# PRESENTATIONS OF REPRESENTATIONS

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Abstract. We give a new technique for constructing presentations by generators and relations for representations of groups like $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{Sp}_{2g}(\mathbb{Z})$. Our results play an important role in recent work of the authors calculating $\mathrm{H}_2$ of the Torelli group.

## 1. Introduction

In this paper, we give a new approach to constructing presentations by generators and relations for representations[1] of groups like $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{Sp}_{2g}(\mathbb{Z})$. The representations we have in mind are finite-dimensional. However, their presentations have infinitely many generators and relations, so this finite-dimensionality is not obvious. Our main goal is to identify a representation we constructed in our work on the second homology group of the Torelli group in [4, 5]. These papers make essential use of Theorems F – G below.

1.1. **Special linear group, standard representation.** We start with an easy example. A set $\{v_1, \ldots, v_k\}$ of vectors in $\mathbb{Z}^n$ is a *partial basis* if it can be extended to a basis $\{v_1, \ldots, x_n\}$. For $v \in \mathbb{Z}^n$, the set $\{v\}$ is partial basis precisely when $v$ is a *primitive vector*, i.e., is not divisible by any integer $d \geq 2$.

**Definition 1.1.** Define $\mathfrak{Q}_n$ to be the $\mathbb{Q}$-vector space with the following presentation:

- **Generators**. A generator $[v]$ for all primitive vectors $v \in \mathbb{Z}^n$. Here $[v]$ should be interpreted as a formal symbol associated to $v$.
- **Relations**. For a partial basis $\{v_1, v_2\}$ of $\mathbb{Z}^n$, the relation $[v_1] + [v_2] = [v_1 + v_2]$. □

The group $\mathrm{SL}_n(\mathbb{Z})$ acts on the set of primitive vectors in $\mathbb{Z}^n$. This induces an action of $\mathrm{SL}_n(\mathbb{Z})$ on $\mathfrak{Q}_n$, so $\mathfrak{Q}_n$ is a representation of $\mathrm{SL}_n(\mathbb{Z})$. Since $\mathfrak{Q}_n$ has infinitely many generators and relations, it is not a priori clear if it is finite-dimensional.

Define $\Phi\colon \mathfrak{Q}_n \to \mathbb{Q}^n$ via the formula $\Phi([v]) = v$. This takes relations to relations, and thus gives a well-defined map that we call the *linearization map*. Similar maps we will define in other contexts will also be called linearization maps. We will prove:

**Theorem A.** *For $n \geq 2$, the linearization map $\Phi\colon \mathfrak{Q}_n \to \mathbb{Q}^n$ is an isomorphism.*

For the proof, let $\mathcal{B} = \{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{Z}^n$ and let $S = \{[e_1], \ldots, [e_n]\}$. The map $\Phi$ takes $S$ bijectively to the basis $\mathcal{B}$ for $\mathbb{Q}^n$, so the restriction of $\Phi$ to $\langle S \rangle$ is an isomorphism. To prove Theorem A, we must prove that $\langle S \rangle = \mathfrak{Q}_n$. For this, let $v \in \mathbb{Z}^n$ be a primitive vector. Write $v = \lambda_1 e_1 + \cdots + \lambda_n e_n$ with $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}$. We must prove that

$$[v] = \lambda_1[e_1] + \cdots + \lambda_n[e_n].$$

We will prove this by studying the action of $\mathrm{SL}_n(\mathbb{Z})$ on $\mathfrak{Q}_n$.

*Remark* 1.2. That $\langle S \rangle = \mathfrak{Q}_n$ can be also be proved directly, and to help the reader appreciate the efficiency of our proof we encourage them to work this out. Our approach is the only one we are aware of that can be adapted to prove the other results in this paper. □

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[1]In this paper, representations are always defined over the field $\mathbb{Q}$, so a representation of a group $G$ is a $\mathbb{Q}$-vector space $V$ equipped with a linear action of $G$.

*Remark* 1.3. Theorem A implies not only that $\mathfrak{Q}_n$ is finite-dimensional, but also that the $\mathrm{SL}_n(\mathbb{Z})$-action on it extends to $\mathrm{GL}_n(\mathbb{Q})$. This is not obvious from the definition of $\mathfrak{Q}_n$. □

*Remark* 1.4. Theorem A is false for $n = 1$. Indeed, $\mathbb{Z}^1$ has two primitive vectors $\pm 1$, so $\mathfrak{Q}_1$ has two generators $[1]$ and $[-1]$ and no relations. It follows that $\mathfrak{Q}_1 \cong \mathbb{Q}^2$. □

*Remark* 1.5. The technique we use to prove Theorem A is very flexible, and for instance can also prove appropriate versions of Theorem A with $\mathbb{Z}$ replaced by a field.[2] Similar remarks apply to our other theorems. Since this paper is already long and technical, we chose to not attempt to state our results in maximal generality. □

1.2. **Adjoint representation.** The proof technique we use for Theorem A can also be used to construct presentations of things like tensor powers, symmetric powers, and exterior powers of $\mathbb{Q}^n$. There are numerous possibilities for the exact form of the relations, so rather than try to prove a general theorem we will give one interesting variant. Recall that the adjoint representation of $\mathrm{SL}_n(\mathbb{Q})$ is the kernel $\mathfrak{sl}_n(\mathbb{Q})$ of the trace map

$$\mathrm{tr}\colon (\mathbb{Q}^n)^* \otimes \mathbb{Q}^n \longrightarrow \mathbb{Q}$$

defined by $\mathrm{tr}(f, v) = f(v)$. The dual space $(\mathbb{Q}^n)^* = \mathrm{Hom}(\mathbb{Q}^n, \mathbb{Q})$ contains the lattice $(\mathbb{Z}^n)^* = \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$. Define the following:

**Definition 1.6.** Define $\mathfrak{A}_n$ to be the $\mathbb{Q}$-vector space with the following presentation:

- **Generators**. A generator $[f, v]_0$ for all primitive vectors $f \in (\mathbb{Z}^n)^*$ and $v \in \mathbb{Z}^n$ such that $f(v) = 0$.
- **Relations**. The following two families of relations:
  - For all primitive vectors $f \in (\mathbb{Z}^n)^*$ and all partial bases $\{v_1, v_2\}$ of $\ker(f)$, the relation $[f, v_1 + v_2]_0 = [f, v_1]_0 + [f, v_2]_0$.
  - For all primitive vectors $v \in \mathbb{Z}^n$ and all partial bases $\{f_1, f_2\}$ of
  $$\ker(v) = \{f \in (\mathbb{Z}^n)^* \mid f(v) = 0\},$$
  the relation $[f_1 + f_2, v]_0 = [f_1, v]_0 + [f_2, v]_0$. □

Define $\Phi\colon \mathfrak{A}_n \to (\mathbb{Q}^n)^* \otimes \mathbb{Q}^n$ via the formula $\Phi([f, v]_0) = f \otimes v$. This takes relations to relations, and thus gives a well-defined linearization map with $\mathrm{Im}(\Phi) \subset \mathfrak{sl}_n(\mathbb{Q})$. We will prove:

**Theorem B.** *For $n \geq 3$, the linearization map $\Phi\colon \mathfrak{A}_n \to \mathfrak{sl}_n(\mathbb{Q})$ is an isomorphism.*

*Remark* 1.7. Theorem B is trivial for $n = 1$ since $\mathfrak{A}_1 = \mathfrak{sl}_1(\mathbb{Q}) = 0$. It is false for $n = 2$ since for primitive $f \in (\mathbb{Z}^2)^*$ and $v \in \mathbb{Z}^2$ we have $\ker(f), \ker(v) \cong \mathbb{Z}^1$. This implies that $\mathfrak{A}_2$ has no relations, and thus is an infinite-dimensional vector space with basis the set of all its generators $[f, v]_0$. □

1.3. **Symplectic group, standard representation.** We next turn to the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$. Set $H = \mathbb{Q}^{2g}$ and $H_{\mathbb{Z}} = \mathbb{Z}^{2g}$. Let $\omega\colon H \times H \longrightarrow \mathbb{Q}$ be the standard symplectic form, so $\mathrm{Sp}_{2g}(\mathbb{Z})$ consists of all $M \in \mathrm{GL}_{2g}(\mathbb{Z})$ such that $\omega(M{\cdot}v, M{\cdot}w) = \omega(v, w)$ for all $v, w \in H$. The following is an $\mathrm{Sp}_{2g}(\mathbb{Z})$-analogue of $\mathfrak{Q}_n$:

**Definition 1.8.** Define $\mathfrak{H}_g$ to be the $\mathbb{Q}$-vector space with the following presentation:

- **Generators**. A generator $[v]_{\mathrm{Sp}}$ for all primitive vectors $v \in H_{\mathbb{Z}}$.
- **Relations**. For a partial basis $\{v_1, v_2\}$ of $H_{\mathbb{Z}}$ with $\omega(v_1, v_2) = 0$, the relation $[v_1]_{\mathrm{Sp}} + [v_2]_{\mathrm{Sp}} = [v_1 + v_2]_{\mathrm{Sp}}$. □

[2]Though for general fields the relations would need to be expanded slightly.

The action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $H_{\mathbb{Z}}$ induces an action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathfrak{H}_g$. Given Theorem A, it is natural to expect that $\mathfrak{H}_g \cong H$. However, identifying $H$ with $\mathbb{Q}^{2g}$ this would imply that $\mathfrak{H}_g \cong \mathfrak{Q}_{2g}$. The vector spaces $\mathfrak{H}_g$ and $\mathfrak{Q}_{2g}$ have the same generators, but $\mathfrak{H}_g$ has fewer relations. It seems hard to directly write each relation in $\mathfrak{Q}_{2g}$ in terms of the relations in $\mathfrak{H}_g$. Nevertheless, define $\Phi\colon \mathfrak{H}_g \to H$ via the formula $\Phi([v]_{\mathrm{Sp}}) = v$. This takes relations to relations, and thus gives a well-defined linearization map. We will prove:

**Theorem C.** *For $g \geq 2$, the linearization map $\Phi\colon \mathfrak{H}_g \to H$ is an isomorphism.*

The proof is similar to that of Theorem A, though the details are harder since the group theory of $\mathrm{Sp}_{2g}(\mathbb{Z})$ is less uniform than $\mathrm{SL}_n(\mathbb{Z})$.

*Remark* 1.9. Theorem C implies that the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathfrak{H}_g$ extends to an action of $\mathrm{Sp}_{2g}(\mathbb{Q})$. In fact, it even extends to an action of $\mathrm{GL}_{2g}(\mathbb{Q})$. This seems hard to see directly from the presentation. □

*Remark* 1.10. Theorem C is false for $g = 1$. Indeed, $\mathfrak{H}_1$ has infinitely many generators but no relations, so $\mathfrak{H}_1$ is infinite-dimensional. □

1.4. **Symplectic kernel.** We now discuss another representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$ that is similar to the adjoint representation $\mathfrak{sl}_n(\mathbb{Q})$. The symplectic form $\omega$ induces a map $\wedge^2 H \to \mathbb{Q}$. Let $\mathcal{Z}_g^a$ be its kernel.[3] Say that $v_1, v_2 \in H$ are *orthogonal* if $\omega(v_1, v_2) = 0$. For $v \in H$, let $v^{\perp}$ be the set of all elements of $H$ that are orthogonal to $v$. For $v \in H_{\mathbb{Z}}$, let $v_{\mathbb{Z}}^{\perp}$ be the set of all element of $H_{\mathbb{Z}}$ that are orthogonal to $v$.

