_{\mu d, \mu n}(R)$ using our fixed total order on $R^{\mu d}$ and the lexicographic order. If under this ordering the columns of $\Phi(f'')$ are less than the columns of $\Phi(g'')$, then $f < g$.
- Otherwise, assume that $n = m$ and that $f'' = g''$. Compare the free rows of $\Phi(f') \in \mathrm{Mat}_{\mu n, \mu d}(R)$ and $\Phi(g') \in \mathrm{Mat}_{\mu n, \mu d}(R)$ using our fixed total order on $R^{\mu d}$ and the lexicographic order. If under this ordering the free rows of $\Phi(f')$ are less than the rows of $\Phi(g')$, then $f < g$.

It is clear that this determines a total order $\leq$ on $\mathfrak{P}(d)$.

Step 2. *We construct the partial order $\preceq$ such that $\leq$ is a refinement of $\preceq$.*

Consider $\mathtt{OVIC}(R)$-morphisms $f\colon [R^d] \to [R^n]$ and $g\colon [R^d] \to [R^m]$ in $\mathfrak{P}(d)$. We then say that $f \prec g$ if $n < m$ and there exists a sequence
$$f = h_0, h_1, \ldots, h_{m-n} = g,$$
where for $i \geq 0$ we have that $h_{i+1}\colon [R^d] \to [R^{n+i+1}]$ is an $\mathtt{OVIC}(R)$-morphism related to $h_i\colon [R^d] \to [R^{n+i}]$ as follows:

- Write $h_i = (h_i', h_i'')$. Regard h_i' and h_i'' as $(n+i) \times d$ and $d \times (n+i)$ matrices, respectively. Pick some $1 \leq a \leq b \leq n+i$ satisfying the following condition:
  - Let $I = \{(a-1)\mu + 1, \ldots, a\mu\}$ be the columns of $\Phi(h_i'') \in \mathrm{Mat}_{\mu d, \mu(n+i)}(R)$ corresponding to the a^{th} column of h_i''. Then I is disjoint from $\mathfrak{S}(h_i'', k)$ for all $1 \leq k \leq q$.

  Writing $h_{i+1} = (h_{i+1}', h_{i+1}'')$, we then have the following:
  - $h_{i+1}'' \in \mathrm{Mat}_{d, n+i+1}(R)$ is obtained from $h_i'' \in \mathrm{Mat}_{d, n+i}(R)$ by inserting a copy of the a^{th} column of h_i'' after the b^{th} column.
  - $h_{i+1}' \in \mathrm{Mat}_{n+i+1, d}(R)$ is obtained from $h_i' \in \mathrm{Mat}_{n+i, d}(R)$ by inserting a copy of the a^{th} row of h_i' after the b^{th} row, and then possibly changing the dependent rows to ensure that $h_{i+1}'' \circ h_{i+1}' = \mathrm{id}$.

This clearly defines a partial ordering $\preceq$ on $\mathfrak{P}(d)$, and since $f \prec g$ required $n < m$ it refines $\leq$.

Step 3. *We prove that $\preceq$ is a well partial order.*

We will embed $(\mathfrak{P}(d), \preceq)$ into a poset $(\Sigma^*, \preceq)$ of words, where Σ is a finite set of letters and $\preceq$ is as in Lemma 5.1. That lemma says that $(\Sigma^*, \preceq)$ is a well partial ordering, so this will imply that $(\mathfrak{P}(d), \preceq)$ is as well.

First, define

$$\widehat{R} = R \sqcup \{\clubsuit\},$$

where $\clubsuit$ is a formal symbol. Though $\widehat{R}$ is not a ring, it still makes sense to speak about the set of matrices with entries in $\widehat{R}$. Define

$$\Sigma = \{(M_1, M_2) \mid M_1 \in \mathrm{Mat}_{\mu,\mu d}(\widehat{R}) \text{ and } M_2 \in \mathrm{Mat}_{d,1}(R)\}.$$

We then define a map $\iota\colon \mathfrak{P}(d) \to \Sigma^*$ in the following way.