**Definition 1.11.** Define $\mathfrak{Z}_g^a$ to be the $\mathbb{Q}$-vector space with the following presentation:

- **Generators**. A generator $(\!| v_1, v_2 |\!)_a$ for all orthogonal primitive vectors $v_1, v_2 \in H_{\mathbb{Z}}$.
- **Relations**. The following two families of relations:
  - For all generators $(\!| v_1, v_2 |\!)_a$, the relation $(\!| v_2, v_1 |\!)_a = -(\!| v_1, v_2 |\!)_a$.
  - For all primitive vectors $v \in H_{\mathbb{Z}}$ and all partial bases $\{w_1, w_2\}$ of $v_{\mathbb{Z}}^{\perp}$, the relation $(\!| v, w_1 + w_2 |\!)_a = (\!| v, w_1 |\!)_a + (\!| v, w_2 |\!)_a$. □

The group $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts on $\mathfrak{Z}_g^a$ via its action on $H_{\mathbb{Z}}$. Define $\Phi\colon \mathfrak{Z}_g^a \to \wedge^2 H$ via the formula $\Phi((\!| w_1, w_2 |\!)_a) = w_1 \wedge w_2$. This takes relations to relations, and thus gives a well-defined linearization map with $\mathrm{Im}(\Phi) \subset \mathcal{Z}_g^a$. We will prove:

**Theorem D.** *For $g \geq 1$, the linearization map $\Phi\colon \mathfrak{Z}_g^a \to \mathcal{Z}_g^a$ is an isomorphism.*

1.5. **Symmetric square.** It is also interesting to replace the anti-symmetric relation in $\mathfrak{Z}_g^a$ with the corresponding symmetric relation:[4]

**Definition 1.12.** Define $\mathfrak{Z}_g^s$ to be the $\mathbb{Q}$-vector space with the following presentation:

- **Generators**. A generator $(\!| v_1, v_2 |\!)_s$ for all orthogonal primitive vectors $v_1, v_2 \in H_{\mathbb{Z}}$.
- **Relations**. The following two families of relations:
  - For all generators $(\!| v_1, v_2 |\!)_s$, the relation $(\!| v_2, v_1 |\!)_s = (\!| v_1, v_2 |\!)_s$.
  - For all primitive vectors $v \in H_{\mathbb{Z}}$ and all partial bases $\{w_1, w_2\}$ of $v_{\mathbb{Z}}^{\perp}$, the relation $(\!| v, w_1 + w_2 |\!)_s = (\!| v, w_1 |\!)_s + (\!| v, w_2 |\!)_s$. □

Again, $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts on $\mathfrak{Z}_g^s$. Define $\Phi\colon \mathfrak{Z}_g^s \to \mathrm{Sym}^2(H)$ via the formula $\Phi((\!| w_1, w_2 |\!)_s) = w_1 \cdot w_2$. This takes relations to relations, and thus gives a well-defined linearization map. Since $\mathrm{Sym}^2(H)$ is an irreducible representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$, it is surjective. We will prove:

**Theorem E.** *For $g \geq 2$, the linearization map $\Phi\colon \mathfrak{Z}_g^s \to \mathrm{Sym}^2(H)$ is an isomorphism.*

[3] The "a" in $\mathcal{Z}_g^a$ stands for "alternating".

[4] The "s" in $\mathfrak{Z}_g^s$ stands for "symmetric".

1.6. **Quotient representation.** Our final theorems[5] are about a subrepresentation of the $\text{Sp}_{2g}(\mathbb{Q})$-representation $(\mathcal{Z}_g^a)^{\otimes 2}$. Above we defined $\mathcal{Z}_g^a$ as the kernel of the map $\wedge^2 H \to \mathbb{Q}$ given by $\omega$. For our final theorems, it is more natural to view it as a quotient of $\wedge^2 H$. The symplectic form $\omega$ on $H$ identifies $H$ with its dual. Using this, we can identify alternating forms on $H$ with elements of $\wedge^2 H$. If $\{a_1, b_1, \ldots, a_g, b_g\}$ is a symplectic basis for $H$, then

$$\omega = a_1 \wedge b_1 + \cdots a_g \wedge b_g.$$

The span of $\omega$ in $\wedge^2 H$ is a copy of $\mathbb{Q}$. The quotient $(\wedge^2 H)/\mathbb{Q}$ is isomorphic to $\mathcal{Z}_g^a$. Since $\omega$ lies in $\wedge^2 H_{\mathbb{Z}}$, the quotient $(\wedge^2 H)/\mathbb{Q}$ has a lattice $(\wedge^2 H_{\mathbb{Z}})/\mathbb{Z}$.

*Remark* 1.13. Except for a few places where clarity will demand we be more careful, our notation will not distinguish elements of $\wedge^2 H$ from their images in $(\wedge^2 H)/\mathbb{Q}$. For instance, for $x, y \in H$ we will often write $x \wedge y$ for the corresponding element of $(\wedge^2 H)/\mathbb{Q}$. □

1.7. **Symmetric contraction.** The *symmetric contraction* is the alternating bilinear map

$$\mathfrak{c}\colon ((\wedge^2 H)/\mathbb{Q}) \times ((\wedge^2 H)/\mathbb{Q}) \longrightarrow \text{Sym}^2(H) \tag{1.1}$$

defined as follows. Start by letting

$$\widehat{\mathfrak{c}}\colon (\wedge^2 H) \times (\wedge^2 H) \to \text{Sym}^2(H)$$

be the alternating bilinear map defined by the formula

$$\widehat{\mathfrak{c}}(x \wedge y, z \wedge w) = \omega(x,z) y{\cdot}w - \omega(x,w) y{\cdot}z - \omega(y,z) x{\cdot}w + \omega(y,w) x{\cdot}z \text{ for } x,y,z,w \in H.$$

This makes sense since the right hand side is alternating in $x$ and $y$ and also alternating in $z$ and $w$. Regarding $\omega$ as an element of $\wedge^2 H$, we have $\widehat{\mathfrak{c}}(\omega, -) = 0$ and $\widehat{\mathfrak{c}}(-, \omega) = 0$. Indeed:

- Both $\widehat{\mathfrak{c}}(\omega, -)$ and $\widehat{\mathfrak{c}}(-, \omega)$ are maps $\wedge^2 H \to \text{Sym}^2(H)$. The representation $\text{Sym}^2(H)$ of $\text{Sp}_{2g}(\mathbb{Q})$ is irreducible and is not isomorphic to either of the two irreducible factors $\mathbb{Q}$ and $(\wedge^2 H)/\mathbb{Q}$ of $\wedge^2 H$. Thus the only map $\wedge^2 H \to \text{Sym}^2(H)$ is the zero map.
- Alternatively, this can be seen directly using the fact that for a symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ of $H$ we have $\omega = a_1 \wedge b_1 + \cdots + a_g \wedge b_g$.

Either way, this implies that $\widehat{\mathfrak{c}}$ induces a map $\mathfrak{c}$ as in (1.1).

1.8. **Symmetric kernel.** The *symmetric kernel*, denoted $\mathcal{K}_g^a$, is the kernel of the map

$$\wedge^2((\wedge^2 H)/\mathbb{Q}) \longrightarrow \text{Sym}^2(H)$$

associated to $\mathfrak{c}$. Say that $\kappa_1, \kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ are *sym-orthogonal* if $\mathfrak{c}(\kappa_1, \kappa_2) = 0$, in which case $\kappa_1 \wedge \kappa_2 \in \mathcal{K}_g^a$. For $\kappa \in (\wedge^2 H)/\mathbb{Q}$, the *symmetric orthogonal complement* of $\kappa$, denoted $\kappa^{\perp}$, consists of all $\kappa' \in (\wedge^2 H)/\mathbb{Q}$ that are sym-orthogonal to $\kappa$.

1.9. **Symplectic pairs.** A *symplectic pair* is an element of $(\wedge^2 H_{\mathbb{Z}})/\mathbb{Z}$ of the form $a \wedge b$, where $a, b \in H_{\mathbb{Z}}$ are such that $\omega(a,b) = 1$. Equivalently, there exists a symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $H_{\mathbb{Z}}$ with $a_1 = a$ and $b_1 = b$. For $X \subset \wedge^2 H$, let $\overline{X}$ be its image in $(\wedge^2 H)/\mathbb{Q}$. Also, for $V \subset H_{\mathbb{Z}}$ let $V_{\mathbb{Q}} = V \otimes \mathbb{Q} \subset H$. We will later prove that for a symplectic pair $a \wedge b$ we have $(a \wedge b)^{\perp} = \overline{\wedge^2 \langle a, b \rangle_{\mathbb{Q}}^{\perp}}$. See Lemma 10.1.

*Remark* 1.14. A symplectic pair is an element of $(\wedge^2 H_{\mathbb{Z}})/\mathbb{Z}$, and can be expressed in many ways as $a \wedge b$ with $a, b \in H_{\mathbb{Z}}$ satisfying $\omega(a,b) = 1$. For instance, if $a \wedge b$ is a symplectic pair, then $a \wedge b = (2a + b) \wedge (a + b)$. □

[5] These are the theorems that are needed for our work on the Torelli group in [4, 5], and thus in some sense are the main point of this paper.

1.10. **Symmetric kernel presentation.** We now make the following definition:

**Definition 1.15.** Define $\mathfrak{K}_g^a$ to be the $\mathbb{Q}$-vector space with the following presentation:

- **Generators**. A generator $[\![\kappa_1, \kappa_2]\!]_a$ for all sym-orthogonal $\kappa_1, \kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ such that either $\kappa_1$ or $\kappa_2$ (or both) is a symplectic pair in $(\wedge^2 H_{\mathbb{Z}})/\mathbb{Z}$.
- **Relations**. The following two families of relations:
  - For all generators $[\![\kappa_1, \kappa_2]\!]_a$, the relation $[\![\kappa_2, \kappa_1]\!]_a = -[\![\kappa_1, \kappa_2]\!]_a$.
  - For all symplectic pairs $a \wedge b \in (\wedge^2 H_{\mathbb{Z}})/\mathbb{Z}$ and all $\kappa_1, \kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ that are sym-orthogonal to $a \wedge b$ and all $\lambda_1, \lambda_2 \in \mathbb{Q}$, the relation

$$[\![a \wedge b, \lambda_1 \kappa_1 + \lambda_2 \kappa_2]\!]_a = \lambda_1 [\![a \wedge b, \kappa_1]\!]_a + \lambda_2 [\![a \wedge b, \kappa_2]\!]_a. \qquad \square$$

The group $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts on $\mathfrak{K}_g^a$ via its action on $H_{\mathbb{Z}}$. Define $\Phi\colon \mathfrak{K}_g^a \to \wedge^2(\wedge^2 H)/\mathbb{Q}$ via the formula $\Phi([\![\kappa_1, \kappa_2]\!]_a) = \kappa_1 \wedge \kappa_2$. This takes relations to relations, and thus gives a well-defined linearization map. Since the $\kappa_i$ are sym-orthogonal, the image of $\Phi$ lies in the symmetric kernel $\mathcal{K}_g^a$. We will prove:

**Theorem F.** *For $g \geq 4$, the linearization map $\Phi\colon \mathfrak{K}_g^a \to \mathcal{K}_g^a$ is an isomorphism.*

Theorem F plays a key role in our work on $\mathrm{H}_2$ of the Torelli group in [4, 5].

*Remark* 1.16. Just like in our previous theorems, it is not obvious from the definitions that $\mathfrak{K}_g^a$ is finite-dimensional or that the $\mathrm{Sp}_{2g}(\mathbb{Z})$-action on it extends to $\mathrm{Sp}_{2g}(\mathbb{Q})$. We will only see these two facts at the very end of our proof. It is unclear if either fact holds for $g \leq 3$. $\square$

1.11. **Symmetric square, II.** Just like we did for $\mathfrak{Z}_g$ in §1.5, it is also interesting to replace the anti-symmetric relation in $\mathfrak{K}_g^a$ with the corresponding symmetric relation:

**Definition 1.17.** Define $\mathfrak{K}_g^s$ to be the $\mathbb{Q}$-vector space with the following presentation:

- **Generators**. A generator $[\![\kappa_1, \kappa_2]\!]_s$ for all sym-orthogonal $\kappa_1, \kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ such that either $\kappa_1$ or $\kappa_2$ (or both) is a symplectic pair in $(\wedge^2 H_{\mathbb{Z}})/\mathbb{Z}$.
- **Relations**. The following two families of relations:
  - For all generators $[\![\kappa_1, \kappa_2]\!]_s$, the relation $[\![\kappa_2, \kappa_1]\!]_s = [\![\kappa_1, \kappa_2]\!]_s$.
  - For all symplectic pairs $a \wedge b \in (\wedge^2 H_{\mathbb{Z}})/\mathbb{Z}$ and all $\kappa_1, \kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ that are sym-orthogonal to $a \wedge b$ and all $\lambda_1, \lambda_2 \in \mathbb{Q}$, the relation

$$[\![a \wedge b, \lambda_1 \kappa_1 + \lambda_2 \kappa_2]\!]_s = \lambda_1 [\![a \wedge b, \kappa_1]\!]_s + \lambda_2 [\![a \wedge b, \kappa_2]\!]_s. \qquad \square$$

Define a linearization map $\Phi\colon \mathfrak{K}_g^s \to \mathrm{Sym}^2((\wedge^2 H)/\mathbb{Q})$ via the formula $\Phi([\![\kappa_1, \kappa_2]\!]_s) = \kappa_1 \cdot \kappa_2$. Unlike $\mathrm{Sym}^2(H)$, the representation $\mathrm{Sym}^2((\wedge^2 H)/\mathbb{Q})$ of $\mathrm{Sp}_{2g}(\mathbb{Z})$ is not irreducible. However, it turns out that $\Phi$ is surjective. In fact:

**Theorem G.** *For $g \geq 4$, the linearization map $\Phi\colon \mathfrak{K}_g^s \to \mathrm{Sym}^2((\wedge^2 H)/\mathbb{Q})$ is an isomorphism.*

Theorem G is also important for our work on the Torelli group.

1.12. **Final remarks.** Innumerable variants and generalizations of Theorems A – G can be proved using our techniques. While these theorems are proved via a common core proof technique, applying this technique requires calculations that seem special to each theorem. Theorems F and G in particular require very elaborate calculations. There should be a common generalization of all these results:

**Question 1.18.** *Does there exist a general abstract theorem that specializes to Theorems A – G, as well as their natural generalizations?*

1.13. **Notation and conventions.** Throughout this paper, we will let $H = \mathbb{Q}^{2g}$ and $H_{\mathbb{Z}} = \mathbb{Z}^{2g}$, and also let $\omega\colon H \times H \to \mathbb{Q}$ be the standard symplectic form on $H$.

1.14. **Outline.** We prove Theorems A – E in Part 1. Next, in Part 2 we expand the presentations for $\mathfrak{K}_g^a$ and $\mathfrak{K}_g^s$ to ones with a larger set of generators. We remark that this expansion uses the same proof technique as our other theorems. We use this expanded presentation to prove Theorems F – G in Parts 3 and 4. We close with Appendix A, which adjusts the presentations given in Theorem F – G to make them match up better with our companion papers [4, 5] on the Torelli group.

## Part 1. Five easy examples

In this part of the paper, we prove Theorems A – E. The proof of Theorem A is in §2. We then extract from that proof an outline of our proof method in §3. We then use this proof technique to prove Theorem B in §4. This is followed by §5 and §6, which prove variants of Theorems A and B that will be needed for our work on symmetric kernel. After the preliminary §7 on generators for $\mathrm{Sp}_{2g}(\mathbb{Z})$, we then prove Theorems C – E in §8 – §9.

### 2. Special linear group I: standard representation

Recall that $\mathfrak{Q}_n$ is the $\mathbb{Q}$-vector space with the following presentation:

- **Generators**. A generator $[v]$ for all primitive vectors $v \in \mathbb{Z}^n$.
- **Relations**. For a partial basis $\{v_1, v_2\}$, the relation $[v_1] + [v_2] = [v_1 + v_2]$.

Define a linearization map $\Phi \colon \mathfrak{Q}_n \to \mathbb{Q}^n$ via the formula $\Phi([v]) = v$. This takes relations to relations, and thus gives a well-defined map. Our goal is to prove:

**Theorem A.** *For $n \geq 2$, the linearization map $\Phi \colon \mathfrak{Q}_n \to \mathbb{Q}^n$ is an isomorphism.*

*Proof.* Let $\mathcal{B} = \{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{Z}^n$. Set $S = \{[e_1], \ldots, [e_n]\}$. The map $\Phi$ takes $S$ bijectively to $\mathcal{B}$. This implies that the restriction of $\Phi$ to $\langle S \rangle$ is an isomorphism. To prove that $\Phi$ is an isomorphism, we must prove that $\langle S \rangle = \mathfrak{Q}_n$.

The group $\mathrm{SL}_n(\mathbb{Z})$ acts on $\mathfrak{Q}_n$. Since $\mathrm{SL}_n(\mathbb{Z})$ acts transitively on primitive vectors,[6] it acts transitively on the generators for $\mathfrak{Q}_n$. It follows that the $\mathrm{SL}_n(\mathbb{Z})$-orbit of $S$ spans $\mathfrak{Q}_n$. To prove that $\langle S \rangle = \mathfrak{Q}_n$, it is therefore enough to prove that $\mathrm{SL}_n(\mathbb{Z})$ takes $\langle S \rangle$ to itself.

For distinct $1 \leq i, j \leq n$, let $E_{ij} \in \mathrm{SL}_n(\mathbb{Z})$ be the elementary matrix obtained from the identity by placing a 1 at position $(i, j)$. These generate $\mathrm{SL}_n(\mathbb{Z})$. Fixing some distinct $1 \leq i, j \leq n$ and some $\epsilon = \pm 1$, it is enough to prove that $E_{ij}^{\epsilon}$ takes $\langle S \rangle$ to itself. Consider some $[e_k] \in S$. We must prove that $[E_{ij}^{\epsilon}(e_k)]$ can be written as a linear combination of elements of $S$. If $k \neq j$, then $E_{ij}^{\epsilon}(e_k) = e_k$ and there is nothing to prove. If $k = j$, there are two cases:

- $\epsilon = 1$. In this case, $[E_{ij}(e_j)] = [e_j + e_i] = [e_j] + [e_i] \in \langle S \rangle$.
- $\epsilon = -1$. In this case, $[E_{ij}^{-1}(e_j)] = [e_j - e_i]$. We would like to prove that this equals $[e_j] - [e_i] \in \langle S \rangle$. For this, since $\{e_j - e_i, e_i\}$ is a partial basis we have

$$[e_j - e_i] + [e_i] = [(e_j - e_i) + e_i] = [e_j]. \qquad \square$$

*Remark* 2.1. Let $n \geq 2$ and let $v \in \mathbb{Z}^n$ be a primitive vector. The above implies that $[-v] = -[v]$. Here is how to prove directly that this holds. Pick $w \in \mathbb{Z}^n$ such that $\{v, w\}$ is a partial basis for $\mathbb{Z}^n$. For $a, b, c, d \in \mathbb{Z}$, the pair $\{av + bw, cv + dw\}$ forms a partial basis for $\mathbb{Z}^n$ precisely when

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1.$$

[6]This is where we use the fact that $n \geq 2$.

Using this, we prove that $[v] + [-v] = 0$ as follows:

$$\begin{aligned}
[v] + [-v] &= [v] + ([v+w] - [v+w]) + [-v] + ([-w] - [-w]) \\
&= ([v] + [v+w]) + ([-v] + [-w]) - ([v+w] + [-w]) \\
&= [2v+w] + [-v-w] - [v] \\
&= [v] - [v] = 0.
\end{aligned}$$

□

## 3. Outline of proof technique

We now abstract a general proof technique from the proof of Theorem A. Let $G$ be a group and let $\mathcal{V}$ be a representation of $G$ that we understand well. Let $\mathfrak{V}$ be a representation of $G$ given by generators and relations that we suspect is isomorphic to $\mathcal{V}$ and let $\Phi\colon \mathfrak{V} \to \mathcal{V}$ be a $G$-equivariant map. The following steps will prove that $\Phi$ is an isomorphism:

**Step 1.** *Construct a subset $S$ of $\mathcal{V}$ such that the restriction of $\Phi$ to $\langle S \rangle$ is an isomorphism.*

One way for this to hold is for $\Phi$ to take $S$ bijectively to a basis for $\mathcal{V}$. However, sometimes it is more natural to use a larger $S$ whose image is a generating set satisfying some relations.

**Step 2.** *Prove that the $G$-orbit of $S$ spans $\mathfrak{V}$.*

Since $\mathfrak{V}$ is given by generators and relations, this is done by making sure that this $G$-orbit contains all the generators.

**Step 3.** *Prove that $G$ takes $\langle S \rangle$ to itself. By Step 2, this will imply that $\langle S \rangle = \mathfrak{V}$, and thus by Step 1 that $\Phi$ is an isomorphism.*

We do this as follows. Let $\Lambda$ be a generating set for $G$. Then it is enough to check that for $f \in \Lambda$ and $s \in S$ the elements $f(x) \in \mathfrak{V}$ and $f^{-1}(x) \in \mathfrak{V}$ can be written as linear combinations of elements of $S$. When we proved Theorem A this step only required the easy identities

$$\begin{aligned}
[e_1 + e_2] &= [e_1] + [e_2], \\
[e_1 - e_2] &= [e_1] - [e_2].
\end{aligned}$$

However, for our other theorems this will be the most calculation heavy step, and the key will be verifying that a large number of explicit elements of $\mathfrak{V}$ lie in $\langle S \rangle$.

## 4. Special linear group II: adjoint representation

Recall that the adjoint representation of $\mathrm{SL}_n(\mathbb{Q})$ is the kernel $\mathfrak{sl}_n(\mathbb{Q})$ of the trace map

$$\mathrm{tr}\colon (\mathbb{Q}^n)^* \otimes \mathbb{Q}^n \longrightarrow \mathbb{Q}$$

defined by $\mathrm{tr}(f, v) = f(v)$. Also, recall that $\mathfrak{A}_n$ is the $\mathbb{Q}$-vector space with the following presentation:

- **Generators**. A generator $[f, v]_0$ for all primitive vectors $f \in (\mathbb{Z}^n)^*$ and $v \in \mathbb{Z}^n$ such that $f(v) = 0$.
- **Relations**. The following two families of relations:
  - For all primitive vectors $f \in (\mathbb{Z}^n)^*$ and all partial bases $\{v_1, v_2\}$ of $\ker(f)$, the relation $[f, v_1 + v_2]_0 = [f, v_1]_0 + [f, v_2]_0$.
  - For all primitive vectors $v \in \mathbb{Z}^n$ and all partial bases $\{f_1, f_2\}$ of
  $$\ker(v) = \{f \in (\mathbb{Z}^n)^* \mid f(v) = 0\},$$
  the relation $[f_1 + f_2, v]_0 = [f_1, v]_0 + [f_2, v]_0$.

Define $\Phi\colon \mathfrak{A}_n \to (\mathbb{Q}^n)^* \otimes \mathbb{Q}^n$ via the formula $\Phi([f,v]_0) = f \otimes v$. This takes relations to relations, and thus gives a well-defined linearization map with $\mathrm{Im}(\Phi) \subset \mathfrak{sl}_n(\mathbb{Q})$. Our goal is to prove:

**Theorem B.** *For $n \geq 3$, the linearization map $\Phi\colon \mathfrak{A}_n \to \mathfrak{sl}_n(\mathbb{Q})$ is an isomorphism.*

*Proof.* We start with the following, which we will use freely throughout the proof:

**Claim.** *In $\mathfrak{A}_n$, the following relations hold for all $m \geq 1$:*

*(i) For all primitive vectors $f \in (\mathbb{Z}^n)^*$ and all primitive vectors $v_1, \ldots, v_m \in \ker(f)$ and all $\lambda_1, \ldots, \lambda_m \in \mathbb{Z}$ such that $\sum_{i=1}^m \lambda_i v_i$ is primitive, we have*

$$[f, \sum_{i=1}^m \lambda_i v_i]_0 = \sum_{i=1}^m \lambda_i [f, v_i]_0.$$

*(ii) For all primitive vectors $v \in \mathbb{Z}^n$ and all primitive vectors $f_1, \ldots, f_m \in \ker(v)$ and all $\lambda_1, \ldots, \lambda_m \in \mathbb{Z}$ such that $\sum_{i=1}^m \lambda_i f_i$ is primitive, we have*

$$[\sum_{i=1}^m \lambda_i f_i, v]_0 = \sum_{i=1}^m \lambda_i [f_i, v]_0.$$

*Proof.* Both are proved the same way, so we will give the details for (i). Let $f \in (\mathbb{Z}^n)^*$ be primitive and let $v_1, \ldots, v_m \in \ker(f)$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{Z}$ be as in the claim. Choose an isomorphism $\mu\colon \mathbb{Z}^{n-1} \to \ker(f)$, and let $w_i = \mu^{-1}(v_i)$. Recall that we defined $\mathfrak{Q}_{n-1}$ in Definition 1.1. Define a map $\psi\colon \mathfrak{Q}_{n-1} \to \mathfrak{A}_n$ via the formula

$$\psi([x]) = [f, \mu(x)]_0 \quad \text{for a primitive } x \in \mathbb{Z}^{n-1}.$$

This takes relations to relations, and thus gives a well-defined map. Recall that Theorem A says that $\mathfrak{Q}_{n-1} \cong \mathbb{Q}^{n-1}$. It follows from this theorem that

$$[\sum_{i=1}^m \lambda_i w_i] = \sum_{i=1}^m \lambda_i [w_i].$$

Plugging this into $\psi$, we see that

$$[f, \sum_{i=1}^m \lambda_i v_i]_0 = \psi\left([\sum_{i=1}^m \lambda_i w_i]\right) = \sum_{i=1}^m \lambda_i \psi\left([w_i]\right) = \sum_{i=1}^m \lambda_i [f, v_i]_0. \qquad \square$$

Let $\mathcal{B} = \{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{Z}^n$ and let $\mathcal{B}^* = \{e_1^*, \ldots, e_n^*\}$ be the corresponding dual basis for $(\mathbb{Z}^n)^*$. We follow the outline from §3, though for readability we divide Step 3 into Steps 3.A and 3.B.