Consider some element $f\colon [R^d] \to [R^n]$ of $\mathfrak{P}(d)$. Write $f = (f', f'')$. Let $c_1, \ldots, c_n \in \mathrm{Mat}_{d,1}(R)$ be the columns of the matrix representing f''. Next, via the following procedure we build modified versions $\widehat{r}_1, \ldots, \widehat{r}_n \in \mathrm{Mat}_{\mu,\mu d}(\widehat{R})$ of the rows of the matrix representing f' so as to ignore the dependent rows. Start with

$$\Phi(f') \in \mathrm{Mat}_{\mu n,\mu d}(R).$$

Define $\widehat{\Phi(f')} \in \mathrm{Mat}_{\mu n,\mu d}(\widehat{R})$ to be the matrix obtained from $\Phi(f')$ by replacing each entry in the rows whose numbers lie in $\mathfrak{S}(f'', k)$ by $\clubsuit$ for all $1 \leq k \leq q$. These are precisely the dependent rows. We then define $\widehat{r}_1, \ldots, \widehat{r}_n$ by letting $\widehat{r}_1 \in \mathrm{Mat}_{\mu,\mu d}(\widehat{R})$ be the submatrix of $\widehat{\Phi(f')}$ consisting of the first μ rows, letting $\widehat{r}_2 \in \mathrm{Mat}_{\mu,\mu d}(\widehat{R})$ be the submatrix of $\widehat{\Phi(f')}$ consisting of the second μ rows, etc. Having done this, we define

$$\iota(f) = (\widehat{r}_1, c_1)(\widehat{r}_2, c_2) \cdots (\widehat{r}_n, c_n) \in \Sigma^*.$$

This is an injection since knowing $\iota(f)$, we can reconstruct f'' and all the free rows of $\Phi(f')$, and this determines f by Lemma 4.6. That ι is order-preserving is immediate from the definitions.

Step 4. *We construct the ϕ satisfying (i).*

Consider $\mathtt{OVIC}(R)$-morphisms $f\colon [R^d] \to [R^n]$ and $g\colon [R^d] \to [R^m]$ with $f \preceq g$. Our goal is to construct an $\mathtt{OVIC}(R)$-morphism $\phi\colon [R^n] \to [R^m]$ such that $g = \phi \circ f$. Examining the definition of the partial ordering $\preceq$ in Step 2, we see that it is enough to deal with the case where $m = n + 1$ since the general case can be dealt with by iterating this $m - n$ times.

Write $f = (f', f'')$ and $g = (g', g'')$. By definition, there exists some $1 \leq a \leq b \leq n + 1$ such that the following three things hold:

- Let $I = \{(a-1)\mu + 1, \ldots, a\mu\}$ be the columns of $\Phi(f'') \in \mathrm{Mat}_{\mu d,\mu n}(R)$ corresponding to the a^{th} column of f''. Then I is disjoint from $\mathfrak{S}(f'', k)$ for all $1 \leq k \leq q$.
- $g'' \in \mathrm{Mat}_{d,n+1}(R)$ is obtained from $f'' \in \mathrm{Mat}_{d,n}(R)$ by inserting a copy of the a^{th} column of f'' after the b^{th} column.
- $g' \in \mathrm{Mat}_{n+1,d}(R)$ is obtained from $f' \in \mathrm{Mat}_{n,d}(R)$ by inserting a copy of the a^{th} row of f' after the b^{th} row, and then possibly changing the dependent rows to ensure that $g'' \circ g' = \mathrm{id}$.

Let $\psi\colon R^d \to R^n$ be the canonical splitting of f'' (see Lemma 4.4). Let $\mathfrak{c} \in R^d$ be the a^{th} column of the matrix representing f'', and set $\widehat{\mathfrak{c}} = \psi(\mathfrak{c}) \in R^n$. We then define $\phi = (\phi', \phi'')$ in the following way:

- $\phi''\colon R^{n+1} \to R^n$ is represented by the matrix obtained by inserting $\widehat{\mathfrak{c}}$ after the b^{th} column of $\text{id}\colon R^n \to R^n$.
- $\phi'\colon R^n \to R^{n+1}$ is represented by the matrix obtained by first subtracting $\widehat{\mathfrak{c}}$ from the a^{th} column of $\text{id}\colon R^n \to R^n$, and then inserting the row $(0,\ldots,0,1,0,\ldots,0)$ with a 1 in position a after the b^{th} row.

For example, for $n = 7$ and $a = 3$ and $b = 4$ we would have

$$\phi'' = \left(\begin{array}{cccc|c|ccc} 1&0&0&0&\widehat{\mathfrak{c}}_1&0&0&0\\ 0&1&0&0&\widehat{\mathfrak{c}}_2&0&0&0\\ 0&0&1&0&\widehat{\mathfrak{c}}_3&0&0&0\\ 0&0&0&1&\widehat{\mathfrak{c}}_4&0&0&0\\ 0&0&0&0&\widehat{\mathfrak{c}}_5&1&0&0\\ 0&0&0&0&\widehat{\mathfrak{c}}_6&0&1&0\\ 0&0&0&0&\widehat{\mathfrak{c}}_7&0&0&1 \end{array}\right) \qquad \phi' = \left(\begin{array}{ccccccc} 1&0&-\widehat{\mathfrak{c}}_1&0&0&0&0\\ 0&1&-\widehat{\mathfrak{c}}_2&0&0&0&0\\ 0&0&1-\widehat{\mathfrak{c}}_3&0&0&0&0\\ 0&0&-\widehat{\mathfrak{c}}_4&1&0&0&0\\ \hline 0&0&1&0&0&0&0\\ \hline 0&0&-\widehat{\mathfrak{c}}_5&0&1&0&0\\ 0&0&-\widehat{\mathfrak{c}}_6&0&0&1&0\\ 0&0&-\widehat{\mathfrak{c}}_7&0&0&0&1 \end{array}\right).$$

It is clear that $\phi'' \circ \phi' = \text{id}$ and that the matrix representing $f'' \circ \phi''\colon R^{n+1} \to R^d$ is obtained by inserting $f''(\widehat{\mathfrak{c}}) = \mathfrak{c}$ after the b^{th} column of the matrix representing f''. Moreover, examining the construction of the canonical splitting in Lemma 4.4, we see that the entries of $\Phi(\widehat{\mathfrak{c}}) \in R^{\mu n}$ lying in the free rows of $\Phi(f')$ are all 0, so the matrix corresponding to $\phi' \circ f'$ is obtained by first inserting a copy of the a^{th} row of the matrix representing f' after the b^{th} row of that matrix, and then possibly modifying the dependent rows.

Step 5. *We prove that the ϕ we constructed satisfy (ii).*

Just like in the previous step, it is enough to deal with the case where $m = n+1$, so $f\colon [R^d] \to [R^n]$ and $\phi\colon [R^n] \to [R^{n+1}]$. Consider some $\texttt{OVIC}(R)$-morphism $h\colon [R^d] \to [R^n]$ such that $h < f$. Our goal is to prove that $\phi \circ h < \phi \circ f$. Write $f = (f', f'')$ and $h = (h', h'')$ and $\phi = (\phi', \phi'')$.

Examining the construction of the total ordering $\leq$ in Step 1, we see that there are three cases we have to deal with. The first is where

$$(\mathfrak{S}(h'',1),\ldots,\mathfrak{S}(h'',q)) < (\mathfrak{S}(f'',1),\ldots,\mathfrak{S}(f'',q)),$$

where the $\mu_k d$-element subsets of $\{1,\ldots,\mu_k n\}$ are ordered using the lexicographic ordering and these tuples are further ordered using the lexicographic ordering. The key fact now is that given *any* column-adapted maps $\zeta_1, \zeta_2\colon R^n \to R^d$ with $\mathfrak{S}(\zeta_1,k) < \mathfrak{S}(\zeta_2,k)$ in the lexicographic order, we have $\mathfrak{S}(\zeta_1 \circ \eta, k) < \mathfrak{S}(\zeta_2 \circ \eta, k)$ for all column-adapted maps $\eta\colon R^{n+1} \to R^n$ (see the proof of Lemma 4.2). It follows that

$$(\mathfrak{S}(h'' \circ \phi'',1),\ldots,\mathfrak{S}(h'' \circ \phi'',q)) < (\mathfrak{S}(f'' \circ \phi'',1),\ldots,\mathfrak{S}(f'' \circ \phi'',q)),$$

so $\phi \circ h < \phi \circ f$.