**Step 1.** *Let $S = S_1 \cup S_2$, where the $S_i$ are:*

$$S_1 = \{[e_i^*, e_j]_0 \mid 1 \leq i, j \leq n \text{ distinct}\},$$
$$S_2 = \left\{[e_i^* + e_{i+1}^*, e_i - e_{i+1}]_0 \mid 1 \leq i < n\right\}.$$

*Like we did here, we will write elements of $S$ in blue. Then the restriction of $\Phi$ to $\langle S \rangle$ is an isomorphism.*

Let $T = T_1 \cup T_2$, where the $T_i$ are:

$$T_1 = \{e_i^* \otimes e_j \mid 1 \leq i, j \leq n \text{ distinct}\},$$
$$T_2 = \left\{e_i^* \otimes e_i - e_{i+1}^* \otimes e_{i+1} \mid 1 \leq i < n\right\}.$$

The set $T$ is a basis for the codimension-1 subspace $\mathfrak{sl}_n(\mathbb{Q})$ of $(\mathbb{Q}^n)^* \otimes \mathbb{Q}^n$. The map $\Phi$ takes $S_1$ bijectively to $T_1$. As for $T_2$, observe that for $1 \leq i < n$ we have

$$\begin{aligned}\Phi([e_i^* + e_{i+1}^*, e_i - e_{i+1}]_0) &= (e_i^* + e_{i+1}^*) \otimes (e_i - e_{i+1}) \\ &= e_i^* \otimes e_i - e_{i+1}^* \otimes e_{i+1} - e_i^* \otimes e_{i+1} + e_{i+1}^* \otimes e_i.\end{aligned}$$

Here we have written elements of $T_1$ in orange. This calculation implies that modulo $T_1$, the map $\Phi$ takes $S_2$ bijectively to $T_2$. Since $T$ is a basis for $\mathfrak{sl}_n(\mathbb{Q})$, we deduce that $\Phi$ takes $S$ bijectively to a basis for $\mathfrak{sl}_n(\mathbb{Q})$. This implies that the restriction of $\Phi$ to $\langle S \rangle$ is an isomorphism.

**Step 2.** *We prove that the* $\mathrm{SL}_n(\mathbb{Z})$*-orbit of* $S$ *spans* $\mathfrak{A}_n$.

Immediate from the fact that $\mathrm{SL}_n(\mathbb{Z})$ acts transitively on the basis elements of $\mathfrak{A}_n$.

**Step 3.A.** *In preparation for proving that* $\mathrm{SL}_n(\mathbb{Z})$ *takes* $\langle S \rangle$ *to* $\langle S \rangle$, *we prove that all elements of*[7]

$$\begin{aligned}E = &\left\{[e_i^* + e_j^*, e_i - e_j]_0 \mid 1 \leq i, j \leq n \text{ distinct}\right\} \\ &\cup \left\{[e_i^* - e_j^*, e_i + e_j]_0 \mid 1 \leq i, j \leq n \text{ distinct}\right\} \\ &\cup \left\{[e_i^* + 2e_j^*, 2e_i - e_j]_0 \mid 1 \leq i, j \leq n \text{ distinct}\right\}.\end{aligned}$$

*lie in* $\langle S \rangle$. *Like we did here, we will write elements of* $E$ *in green.*

Above we wrote elements of $S_1$ in blue. We extend this to certain elements that "obviously" lie in $\langle S_1 \rangle$ as follows:

- Consider a generator $[f, v]_0$. Assume there exist $\mathcal{B}_1^* \subset \mathcal{B}^*$ and $\mathcal{B}_2 \subset \mathcal{B}$ such that $f \in \langle \mathcal{B}_1^* \rangle$ and $v \in \langle \mathcal{B}_2 \rangle$ and such that $g(w) = 0$ for all $g \in \mathcal{B}_1^*$ and $w \in \mathcal{B}_2$. It is then immediate that $[f, v]_0 \in \langle S_1 \rangle$. This is most easily seen by example:

$$\begin{aligned}[7e_1^* + 3e_3^*, 2e_2 - 5e_4]_0 &= 2[7e_1^* + 3e_3^*, e_2]_0 - 5[7e_1^* + 3e_3^*, e_4]_0 \\ &= 14[e_1^*, e_2]_0 + 21[e_3^*, e_1]_0 - 35[e_1^*, e_4]_0 - 15[e_3^*, e_4]_0.\end{aligned}$$

  Previously we were only writing elements of $S_1$ in blue, but now we will write these elements in blue as well. For instance, we will write $[7e_1^* + 3e_3^*, 2e_2 - 5e_4]_0$.

Let $\equiv$ denote equality modulo $\langle S \rangle$. During the proof, we will underline elements of $E$ that we have not yet proven lie in $\langle S \rangle$. We divide the proof into three claims.

**Claim 3.A.1.** *For distinct* $1 \leq i, j \leq n$, *we have* $\underline{[e_i^* + e_j^*, e_i - e_j]_0} \equiv \underline{[e_i^* - e_j^*, e_i + e_j]_0}$.

Since $n \geq 3$, we can pick some $1 \leq k \leq n$ that is distinct from $i$ and $j$. For $\epsilon \in \{\pm 1\}$, let $x_\epsilon = \underline{[e_i^* + \epsilon e_k^*, e_i - \epsilon e_k]_0}$. For $c \in \{\pm 1\}$, we have

$$\begin{aligned}x_\epsilon \equiv &\underline{[e_i^* + \epsilon e_k^*, e_i - \epsilon e_k]_0} + [ce_j^*, e_i - \epsilon e_k]_0 = [e_i^* + ce_j^* + \epsilon e_k^*, e_i - \epsilon e_k]_0 \\ = &[e_i^* + ce_j^* + \epsilon e_k^*, (e_i - ce_j) + (ce_j - \epsilon e_k)]_0 \\ = &[e_i^* + ce_j^* + \epsilon e_k^*, e_i - ce_j]_0 + [e_i^* + ce_j^* + \epsilon e_k^*, ce_j - \epsilon e_k]_0 \\ = &\underline{[e_i^* + ce_j^*, e_i - ce_j]_0} + [\epsilon e_k^*, e_i - ce_j]_0 + [e_i^*, ce_j - \epsilon e_k]_0 + c[e_j^* + \epsilon c e_k^*, ce_j - \epsilon e_k]_0 \\ \equiv &\underline{[e_i^* + ce_j^*, e_i - ce_j]_0} + \underline{[e_j^* + \epsilon c e_k^*, e_j - \epsilon c e_k]_0}.\end{aligned}$$

We can equate these expressions for $x_1$ for $c = 1$ and $c = -1$ to get

$$\underline{[e_i^* + e_j^*, e_i - e_j]_0} + \underline{[e_j^* + e_k^*, e_j - e_k]_0} \equiv \underline{[e_i^* - e_j^*, e_i + e_j]_0} + \underline{[e_j^* - e_k^*, e_j + e_k]_0}. \tag{4.1}$$

[7] The letter $E$ stands for "extra elements".

Similarly, equating these expressions for $x_{-1}$ for $c = 1$ and $c = -1$ we get

$$[e_i^* + e_j^*, e_i - e_j]_0 + [e_j^* - e_k^*, e_j + e_k]_0 \equiv [e_i^* - e_j^*, e_i + e_j]_0 + [e_j^* + e_k^*, e_j - e_k]_0. \tag{4.2}$$

Combining (4.1) and (4.2), we conclude that $[e_i^* + e_j^*, e_i - e_j]_0 \equiv [e_i^* - e_j^*, e_i + e_j]_0$.

**Claim 3.A.2.** *For distinct $1 \leq i, j \leq n$, we have $[e_i^* + e_j^*, e_i - e_j]_0 \in \langle S \rangle$. In light of Claim 3.A.1, this will imply that $[e_i^* - e_j^*, e_i + e_j]_0 \in \langle S \rangle$ as well.*

Swapping $i$ and $j$ multiplies $[e_i^* + e_j^*, e_i - e_j]_0$ by $-1$, so we can assume without loss of generality that $i < j$. The proof will be by induction on $j - i$. The base case is when $j - i = 1$, in which case $[e_i^* + e_j^*, e_i - e_j]_0$ lies in $S$ and there is nothing to prove. Assume, therefore, that $j - i > 1$ and that the claim is true whenever $j - i$ is smaller. Pick $k$ with $i < k < j$. The element $[e_i^* + e_k^* + e_j^*, e_i - e_j]_0$ equals

$$[e_i^* + e_j^*, e_i - e_j]_0 + [e_k^*, e_i - e_j]_0 \equiv [e_i^* + e_j^*, e_i - e_j]_0.$$

On the other hand, it also equals

$$\begin{aligned}
&[e_i^* + e_k^* + e_j^*, (e_i - e_k) + (e_k - e_j)]_0 \\
&\quad = [e_i^* + e_k^* + e_j^*, e_i - e_k]_0 + [e_i^* + e_k^* + e_j^*, e_k - e_j]_0 \\
&\quad = [e_i^* + e_k^*, e_i - e_k]_0 + [e_j^*, e_i - e_k]_0 + [e_k^* + e_j^*, e_k - e_j]_0 + [e_i^*, e_k - e_j]_0 \equiv 0.
\end{aligned}$$

Here the non-underlined green terms lie in $\langle S \rangle$ by our inductive hypothesis. Combining these, we conclude that $[e_i^* + e_j^*, e_i - e_j]_0 \equiv 0$.

**Claim 3.A.3.** *For distinct $1 \leq i, j \leq n$, we have $[e_i^* + 2e_j^*, 2e_i - e_j]_0 \in \langle S \rangle$.*

Since $n \geq 3$, we can pick some $1 \leq k \leq n$ that is distinct from $i$ and $j$. The element $[e_i^* + 2e_j^* + e_k^*, 2e_i - e_j]_0$ equals

$$[e_i^* + 2e_j^*, 2e_i - e_j]_0 + [e_k^*, 2e_i - e_j]_0 \equiv [e_i^* + 2e_j^*, 2e_i - e_j]_0.$$

On the other hand, it also equals

$$\begin{aligned}
&[e_i^* + 2e_j^* + e_k^*, (e_i - e_k) + (e_i + e_k - e_j)]_0 \\
&\quad = [e_i^* + 2e_j^* + e_k^*, e_i - e_k]_0 + [e_i^* + 2e_j^* + e_k^*, e_i + e_k - e_j]_0 \\
&\quad = [e_i^* + e_k^*, e_i - e_k]_0 + 2[e_j^*, e_i - e_k]_0 + [e_i^* + e_j^*, e_i + e_k - e_j]_0 + [e_j^* + e_k^*, e_i + e_k - e_j]_0 \\
&\quad \equiv [e_i^* + e_j^*, e_i - e_j]_0 + [e_i^* + e_j^*, e_k]_0 + [e_j^* + e_k^*, e_k - e_j]_0 + [e_j^* + e_k^*, e_i]_0 \equiv 0.
\end{aligned}$$

Combining these, we conclude that $[e_i^* + 2e_j^*, 2e_i - e_j]_0 \equiv 0$.

**Step 3.B.** *We prove that $\mathrm{SL}_n(\mathbb{Z})$ takes $\langle S \rangle$ to itself. By Step 2 this will imply that $\langle S \rangle = \mathfrak{A}_n$, and thus by Step 1 that $\Phi$ is an isomorphism.*

For distinct $1 \leq i, j \leq n$, let $E_{ij} \in \mathrm{SL}_n(\mathbb{Z})$ be the elementary matrix obtained from the identity by placing a 1 at position $(i, j)$. These generate $\mathrm{SL}_n(\mathbb{Z})$. Fixing some distinct $1 \leq i, j \leq n$ and some $\epsilon = \pm 1$, it is enough to prove that $E_{ij}^{\epsilon}$ takes $\langle S \rangle$ to itself. To do this, we must prove that for all $s \in S$ the image $E_{ij}^{\epsilon}(s)$ can be written as a linear combination of elements of $S$. The matrix $E_{ij}$ satisfies

$$\begin{aligned}
E_{ij}(e_i^*) &= e_i^* - e_j^* \quad & E_{ij}(e_j) &= e_j + e_i, \\
E_{ij}^{-1}(e_i^*) &= e_i^* + e_j^* \quad & E_{ij}^*(e_j) &= e_j - e_i.
\end{aligned}$$

It fixes all other elements of $\mathcal{B}$ and $\mathcal{B}^*$. Consider $s \in S$. If $E_{ij}^{\epsilon}(s) = s$, there is nothing to prove. We can therefore assume that $E_{ij}\epsilon(s) \neq s$. There are two cases.