The second case is where $\mathfrak{S}(h'',k) = \mathfrak{S}(f'',k)$ for all k, but the columns of $\Phi(h'')$ are less than the columns of $\Phi(f'')$ in the lexicographic ordering (using our fixed total ordering on

$R^{\mu d}$). In this case, it follows from our construction of ϕ that the matrix representing $h'' \circ \phi''$ is obtained from the matrix representing h'' by inserting a copy of the a^{th} column of the matrix representing f'' after the b^{th} column, and similarly for $f'' \circ \phi''$. This implies that the columns of $\Phi(h'' \circ \phi'')$ remain less than the columns of $\Phi(f'' \circ \phi'')$, so $\phi \circ h < \phi \circ f$.

The final case is where $h'' = f''$, but the free rows of $\Phi(h')$ are less than the free rows of $\Phi(f')$ in the lexicographic ordering. In this case, $\Phi(\phi' \circ h')$ is obtained from $\Phi(h')$ by taking a bunch of free rows and duplicating them lower in the matrix, and similarly for $\Phi(\phi' \circ f')$ (with the same rows). It follows that the free rows of $\Phi(\phi' \circ h')$ remain less than the free rows of $\Phi(\phi' \circ f')$, so $\phi \circ h < \phi \circ f$. □

5.3 Local Noetherianity

We now prove the following, which shows that $\mathtt{OVIC}(R)$ satisfies part (c) of Theorem 2.1:

Proposition 5.4. *Let R be a finite ring and let $\mathbf{k}$ be a left Noetherian ring. Then the category of $\mathtt{OVIC}(R)$-modules over $\mathbf{k}$ is locally Noetherian.*

Proof. Just like in the proof of Theorem A in §2, we will prove this by studying representable modules. For $d \geq 0$, let $P(d)$ be the $\mathtt{OVIC}(R)$-module defined via the formula

$$P(d)_n = \mathbf{k}[\mathrm{Hom}_{\mathtt{OVIC}(R)}(R^d, R^n)] \qquad (n \geq 0).$$

As we discussed in the proof of Theorem A, every finitely generated $\mathtt{OVIC}(R)$-module over $\mathbf{k}$ is the surjective image of a direct sum of finitely many $P(d)$ (for differing choices of d). To prove that every submodule of such a finitely generated module is finitely generated, it is thus enough to prove this for $P(d)$.

We start with some preliminaries. Let $\preceq$ and $\leq$ be the orderings on

$$\mathfrak{P}(d) = \bigsqcup_{n=0}^{\infty} \mathrm{Hom}_{\mathtt{OVIC}(R)}(R^d, R^n)$$

provided by Lemma 5.3. For a nonzero $x \in P(d)_n$, define the *initial term* of x, denoted $\mathrm{init}(x)$, as follows. Write

$$x = \alpha_1 f_1 + \cdots + \alpha_k f_k \quad \text{with } \alpha_1, \ldots, \alpha_k \in \mathbf{k} \setminus \{0\} \text{ and } f_1, \ldots, f_k \in \mathrm{Hom}_{\mathtt{OVIC}(R)}(R^d, R^n).$$

Order these terms such that $f_1 < f_2 < \cdots < f_k$. Then $\mathrm{init}(x) = \alpha_k f_k$.

Next, for an $\mathtt{OVIC}(R)$-submodule M of $P(d)$, define the *initial module* $\mathcal{I}(M)_\bullet$ of M to be the ordered sequence of $\mathbf{k}$-modules defined via the formula

$$\mathcal{I}(M)_n = \mathbf{k}\{\mathrm{init}(x) \mid x \in M_n\} \qquad (n \geq 1).$$

Be warned that this need not be an $\mathtt{OVIC}(R)$-submodule of $P(d)$. However, we do have the following.

Claim. *If N and M are $\mathtt{OVIC}(R)$-submodules of $P(d)$ with $N \subset M$ and $\mathcal{I}(N)_\bullet = \mathcal{I}(M)_\bullet$, then $N = M$.*

Proof of claim. Assume otherwise, and let $n \geq 0$ be such that $N_n \subsetneq M_n$. Let $f\colon [R^d] \to [R^n]$ be the $\leq$-minimal element of the set

$$\{f \mid \text{there exists } x \in M_n \setminus N_n \text{ and } \alpha \in \mathbf{k} \text{ such that } \mathrm{init}(x) = \alpha f\}.$$

Let $x \in M_n \setminus N_n$ satisfy $\mathrm{init}(x) = \alpha f$ with $\alpha \in \mathbf{k}$. By assumption, there exists some $y \in N_n$ such that $\mathrm{init}(y) = \alpha f$. The αf terms cancel in $x - y$, so $\mathrm{init}(x - y) = \beta g$ with $\beta \in \mathbf{k}$ and $g < f$. Since $x \in M_n \setminus N_n$ and $y \in N_n$, we have $x - y \in M_n \setminus N_n$, so this contradicts the minimality of f. □

We now commence with the proof that every $\mathtt{OVIC}(R)$-submodule of $P(d)$ is finitely generated. Assume otherwise, so there exists a strictly increasing chain

$$M^0 \subsetneq M^1 \subsetneq M^2 \subsetneq \cdots$$

of $\mathtt{OVIC}(R)$-submodules of $P(d)$. By the above claim, the sequences $\mathcal{I}(M^i)_\bullet$ must all be distinct, so for all $i \geq 1$ we can find some $n_i \geq 0$ such that there exists some

$$\alpha_i f_i \in \mathcal{I}(M^i)_{n_i} \setminus \mathcal{I}(M^{i-1})_{n_i} \quad \text{with } \alpha_i \in \mathbf{k} \text{ and } f_i\colon [R^d] \to [R^{n_i}].$$

Let $x_i \in M^i_{n_i}$ be an element with $\mathrm{init}(x_i) = \alpha_i f_i$.

Since $\preceq$ is a well partial ordering, there exists some increasing sequence $i_1 < i_2 < i_3 < \cdots$ of indices such that

$$f_{i_1} \preceq f_{i_2} \preceq f_{i_3} \preceq \cdots .$$

Since $\mathbf{k}$ is a left Noetherian ring, there exists some $m \geq 1$ such that α_{m+1} is in the left $\mathbf{k}$-ideal generated by $\alpha_{i_1}, \ldots, \alpha_{i_m}$, i.e. we can write

$$\alpha_{m+1} = c_1 \alpha_{i_1} + \cdots + c_m \alpha_{i_m} \quad \text{with } c_1, \ldots, c_m \in \mathbf{k}.$$

For $1 \leq j \leq m$, the fact that $f_{i_j} \preceq f_{i_{m+1}}$ implies by conclusion (i) of Lemma 5.3 that there exists some $\mathtt{OVIC}(R)$-morphism $\phi_j\colon [R^{n_{i_j}}] \to [R^{n_{i_{m+1}}}]$ such that $f_{i_{m+1}} = \phi_j \circ f_{i_j}$. Conclusion (ii) of Lemma 5.3 implies that $\mathrm{init}(\phi_j \circ x) = \alpha_j f_{i_{m+1}}$. Setting

$$y = \sum_{j=1}^{m} c_j (\phi_j \circ x_{i_j}) \in M^{i_m}_{n_{i_{m+1}}},$$

we thus see that

$$\mathrm{init}(y) = \sum_{j=1}^{m} c_j \alpha_j f_{i_{m+1}} = \alpha_{m+1} f_{i_{m+1}} = \mathrm{init}(x_{i_{m+1}}).$$