**Case 3.B.1.** *$s \in S_1 = \{[e_i^*, e_j]_0 \mid 1 \le i, j \le n$ distinct$\}$.*

The matrix $E_{ij}^{\epsilon}$ fixes all elements of $S_1$ except for the following:

- $[e_i^*, e_k]_0$ with $k \neq i, j$. For these, we have
$$E_{ij}([e_i^*, e_k]_0) = [e_i^* - e_j^*, e_k]_0 = [e_i^*, e_k]_0 - [e_j^*, e_k]_0 \in \langle S_1 \rangle,$$
$$E_{ij}^{-1}([e_i^*, e_k]_0) = [e_i^* + e_j^*, e_k]_0 = [e_i^*, e_k]_0 + [e_j^*, e_k]_0 \in \langle S_1 \rangle.$$
- $[e_k^*, e_j]_0$ with $k \neq i, j$. For these, we have
$$E_{ij}([e_k^*, e_j]_0) = [e_k^*, e_j + e_i]_0 = [e_k^*, e_j]_0 + [e_k^*, e_i]_0 \in \langle S_1 \rangle,$$
$$E_{ij}^{-1}([e_k^*, e_j]_0) = [e_k^*, e_j - e_i]_0 = [e_k^*, e_j]_0 - [e_k^*, e_i]_0 \in \langle S_1 \rangle.$$
- $[e_j^*, e_i]_0$. For this, we have
$$E_{ij}([e_j^*, e_i]_0) = [e_j^* - e_i^*, e_i + e_j]_0 \in E \subset \langle S \rangle,$$
$$E_{ij}^{-1}([e_j^*, e_i]_0) = [e_j^* + e_i^*, e_i - e_j]_0 \in E \subset \langle S \rangle.$$

**Case 3.B.2.** *$s \in S_2 = \left\{[e_i^* + e_{i+1}^*, e_i - e_{i+1}]_0 \mid 1 \le i < n\right\}$.*

To decrease the number of special cases, we will deal more generally with elements of the form $[e_a^* + e_b^*, e_a - e_b]_0$ for distinct $1 \le a, b \le n$. The matrix $E_{ij}^{\epsilon}$ fixes these except when:

- $(a, b) = (i, k)$ or $(a, b) = (k, i)$ for some $1 \le k \le n$ with $k \neq i, j$. Swapping $a$ and $b$ multiplies $[e_a^* + e_b^*, e_a - e_b]_0$ by a sign, so it is enough to deal with $(a, b) = (i, k)$:
$$E_{ij}([e_i^* + e_k^*, e_i - e_k]_0) = [e_i^* - e_j^* + e_k^*, e_i - e_k]_0 = [e_i^* + e_k^*, e_i - e_k]_0 - [e_j^*, e_i - e_k]_0 \in \langle S \rangle,$$
$$E_{ij}^{-1}([e_i^* + e_k^*, e_i - e_k]_0) = [e_i^* + e_j^* + e_k^*, e_i - e_k]_0 = [e_i^* + e_k^*, e_i - e_k]_0 + [e_j^*, e_i - e_k]_0 \in \langle S \rangle.$$
- $(a, b) = (j, k)$ or $(a, b) = (k, j)$ for some $1 \le k \le n$ with $k \neq i, j$. Again, it is enough to deal with $(a, b) = (j, k)$:
$$E_{ij}([e_j^* + e_k^*, e_j - e_k]_0) = [e_j^* + e_k^*, e_j + e_i - e_k]_0 = [e_j^* + e_k^*, e_j - e_k]_0 + [e_j^* + e_k^*, e_i]_0 \in \langle S \rangle,$$
$$E_{ij}^{-1}([e_j^* + e_k^*, e_j - e_k]_0) = [e_j^* + e_k^*, e_j - e_i - e_k]_0 = [e_j^* + e_k^*, e_j - e_k]_0 - [e_j^* + e_k^*, e_i]_0 \in \langle S \rangle.$$
- $(a, b) = (i, j)$ or $(a, b) = (j, i)$. Again it is enough to deal with $(a, b) = (i, j)$:
$$E_{ij}([e_i^* + e_j^*, e_i - e_j]_0) = [e_i^* + 2e_j^*, 2e_i - e_j]_0 \in E,$$
$$E_{ij}^{-1}([e_i^* + e_j^*, e_i - e_j]_0) = [e_i^*, -e_j]_0 =\in \langle S_1 \rangle.$$
□

## 5. Special linear group I′: variant presentation of standard representation

Before proceeding with proofs of our main results, this section and the next give alternate presentations of the standard and adjoint representations of $\mathrm{SL}_n(\mathbb{Q})$ that are needed in Part 3 for our work on the symmetric kernel. We start with the standard representation. Let $\mathcal{B} = \{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{Z}^n$. Say that $v \in \mathbb{Z}^n$ is *$e_i$-standard* (resp. *$e_i$-vanishing*) if the $e_i$-coordinate of $v$ lies in $\{-1, 0, 1\}$ (resp. is 0). If $v_1 \in \mathbb{Z}^n$ is $e_i$-standard and $v_2 \in \mathbb{Z}^n$ is $e_i$-vanishing, then $v_1 + v_2$ is $e_i$-standard. Define the following:

**Definition 5.1.** Define $\mathfrak{Q}'_n$ to be the $\mathbb{Q}$-vector space with the following presentation:

- **Generators**. A generator $[v]'$ for all primitive vectors $v \in \mathbb{Z}^n$ that are $e_1$-standard.
- **Relations**. For a partial basis $\{v_1, v_2\}$ of $\mathbb{Z}^n$ such that $v_1$ is $e_1$-standard and $e_2$ is $e_1$-vanishing, the relation $[v_1]' + [v_2]' = [v_1 + v_2]'$. □

Define a linearization map $\Phi \colon \mathfrak{Q}'_n \to \mathbb{Q}^n$ via the formula $\Phi([v]') = v$. This takes relations to relations, and thus gives a well-defined map. Our goal is to prove:

**Theorem A′.** *For $n \ge 3$, the linearization map $\Phi \colon \mathfrak{Q}'_n \to \mathbb{Q}^n$ is an isomorphism.*

*Proof.* Recall that $\mathcal{B} = \{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{Z}^n$. Let $\mathcal{B}^* = \{e_1^*, \ldots, e_n^*\}$ be the corresponding dual basis. Let $\mathrm{SL}_n(\mathbb{Z}, e_1^*)$ be the stabilizer of $e_1^*$ in $\mathrm{SL}_n(\mathbb{Z})$. The action of $\mathrm{SL}_n(\mathbb{Z}, e_1^*)$ on $\mathbb{Z}^n$ takes $e_1$-standard vectors to $e_1$-standard vectors. It follows that $\mathrm{SL}_n(\mathbb{Z}, e_1^*)$ acts on $\mathfrak{Q}_n'$, and we will use this action to prove our theorem. We follow the outline from §3.

**Step 1.** *Let $S = \{[e_1]', \ldots, [e_n]'\}$. Like we did here, we will write elements of $S$ in blue. Then the restriction of $\Phi$ to $\langle S \rangle$ is an isomorphism.*

Immediate from the fact that $\Phi$ takes $S$ to a basis for $\mathbb{Q}^n$.

**Step 2.** *We prove that the $\mathrm{SL}_n(\mathbb{Z}, e_1^*)$-orbit of $S$ spans $\mathfrak{Q}_n'$.*

Let $W$ be the span of the $\mathrm{SL}_n(\mathbb{Z}, e_1^*)$-orbit of $S$. The action of $\mathrm{SL}_n(\mathbb{Z}, e_1^*)$ on the set of generators for $\mathfrak{Q}_n'$ has three orbits:[8]

$$\begin{aligned}
\mathcal{O}_{-1} &= \left\{ [v]' \mid v \in \mathbb{Z}^n \text{ is primitive and the } e_1\text{-coordinate of } v \text{ is } -1 \right\}, \\
\mathcal{O}_0 &= \left\{ [v]' \mid v \in \mathbb{Z}^n \text{ is primitive and the } e_1\text{-coordinate of } v \text{ is } 0 \right\}, \\
\mathcal{O}_1 &= \left\{ [v]' \mid v \in \mathbb{Z}^n \text{ is primitive and the } e_1\text{-coordinate of } v \text{ is } 1 \right\}.
\end{aligned}$$

It is enough to prove that $W$ contains elements from all three of these orbits. We have $[e_1]' \in \mathcal{O}_1$ and $[e_2]' \in \mathcal{O}_0$, so the only nontrivial case is $\mathcal{O}_{-1}$. Since $W$ contains $[e_1]' \in \mathcal{O}_1$ and $[e_2]' \in \mathcal{O}_0$, it follows that $W$ contains all elements of $\mathcal{O}_0$ and $\mathcal{O}_1$. In $\mathfrak{Q}_n'$, we have the relation

$$[e_1 + e_2]' + [-e_1]' = [e_2]'.$$

Since $W$ contains $[e_1 + e_2]' \in \mathcal{O}_1$ and $[e_2]' \in \mathcal{O}_0$, it also contains $[-e_1]' \in \mathcal{O}_{-1}$. The step follows.

**Step 3.** *We prove that $\mathrm{SL}_n(\mathbb{Z}, e_1^*)$ takes $\langle S \rangle$ to itself. By Step 2 this will imply that $\langle S \rangle = \mathfrak{Q}_n'$, and thus by Step 1 that $\Phi$ is an isomorphism.*

For distinct $1 \leq i, j \leq n$, let $E_{ij} \in \mathrm{SL}_n(\mathbb{Z})$ be the elementary matrix obtained from the identity by placing a 1 at position $(i, j)$. The matrix $E_{ij}$ lies in $\mathrm{SL}_n(\mathbb{Z}, e_1^*)$ if $i \neq 1$, and the set

$$\Lambda = \{E_{ij} \mid 1 \leq i, j \leq n \text{ distinct}, \ i \neq 1\}$$

generates $\mathrm{SL}_n(\mathbb{Z}, e_1^*)$. Fixing some $E_{ij} \in \Lambda$ and some $\epsilon = \pm 1$, it is enough to prove that $E_{ij}^{\epsilon}$ takes $\langle S \rangle$ to itself. Consider some $[e_k]' \in S$. We must prove that $[E_{ij}^{\epsilon}(e_k)]'$ can be written as a linear combination of elements of $S$. If $k \neq j$, then $E_{ij}^{\epsilon}(e_k) = e_k$ and there is nothing to prove. If $k = j$, there are two cases:

- $\epsilon = 1$. In this case, since $i \neq 1$ it follows that $e_i$ is $e_1$-vanishing, so $[E_{ij}(e_j)]' = [e_j + e_i]' = [e_j]' + [e_i]' \in \langle S \rangle$.
- $\epsilon = -1$. In this case, $[E_{ij}^{-1}(e_j)]' = [e_j - e_i]'$. We would like to prove that this equals $[e_j] - [e_i] \in \langle S \rangle$. For this, since $i \neq 1$ the set $\{e_j - e_i, e_i\}$ is a partial basis such that $e_j - e_i$ is $e_1$-standard and $e_i$ is $e_1$-vanishing. We have

$$[e_j - e_i] + [e_i] = [(e_j - e_i) + e_i] = [e_j]. \qquad \square$$

[8]For instance, to see that $\mathrm{SL}_n(\mathbb{Z}, e_1^*)$ acts transitively on $\mathcal{O}_1$, consider some primitive $v \in \mathbb{Z}^n$ with $e_1$-coordinate 1. We must find $M \in \mathrm{SL}_n(\mathbb{Z}, e_1^*)$ with $M([e_1]') = [v]'$. Since $v$ is primitive, we can find a basis $\{v_1, \ldots, v_n\}$ for $\mathbb{Z}^n$ with $v_1 = v$. Adding multiples of $v_1$ to the other $v_i$, we can assume that the $e_1$-coordinate of $v_i$ is 0 for $2 \leq i \leq n$. Let $M \in \mathrm{GL}_n(\mathbb{Z})$ be the matrix whose columns are $\{v_1, \ldots, v_n\}$. Multiplying $v_2$ by $-1$ if necessary, we can assume that $\det(M) = 1$. Since the $e_1$-coordinate of $v_1$ is 1 and the $e_1$-coordinate of $v_i$ is 0 for $2 \leq i \leq n$, we have $M \in \mathrm{SL}_n(\mathbb{Z}, e_1^*)$. By construction, $M([e_1]') = [v_1]' = [v]'$, as desired.

## 6. Special linear group II′: variant presentation of adjoint representation

We now give a variant of our presentation for the adjoint representation. Let $\mathcal{B} = \{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{Z}^n$ and let $\mathcal{B}^* = \{e_1^*, \ldots, e_n^*\}$ be the corresponding dual basis. As in §5, an element $f \in (\mathbb{Z}^n)^*$ is *$e_i^*$-standard* (resp. $e_i^*$-vanishing) if the $e_i^*$-coordinate of $f$ lies in $\{-1, 0, 1\}$ (resp. is 0). We define $v \in \mathbb{Z}^n$ being $e_i$-standard and $e_i$-vanishing similarly. Define:

**Definition 6.1.** Define $\mathfrak{A}'_n$ to be the $\mathbb{Q}$-vector space with the following presentation:

- **Generators**. A generator $[f, v]'_0$ for all primitive vectors $f \in (\mathbb{Z}^n)^*$ and $v \in \mathbb{Z}^n$ such that $f(v) = 0$ and such that $f$ is $e_1^*$-standard and $v$ is $e_2$-standard.
- **Relations**. The following two families of relations:
  - For all $e_1^*$-primitive vectors $f \in (\mathbb{Z}^n)^*$ and all partial bases $\{v_1, v_2\}$ of $\ker(f)$ such that $v_1$ is $e_2$-standard and $v_2$ is $e_2$-vanishing, the relation $[f, v_1 + v_2]'_0 = [f, v_1]'_0 + [f, v_2]'_0$.
  - For all $e_2$-standard primitive vectors $v \in \mathbb{Z}^n$ and all partial bases $\{f_1, f_2\}$ of
$$\ker(v) = \{f \in (\mathbb{Z}^n)^* \mid f(v) = 0\},$$
such that $f_1$ is $e_1^*$-standard and $f_2$ is $e_1^*$-vanishing, the relation $[f_1 + f_2, v]_0 = [f_1, v]_0 + [f_2, v]_0$. □

Define $\Phi\colon \mathfrak{A}'_n \to (\mathbb{Q}^n)^* \otimes \mathbb{Q}^n$ via the formula $\Phi([f, v]'_0) = f \otimes v$. This takes relations to relations, and thus gives a well-defined linearization map with $\mathrm{Im}(\Phi) \subset \mathfrak{sl}_n(\mathbb{Q})$. Our goal is to prove:

**Theorem B′.** *For $n \geq 4$, the linearization map $\Phi\colon \mathfrak{A}'_n \to \mathfrak{sl}_n(\mathbb{Q})$ is an isomorphism.*

*Proof.* We start with the following, which we will use freely throughout the proof:

**Claim.** *In $\mathfrak{A}'_n$, the following relations hold for all $m \geq 1$:*

*(i) For all $e_1^*$-standard primitive vectors $f \in (\mathbb{Z}^n)^*$ and all $e_2$-standard primitive vectors $v_1, \ldots, v_m \in \ker(f)$ and all $\lambda_1, \ldots, \lambda_m \in \mathbb{Z}$ such that $\sum_{i=1}^m \lambda_i v_i$ is primitive and $e_2$-standard, we have*
$$[f, \sum_{i=1}^m \lambda_i v_i]'_0 = \sum_{i=1}^m \lambda_i [f, v_i]'_0.$$

*(ii) For all $e_2$-standard primitive vectors $v \in \mathbb{Z}^n$ and all $e_1^*$-standard primitive vectors $f_1, \ldots, f_m \in \ker(v)$ and all $\lambda_1, \ldots, \lambda_m \in \mathbb{Z}$ such that $\sum_{i=1}^m \lambda_i f_i$ is primitive and $e_1^*$-standard, we have*
$$[\sum_{i=1}^m \lambda_i f_i, v]'_0 = \sum_{i=1}^m \lambda_i [f_i, v]'_0.$$

*Proof.* Both are proved the same way, so we will give the details for (i). Let $f \in (\mathbb{Z}^n)^*$ and $v_1, \ldots, v_m \in \ker(f)$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{Z}$ be as in the claim. There are now two cases.

The first is that the restriction of $e_2^*$ to $\ker(f)$ is surjective. This implies that $\ker(f)$ contains $e_2$-standard vectors that are not $e_2$-vanishing. We can then choose an isomorphism $\mu\colon \mathbb{Z}^{n-1} \to \ker(f)$ with the following property:

- Let $\{x_1, \ldots, x_{n-1}\}$ be the standard basis for $\mathbb{Z}^{n-1}$ and let $\{x_1^*, \ldots, x_{n-1}^*\}$ be the corresponding dual basis. Then the induced map $\mu^*\colon \ker(f)^* \to (\mathbb{Z}^{n-1})^*$ takes the restriction of $e_2^*$ to $x_1^*$. This condition ensures that $\mu$ gives a bijection between $x_1$-standard vectors in $\mathbb{Z}^{n-1}$ and $e_2$-standard vectors in $\ker(f)$.

Let $u_i = \mu^{-1}(v_i)$. Recall that we defined $\mathfrak{Q}'_{n-1}$ in Definition 5.1. Define a map $\psi\colon \mathfrak{Q}'_{n-1} \to \mathfrak{A}'_n$ via the formula

$$\psi([w]') = [f, \mu(w)]'_0 \quad \text{for an } x_1\text{-standard primitive } w \in \mathbb{Z}^{n-1}.$$

This takes relations to relations, and thus gives a well-defined map. Since $n \geq 3$, Theorem A′ says that $\mathfrak{Q}'_{n-1} \cong \mathbb{Q}^{n-1}$. It follows from this theorem that

$$[\sum_{i=1}^{m} \lambda_i u_i]' = \sum_{i=1}^{m} \lambda_i [u_i]'.$$

Plugging this into $\psi$, we see that

$$[f, \sum_{i=1}^{m} \lambda_i v_i]'_0 = \psi\left([\sum_{i=1}^{m} \lambda_i u_i]'\right) = \sum_{i=1}^{m} \lambda_i \psi\left([u_i]'\right) = \sum_{i=1}^{m} \lambda_i [f, v_i]'_0,$$

as desired.

The second case is that the restriction of $e_2^*$ to $\ker(f)$ is not surjective, so all $e_2$-standard vectors in $\ker(f)$ are $e_2$-vanishing. In particular, all the $v_i$ are $e_2$-vanishing. Letting $U = \ker(e_2^*) \cap \ker(f)$, this implies that all the $v_i$ lie in $U$. Since $U$ is the intersection of two direct summands of $\mathbb{Z}^n$, it follows that $U$ is also a direct summand of $\mathbb{Z}^n$. Let $r$ be the rank of $U$. Since $n \geq 4$, we have $r \geq 2$. Choose an isomorphism $\mu\colon \mathbb{Z}^r \to U$. Let $u_i = \mu^{-1}(v_i)$. Recall that we defined $\mathfrak{Q}_r$ in Definition 1.1. Define a map $\psi\colon \mathfrak{Q}_r \to \mathfrak{A}'_n$ via the formula

$$\psi([w]) = [f, \mu(w)]'_0 \quad \text{for a primitive } w \in \mathbb{Z}^{n-1}.$$

This makes sense since each $\mu(w)$ is $e_2$-vanishing, and hence also $e_2$-standard. The map $\psi$ takes relations to relations, and thus gives a well-defined map. Since $r \geq 2$, Theorem A says that $\mathfrak{Q}_{n-1} \cong \mathbb{Q}^r$. It follows from this theorem that

$$[\sum_{i=1}^{m} \lambda_i u_i] = \sum_{i=1}^{m} \lambda_i [u_i].$$

Plugging this into $\psi$, we see that

$$[f, \sum_{i=1}^{m} \lambda_i v_i]'_0 = \psi\left([\sum_{i=1}^{m} \lambda_i u_i]\right) = \sum_{i=1}^{m} \lambda_i \psi\left([u_i]\right) = \sum_{i=1}^{m} \lambda_i [f, v_i]'_0,$$

as desired. □

Recall that $\mathcal{B} = \{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{Z}^n$ and $\mathcal{B}^* = \{e_1^*, \ldots, e_n^*\}$ is the corresponding dual basis for $(\mathbb{Z}^n)^*$. Let $\Gamma = \mathrm{SL}_n(\mathbb{Z}, e_1, e_2^*)$ be the stabilizer of $e_1$ and $e_2^*$ in $\mathrm{SL}_n(\mathbb{Z})$. The action of $\Gamma$ on $(\mathbb{Z}^n)^*$ fixes the $e_1^*$-coordinate, and the action of $\Gamma$ on $\mathbb{Z}^n$ fixes the $e_2$-coordinate. It follows that $\Gamma$ acts on on $\mathfrak{A}'_n$, and we will use this action to prove our theorem. We follow the outline from §3, though for readability we divide Step 3 into Steps 3.A and 3.B.

**Step 1.** *Let $S = S_1 \cup S_2$, where the $S_i$ are:*

$$S_1 = \left\{[e_i^*, e_j]'_0 \mid 1 \leq i, j \leq n \text{ distinct}\right\},$$
$$S_2 = \left\{[e_i^* + e_{i+1}^*, e_i - e_{i+1}]'_0 \mid 1 \leq i < n\right\}.$$

*Note that all of these are generators of $\mathfrak{A}'_n$. Like we did here, we will write elements of $S$ in blue. Then the restriction of $\Phi$ to $\langle S \rangle$ is an isomorphism.*

The proof is identical to that of Step 1 in the proof of Theorem B, so we omit the details.

**Step 2.** *Recall that $\Gamma = \mathrm{SL}_n(\mathbb{Z}, e_1, e_2^*)$. We prove that the $\Gamma$-orbit of $S$ spans $\mathfrak{A}'_n$.*

Consider a generator $[f, v]'_0$ of $\mathfrak{A}'_n$, so:

- $f \in (\mathbb{Z}^n)^*$ is primitive and $e_1^*$-standard; and
- $v \in \mathbb{Z}^n$ is primitive and $e_2$-standard; and
- $f(v) = 0$.

We want to prove that $[f, v]'_0$ is in the span of the $\Gamma$-orbit of $S$. There are three (nonexclusive) cases:

**Case 2.1.** *The $e_1^*$-coordinate of $f$ is $\pm 1$.*

Using the claim from the beginning of the proof, we have $[f, v]'_0 = -[-f, v]'_0$, so replacing $[f, v]'_0$ with $[-f, v]'_0$ if necessary we can assume that the $e_1^*$-coordinate of $f$ is 1. Let $M$ be the matrix whose rows are $\{f, e_2^*, \ldots, e_n^*\}$. Since the $e_1^*$-coordinate of $f$ is 1, we have $\det(M) = 1$. By construction we have have $M \in \Gamma = \mathrm{SL}_n(\mathbb{Z}, e_1, e_2^*)$ and $M(e_1^*) = f$. Replacing $[f, v]'_0$ by $M^{-1}([f, v]'_0)$, we can assume that $f = e_1^*$. Write

$$v = \lambda_1 e_1 + \cdots + \lambda_n e_n \quad \text{with } \lambda_1, \ldots, \lambda_n \in \mathbb{Z}.$$

Since $v$ is in the kernel of $f = e_1^*$, we have $\lambda_1 = 0$. Using the claim from the beginning of the proof, we have

$$[f, v]'_0 = [e_1^*, \lambda_2 e_2 + \cdots + \lambda_n e_n]'_0 = \lambda_2 [e_1^*, e_2]'_0 + \cdots + \lambda_n [e_1^*, e_n]'_0 \in \langle S \rangle,$$

as desired.

**Case 2.2.** *The $e_2$-coordinate of $v$ is $\pm 1$.*

Similar to Case 2.1 except that after possibly replacing $[f, v]'_0$ with $[f, -v]'_0$ we apply an element of $\Gamma = \mathrm{SL}_n(\mathbb{Z}, e_1, e_2^*)$ to change $v$ to $e_2$.