This contradicts the fact that

$$\mathrm{init}(x_{i_{m+1}}) \in \mathcal{I}(M^{i_{m+1}})_{n_{i_{m+1}}} \setminus \mathcal{I}(M^{i_m})_{n_{i_{m+1}}}.$$

The proposition follows. □

References

[1] T. Church, J. S. Ellenberg and B. Farb, FI-modules and stability for representations of symmetric groups, Duke Math. J. 164 (2015), no. 9, 1833–1910, arXiv:1204.4533v4.

[2] T. Church, J. S. Ellenberg, B. Farb and R. Nagpal, FI-modules over Noetherian rings, Geom. Topol. 18 (2014), no. 5, 2951–2984, arXiv:1210.1854v2.

[3] T. Church and B. Farb, Representation theory and homological stability, Adv. Math. 245 (2013), 250–314, arXiv:1008.1368v3.

[4] J. Draisma and J. Kuttler, Bounded-rank tensors are defined in bounded degree, Duke Math. J. 163 (2014), no. 1, 35–63, arXiv:1103.5336v2.

[5] W. G. Dwyer, Twisted homological stability for general linear groups, Ann. of Math. (2) 111 (1980), no. 2, 239–251.

[6] B. Farb, Representation stability, Proceedings of the 2014 International Congress of Mathematicians. Volume II, 1173–1196, arXiv:1404.4065v1.

[7] V. Franjou and A. Touzé (eds.), *Lectures on functor homology*, Progress in Mathematics, 311, Birkhäuser/Springer, Cham, 2015.

[8] W. L. Gan and L. Li, Noetherian property of infinite EI categories, New York J. Math. **21** (2015), 369–382, arXiv:1407.8235v3.

[9] N. Harman, Effective and Infinite-Rank Superrigidity in the Context of Representation Stability, preprint 2019, arXiv:1902.05603v1.

[10] G. Higman, Ordering by divisibility in abstract algebras, Proc. London Math. Soc. (3) 2 (1952), 326–336.

[11] N. J. Kuhn, Generic representations of the finite general linear groups and the Steenrod algebra. II, K-Theory 8 (1994), no. 4, 395–428.

[12] T. Y. Lam, *A first course in noncommutative rings*, second edition, Graduate Texts in Mathematics, 131, Springer-Verlag, New York, 2001.

[13] T. Y. Lam, *Serre's problem on projective modules*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2006.

[14] A. Putman and S. V. Sam, Representation stability and finite linear groups, Duke Math. J. **166** (2017), no. 13, 2521–2598, arXiv:1408.3694v3.

[15] A. Putman, S. V. Sam and A. Snowden, Stability in the homology of unipotent groups, Algebra Number Theory **14** (2020), no. 1, 119–154, arXiv:1711.11080v4.

[16] G. Richter, Noetherian semigroup rings with several objects, in *Group and semigroup rings (Johannesburg, 1985)*, 231–246, North-Holland Math. Stud., 126, Notas Mat., 111, North-Holland, Amsterdam.

[17] S. V Sam and A. Snowden, Gröbner methods for representations of combinatorial categories, J. Amer. Math. Soc. 30 (2017), 159–203, arXiv:1409.1670v3.

[18] A. Scorichenko, Stable K-theory and functor homology over a ring, thesis 2000.

[19] W. van der Kallen, Homology stability for linear groups, Invent. Math. 60 (1980), no. 3, 269–295.

[20] J. C. H. Wilson, FI_{W}-modules and stability criteria for representations of classical Weyl groups, J. Algebra **420** (2014), 269–332, arXiv:1309.3817v2.

Andrew Putman
Department of Mathematics
University of Notre Dame
279 Hurley Hall
South Bend, IN 46556
andyp@nd.edu

Steven V Sam
Department of Mathematics
University of California, San Diego
9500 Gilman Drive
La Jolla, CA 92093-0112
ssam@ucsd.edu