**Case 2.3.** *The $e_1^*$-coordinate of $f$ is $0$ and the $e_2$-coordinate of $v$ is $0$.*

Write $f = p e_2^* + q g$ with $p, q \in \mathbb{Z}$ and $g \in \langle e_3^*, \ldots, e_n^* \rangle$ primitive. Since $f$ is primitive, we have $\gcd(p, q) = 1$. Since $n \geq 4$, it follows that the action of $\mathrm{SL}(\langle e_3^*, \ldots, e_n^* \rangle)$ on $\langle e_3^*, \ldots, e_n^* \rangle$ is transitive on primitive vectors. Using this, we can find $M \in \mathrm{SL}_n(\mathbb{Z})$ such that $M(g) = e_3^*$ and such that $M$ acts as the identity on $\langle e_1^*, e_2^* \rangle$ and $\langle e_1, e_2 \rangle$. It follows that $M \in \Gamma = \mathrm{SL}_n(\mathbb{Z}, e_1, e_2^*)$. Replacing $[f, v]'_0 = [p e_2^* + q g, v]'_0$ with $M([f, v]'_0)$, we can assume that $f = p e_2^* + q e_3^*$. Write

$$v = \lambda_1 e_1 + \cdots + \lambda_n e_n \quad \text{with } \lambda_1, \ldots, \lambda_n \in \mathbb{Z}.$$

By assumption, we have $\lambda_2 = 0$. Since $f(v) = 0$ and $f = p e_2^* + q e_3^*$, we have

$$p\lambda_2 + q\lambda_3 = q\lambda_3 = 0.$$

It follows that either $q$ or $\lambda_3$ must vanish. If $q = 0$, then since $f = p e_2^* + q e_3^*$ is primitive we must have $p \in \{\pm 1\}$ and we can use the claim from the beginning of the proof to see that

$$\begin{aligned} [f, v]'_0 &= [p e_2^*, \lambda_1 e_1 + \lambda_3 e_3 + \cdots + \lambda_n e_n]'_0 \\ &= p\lambda_1 [e_2^*, e_1]'_0 + p\lambda_3 [e_2^*, e_3]'_0 + \cdots + p\lambda_n [e_2^*, e_n]'_0 \in \langle S \rangle, \end{aligned}$$

as desired. If instead $\lambda_3 = 0$, then we can again use the claim from beginning of the proof to see that

$$\begin{aligned} [f, v]'_0 &= [p e_2^* + q e_3^*, \lambda_1 e_1 + \lambda_4 e_4 + \cdots + \lambda_n e_n]'_0 \\ &= \sum_{\substack{1 \leq i \leq n \\ i \neq 2,3}} \lambda_i [p e_2^* + q e_3^*, e_i]'_0 = \sum_{\substack{1 \leq i \leq n \\ i \neq 2,3}} \left( p\lambda_i [e_2^*, e_i]'_0 + q\lambda_i [e_3^*, e_i]'_0 \right) \in \langle S \rangle, \end{aligned}$$

again as desired.

**Step 3.A.** *In preparation for proving that $\Gamma = \mathrm{SL}_n(\mathbb{Z}, e_1, e_2^*)$ takes $\langle S \rangle$ to $\langle S \rangle$, we prove that all elements of*[9]

$$\begin{aligned} E = &\left\{ [e_i^* + e_j^*, e_i - e_j]_0' \mid 1 \leq i,j \leq n \text{ distinct} \right\} \\ &\cup \left\{ [e_i^* - e_j^*, e_i + e_j]_0' \mid 1 \leq i,j \leq n \text{ distinct} \right\} \\ &\cup \left\{ [e_i^* + 2e_j^*, 2e_i - e_j]_0' \mid 1 \leq i,j \leq n \text{ distinct, } j \neq 1,\ i \neq 2 \right\}. \end{aligned}$$

*lie in $\langle S \rangle$. Like we did here, we will write elements of $E$ in green.*

The proof is identical to that of Step 3.A of the proof of Theorem B; indeed, a careful examination of that proof shows that every term $[f, v]_0'$ that appears when handling the elements of $E$ listed above has the property that $f$ is $e_1^*$-standard and $v$ is $e_2$-standard. We therefore omit the details.

**Step 3.B.** *We prove that $\Gamma = \mathrm{SL}_n(\mathbb{Z}, e_1, e_2^*)$ takes $\langle S \rangle$ to itself. By Step 2 this will imply that $\langle S \rangle = \mathfrak{A}_n'$, and thus by Step 1 that $\Phi$ is an isomorphism.*

For distinct $1 \leq i, j \leq n$, let $E_{ij} \in \mathrm{SL}_n(\mathbb{Z})$ be the elementary matrix obtained from the identity by placing a 1 at position $(i, j)$. The matrix $E_{ij}$ lies in $\Gamma$ if $j \neq 1$ and $i \neq 2$, and the set

$$\Lambda = \{E_{ij} \mid 1 \leq i,j \leq n \text{ distinct, } j \neq 1,\ i \neq 2\}$$

generates $\Gamma$. Fixing some $E_{ij} \in \Lambda$ and some $\epsilon = \pm 1$, it is enough to prove that $E_{ij}^{\epsilon}$ takes $\langle S \rangle$ to itself. The proof of this is identical to that of Step 3.B of the proof of Theorem B. We therefore omit the details. □

## 7. Generating the symplectic group

To prove our theorems about $\mathrm{Sp}_{2g}(\mathbb{Z})$, we need generators for $\mathrm{Sp}_{2g}(\mathbb{Z})$. The most convenient generating set was constructed by Hua–Reiner [3], which we now describe. Recall from §1.13 that $H = \mathbb{Q}^{2g}$ and $H_{\mathbb{Z}} = \mathbb{Z}^{2g}$, which are equipped with the standard symplectic form $\omega$. Let $\mathcal{B} = \{a_1, b_1, \ldots, a_g, b_g\}$ be a symplectic basis for $H_{\mathbb{Z}}$.

Define $\mathrm{SymSp}_g$ to be the subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$ consisting of all $f \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that for all $x \in \mathcal{B}$, we have either $f(x) \in \mathcal{B}$ or $-f(x) \in \mathcal{B}$. This is a finite group. Associated to each $f \in \mathrm{Sp}_{2g}(\mathbb{Z})$ is a permutation $p$ of $\{1, \ldots, g\}$ such that for all $1 \leq i \leq g$ the pair $(f(a_i), f(b_i))$ is one of the following:

$$(a_{p(i)}, b_{p(i)}), \quad \text{or} \quad (-a_{p(i)}, -b_{p(i)}), \quad \text{or} \quad (b_{p(i)}, -a_{p(i)}), \quad \text{or} \quad (-b_{p(i)}, a_{p(i)}).$$

Next, for $1 \leq i \leq g$ let $X_i \in \mathrm{Sp}_{2g}(\mathbb{Z})$ be the element defined by

$$X_i(a_i) = a_i + b_i \quad \text{and} \quad X_i(x) = x \text{ for } x \in \mathcal{B} \setminus \{a_i\}.$$

Finally, for distinct $1 \leq i, j \leq g$ let $Y_{ij} \in \mathrm{Sp}_{2g}(\mathbb{Z})$ be the element defined by

$$Y_{ij}(a_i) = a_i + b_j \quad \text{and} \quad Y_{ij}(a_j) = a_j + b_i \quad \text{and} \quad Y_{ij}(x) = x \text{ for } x \in \mathcal{B} \setminus \{a_i, a_j\}.$$

It follows from the calculations in [3] that:[10]

**Theorem 7.1** (Hua–Reiner, [3])**.** *For all $g \geq 1$, the group $\mathrm{Sp}_{2g}(\mathbb{Z})$ is generated by $\mathrm{SymSp}_g$ and $X_1$ and $Y_{12}$.*

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[9]Note that all of these lie in $\mathfrak{A}_n'$; indeed, the only $[f, v]_0' \in E$ where there is any possibility that either $f$ is not $e_1^*$-standard or $v$ is not $e_2$-standard are those of the form $[e_i^* + 2e_j^*, 2e_i - e_j]_0'$, and our assumptions that $j \neq 1$ and $i \neq 2$ ensure that indeed $[f, v]_0' \in \mathfrak{A}_n'$.

[10]This is not identical to the generating set from [3], but it is easily seen to be equivalent to it and fits better into our calculations.

*Remark* 7.2. The other $X_i$ are not needed since they are all conjugate to $X_1$ by elements of $\mathrm{SymSp}_g$. Similarly, the other $Y_{ij}$ are not needed. □

The most complicated generator is $Y_{12}$. The following observation will simplify our calculations by allowing us to not worry about $Y_{12}^{-1}$:

**Corollary 7.3.** *For all $g \geq 1$, the group $\mathrm{Sp}_{2g}(\mathbb{Z})$ is generated as a monoid by $\mathrm{SymSp}_g$ and $\{X_1, X_1^{-1}, Y_{12}\}$.*

*Proof.* Let $\sigma \in \mathrm{SymSp}_{2g}$ be the element that multiplies $a_1$ and $b_1$ by $-1$ while fixing all other elements of $\mathcal{B}$. We then have $Y_{12}^{-1} = \sigma Y_{12} \sigma$. The corollary follows. □

## 8. Symplectic group I: standard representation

Recall that $\mathfrak{H}_n$ is the $\mathbb{Q}$-vector space with the following presentation:

- **Generators**. A generator $[v]_{\mathrm{Sp}}$ for all primitive vectors $v \in H_{\mathbb{Z}}$.
- **Relations**. For a partial basis $\{v_1, v_2\}$ of $H_{\mathbb{Z}}$ with $\omega(v_1, v_2) = 0$, the relation $[v_1]_{\mathrm{Sp}} + [v_2]_{\mathrm{Sp}} = [v_1 + v_2]_{\mathrm{Sp}}$.

Define a linearization map $\Phi\colon \mathfrak{H}_g \to H$ via the formula $\Phi([v]_{\mathrm{Sp}}) = v$. This takes relations to relations, and thus gives a well-defined map. Our goal is to prove:

**Theorem C.** *For $g \geq 2$, the linearization map $\Phi\colon \mathfrak{H}_g \to H$ is an isomorphism.*

*Proof.* Let $\mathcal{B} = \{a_1, b_1, \ldots, a_g, b_g\}$ be a symplectic basis for $H_{\mathbb{Z}}$. We follow the outline from §3, though for readability we divide Step 3 into Steps 3.A and 3.B.

**Step 1.** *Let $S = \{[x]_{\mathrm{Sp}} \mid x \in \mathcal{B}\}$. Then the restriction of $\Phi$ to $\langle S \rangle$ is an isomorphism.*

Immediate.

**Step 2.** *We prove that the $\mathrm{Sp}_{2g}(\mathbb{Z})$-orbit of $S$ spans $\mathfrak{H}_g$.*

Since $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts transitively on primitive vectors, it acts transitively on the generators for $\mathfrak{H}_g$. It follows that the $\mathrm{Sp}_{2g}(\mathbb{Z})$-orbit of $S$ spans $\mathfrak{H}_g$.

**Step 3.A.** *In preparation for proving that $\mathrm{Sp}_{2g}(\mathbb{Z})$ takes $\langle S \rangle$ to $\langle S \rangle$, we prove that all elements of $E = \{[-x]_{\mathrm{Sp}} \mid x \in \mathcal{B}\} \cup \{[a_1 + b_1]_{\mathrm{Sp}}, [a_1 - b_1]_{\mathrm{Sp}}\}$ lie in $\langle S \rangle$.*

We divide this into three claims:

**Claim 3.A.1.** *For all primitive $v \in H_{\mathbb{Z}}$, we have $[-v]_{\mathrm{Sp}} = -[v]_{\mathrm{Sp}}$. In particular, all elements of $\{[-x]_{\mathrm{Sp}} \mid x \in \mathcal{B}\}$ lie in $S$.*

Since $g \geq 2$, we can find $w \in H_{\mathbb{Z}}$ with $\omega(v, w) = 0$ such that $\langle v, w \rangle$ is a rank-2 direct summand of $\mathbb{Z}^{2g}$. Let $\mu\colon \mathbb{Z}^2 \to \langle v, w \rangle$ be the isomorphism taking the standard basis $\{e_1, e_2\}$ of $\mathbb{Z}^2$ to $\{v, w\}$. Recall that we defined $\mathfrak{Q}_n$ in Definition 1.1. Define a map $\psi\colon \mathfrak{Q}_2 \to \mathfrak{H}_g$ via the formula

$$\psi([x]) = [\mu(x)]_{\mathrm{Sp}} \quad \text{for a primitive } x \in \mathbb{Z}^2.$$

Since $\omega(-,-)$ vanishes identically on $\langle v, w \rangle$, this formula takes generators to generators. Theorem A says that $\mathfrak{Q}_2 \cong \mathbb{Q}^2$. This implies that $[-x] = -[x]$ for all primitive $x \in \mathbb{Z}^2$. In particular,

$$[-v]_{\mathrm{Sp}} = [-\mu(e_1)]_{\mathrm{Sp}} = \psi([-e_1]) = \psi(-[e_1]) = -[v]_{\mathrm{Sp}}.$$

**Claim 3.A.2.** *We have $[a_1 + b_1]_{\mathrm{Sp}} \in \langle S \rangle$.*

The element $[(a_1+b_1)+(a_2+b_2)]_{\mathrm{Sp}}$ equals $[a_1+b_1]_{\mathrm{Sp}}+[a_2+b_2]_{\mathrm{Sp}}$. It also equals

$$[(a_1+b_2)+(a_2+b_1)]_{\mathrm{Sp}}=[a_1+b_2]_{\mathrm{Sp}}+[a_2+b_1]_{\mathrm{Sp}}=[a_1]_{\mathrm{Sp}}+[b_2]_{\mathrm{Sp}}+[a_2]_{\mathrm{Sp}}+[b_1]_{\mathrm{Sp}}.$$

We deduce that

$$[a_1+b_1]_{\mathrm{Sp}}+[a_2+b_2]_{\mathrm{Sp}}=[a_1]_{\mathrm{Sp}}+[b_2]_{\mathrm{Sp}}+[a_2]_{\mathrm{Sp}}+[b_1]_{\mathrm{Sp}}. \tag{8.1}$$

Similarly, the element $[(a_1+b_1)-(a_2+b_2)]_{\mathrm{Sp}}$ equals $[a_1+b_1]_{\mathrm{Sp}}-[a_2+b_2]_{\mathrm{Sp}}$. It also equals

$$[(a_1-b_2)+(-a_2+b_1)]_{\mathrm{Sp}}=[a_1-b_2]_{\mathrm{Sp}}+[-a_2+b_1]_{\mathrm{Sp}}=[a_1]_{\mathrm{Sp}}-[b_2]_{\mathrm{Sp}}-[a_2]_{\mathrm{Sp}}+[b_1]_{\mathrm{Sp}}.$$

We deduce that

$$[a_1+b_1]_{\mathrm{Sp}}-[a_2+b_2]_{\mathrm{Sp}}=[a_1]_{\mathrm{Sp}}-[b_2]_{\mathrm{Sp}}-[a_2]_{\mathrm{Sp}}+[b_1]_{\mathrm{Sp}}. \tag{8.2}$$

Adding (8.1) and (8.2) and dividing by 2 yield the claim.

**Claim 3.A.3.** *We have* $[a_1-b_1]_{\mathrm{Sp}}\in\langle S\rangle$.

The element $[(a_1-b_1)+(a_2-b_2)]_{\mathrm{Sp}}$ equals $[a_1-b_1]_{\mathrm{Sp}}+[a_2-b_2]_{\mathrm{Sp}}$. It also equals

$$\begin{aligned}[(a_1-b_2)+(a_2-b_1)]_{\mathrm{Sp}}&=[a_1-b_2]_{\mathrm{Sp}}+[a_2-b_1]_{\mathrm{Sp}}\\&=[a_1]_{\mathrm{Sp}}-[b_2]_{\mathrm{Sp}}+[a_2]_{\mathrm{Sp}}-[b_1]_{\mathrm{Sp}}.\end{aligned}$$

We deduce that

$$[a_1-b_1]_{\mathrm{Sp}}+[a_2-b_2]_{\mathrm{Sp}}=[a_1]_{\mathrm{Sp}}-[b_2]_{\mathrm{Sp}}+[a_2]_{\mathrm{Sp}}-[b_1]_{\mathrm{Sp}}. \tag{8.3}$$

Similarly, the element $[(a_1-b_1)-(a_2-b_2)]_{\mathrm{Sp}}$ equals $[a_1-b_1]_{\mathrm{Sp}}-[a_2-b_2]_{\mathrm{Sp}}$. It also equals

$$\begin{aligned}[(a_1+b_2)+(-a_2-b_1)]_{\mathrm{Sp}}&=[a_1+b_2]_{\mathrm{Sp}}+[-a_2-b_1]_{\mathrm{Sp}}\\&=[a_1]_{\mathrm{Sp}}+[b_2]_{\mathrm{Sp}}-[a_2]_{\mathrm{Sp}}-[b_1]_{\mathrm{Sp}}.\end{aligned}$$

We deduce that

$$[a_1-b_1]_{\mathrm{Sp}}-[a_2-b_2]_{\mathrm{Sp}}=[a_1]_{\mathrm{Sp}}+[b_2]_{\mathrm{Sp}}-[a_2]_{\mathrm{Sp}}-[b_1]_{\mathrm{Sp}}. \tag{8.4}$$

Adding (8.3) and (8.4) and dividing by 2 yields the claim.

**Step 3.B.** *We prove that* $\mathrm{Sp}_{2g}(\mathbb{Z})$ *takes* $\langle S\rangle$ *to itself. By Step 2 this will imply that* $\langle S\rangle=\mathfrak{H}_g$*, and thus by Step 1 that* $\Phi$ *is an isomorphism.*

Corollary 7.3 says that $\mathrm{Sp}_{2g}(\mathbb{Z})$ is generated as a monoid by $\mathrm{SymSp}_g\cup\{X_1,X_1^{-1},Y_{12}\}$. Let $f\in\mathrm{SymSp}_g\cup\{X_1,X_1^{-1},Y_{12}\}$ and let $s\in S$. Using Step 3.A, it is enough to check that $f(s)$ can both be written as a linear combination of elements of $S$ and $E$.

The first case is $f\in\mathrm{SymSp}_g$. This case is easy: we have $s=[x]_{\mathrm{Sp}}$ for some $x\in\mathcal{B}$, and for some $y\in\mathcal{B}$ the element $f(s)=[f(x)]_{\mathrm{Sp}}$ is either $[y]_{\mathrm{Sp}}\in S$ or $[-y]_{\mathrm{Sp}}\in E$. The second case is $f=X_1$ or $f=X_1^{-1}$. Recall that $X_1$ takes $a_1$ to $a_1+b_1$ and fixes all other elements of $\mathcal{B}$. Both $X_1$ and $X_1^{-1}$ fix all elements of $S$ except for $[a_1]_{\mathrm{Sp}}$, and for this we have

$$\begin{aligned}X_1([a_1]_{\mathrm{Sp}})&=[a_1+b_1]_{\mathrm{Sp}}\in E,\\X_1^{-1}([a_1]_{\mathrm{Sp}})&=[a_1-b_1]_{\mathrm{Sp}}\in E.\end{aligned}$$

The final case is $f=Y_{12}$. Recall that this takes $a_1$ to $a_1+b_2$ and $a_2$ to $b_1+a_2$ and fixes all other elements of $\mathcal{B}$. The element $Y_{12}$ fixes all elements of $S$ except for $[a_1]_{\mathrm{Sp}}$ and $[a_2]_{\mathrm{Sp}}$, for these we have

$$\begin{aligned}Y_{12}([a_1]_{\mathrm{Sp}})&=[a_1+b_2]_{\mathrm{Sp}}=[a_1]_{\mathrm{Sp}}+[b_2]_{\mathrm{Sp}}\in\langle S\rangle,\\Y_{12}([a_2]_{\mathrm{Sp}})&=[a_2+b_1]_{\mathrm{Sp}}=[a_2]_{\mathrm{Sp}}+[b_1]_{\mathrm{Sp}}\in\langle S\rangle.\end{aligned}$$

□

## 9. Symplectic group II: kernel and symmetric representations

We could prove Theorems D and E using our now-standard proof technique, but to illustrate another useful tool we show how to deduce them from Theorem B.

9.1. **Non-symmetric presentation.** We defined $\mathfrak{Z}_g^s$ and $\mathfrak{Z}_g^a$ in Definitions 1.12 and 1.11. We now define the following, which is similar to these but does not include their symmetric/anti-symmetric relations:

**Definition 9.1.** Define $\mathfrak{Z}_g$ to be the $\mathbb{Q}$-vector space with the following presentation:

- **Generators**. A generator $(\!|v_1, v_2|\!)$ for all orthogonal primitive vectors $v_1, v_2 \in H_{\mathbb{Z}}$.
- **Relations**. For all primitive vectors $v \in H_{\mathbb{Z}}$ and all partial bases $\{w_1, w_2\}$ of $v_{\mathbb{Z}}^{\perp}$, the relations

$$(\!|v, w_1 + w_2|\!) = (\!|v, w_1|\!) + (\!|v, w_2|\!), \quad \text{and}$$
$$(\!|w_1 + w_2, v|\!) = (\!|w_1, v|\!) + (\!|w_2, v|\!). \qquad \square$$

This combines $\mathfrak{Z}_g^s$ and $\mathfrak{Z}_g^a$ in the following way. There is an involution $I\colon \mathfrak{Z}_g \to \mathfrak{Z}_g$ defined by $I((\!|v_1, v_2|\!)) = (\!|v_2, v_1|\!)$ that we will call the *canonical involution*. We then have:

**Lemma 9.2.** *We have $\mathfrak{Z}_g = \mathfrak{Z}_g^s \oplus \mathfrak{Z}_g^a$, where $\mathfrak{Z}_g^s$ and $\mathfrak{Z}_g^a$ are identified with the $+1$ and $-1$ eigenspaces of the canonical involution.*

*Proof.* Define a map $\pi\colon \mathfrak{Z}_g \to \mathfrak{Z}_g^s \oplus \mathfrak{Z}_g^a$ via the formula

$$\pi((\!|v_1, v_2|\!)) = ((\!|v_1, v_2|\!)_s, (\!|v_1, v_2|\!)_a).$$

This take relations to relations, and thus gives a well-defined map. Next, define $\iota\colon \mathfrak{Z}_g^s \oplus \mathfrak{Z}_g^a$ as $\iota = \iota_s + \iota_a$, where $\iota_s\colon \mathfrak{Z}_g^s \to \mathfrak{Z}_g$ and $\iota_a\colon \mathfrak{Z}_g^a \to \mathfrak{Z}_g$ are the maps defined via the formulas

$$\iota_s((\!|v_1, v_2|\!)_s) = \frac{1}{2}\left((\!|v_1, v_2|\!) + (\!|v_2, v_1|\!)\right),$$
$$\iota_a((\!|v_1, v_2|\!)_a) = \frac{1}{2}\left((\!|v_1, v_2|\!) - (\!|v_2, v_1|\!)\right).$$

Again, these take relations to relations and thus give well-defined maps. The maps $\pi$ and $\iota$ are inverses, so both are isomorphisms. This gives the decomposition $\mathfrak{Z}_g = \mathfrak{Z}_g^s \oplus \mathfrak{Z}_g^a$, and the fact that $\mathfrak{Z}_g^s$ and $\mathfrak{Z}_g^a$ are identified with the $+1$ and $-1$ eigenspaces of the canonical involution follows from the above formulas. $\square$

9.2. **Identifying the non-symmetric presentation.** Define a linearization map $\Phi\colon \mathfrak{Z}_g \to H^{\otimes 2}$ via the formula $\Phi((\!|v_1, v_2|\!)) = v_1 \otimes v_2$. This takes relations to relations, and thus gives a well-defined map. Its image lies in the kernel $\mathcal{Z}_g$ of the map

$$H^{\otimes 2} \longrightarrow \wedge^2 H \xrightarrow{\ \omega\ } \mathbb{Q},$$

where the second map is the one induced by the symplectic form $\omega$. We will prove:

**Theorem 9.3.** *For $g \geq 2$, the linearization map $\Phi\colon \mathfrak{Z}_g \to \mathcal{Z}_g$ is an isomorphism.*

*Proof.* Pick an isomorphism $\mu_2\colon H_{\mathbb{Z}} \to \mathbb{Z}^{2g}$. Since $\omega$ identifies $H_{\mathbb{Z}}$ with its dual $H_{\mathbb{Z}}^* = \text{Hom}(H, \mathbb{Z})$, we can find a corresponding isomorphism $\mu_1\colon H_{\mathbb{Z}} \to (\mathbb{Z}^{2g})^*$ such that

$$\omega(v_1, v_2) = \mu_1(v_1)(\mu_2(v_2)) \quad \text{for all } v_1, v_2 \in H_{\mathbb{Z}}. \tag{9.1}$$

The map $\mu_1 \otimes \mu_2\colon H_{\mathbb{Z}}^{\otimes 2} \to (\mathbb{Z}^{2g})^* \otimes \mathbb{Z}^{2g}$ is then an $\text{Sp}_{2g}(\mathbb{Z})$-equivariant isomorphism.

Recall that we defined the vector space $\mathfrak{A}_{2g}$ in Definition 1.6. Using $\mu_1$ and $\mu_2$, we can define a map $M\colon \mathfrak{Z}_g \to \mathfrak{A}_{2g}$ via the formula $M((\!|v_1, v_2|\!)) = [\mu_1(v_1), \mu_2(v_2)]_0$. The identity (9.1) implies that $[\mu_1(v_1), \mu_2(v_2)]_0$ is a generator for $\mathfrak{A}_{2g}$. The map $M$ takes relations to relations, and thus gives a well-defined map. In fact, even more is true: $M$ is an isomorphism with inverse the map taking $[f,v]_0$ to $\llparenthesis \mu_1^{-1}(f), \mu_2^{-1}(v) \rrparenthesis$.

The vector space $\mathfrak{A}_{2g}$ is equipped with a linearization map $\Phi\colon \mathfrak{A}_{2g} \to (\mathbb{Q}^{2g})^* \otimes \mathbb{Q}^{2g}$ defined via the formula $\Phi([f,v]_0) = f \otimes v$. The image of this map is the kernel $\mathfrak{sl}_{2g}(\mathbb{Q})$ of the trace map

$$\text{tr}\colon (\mathbb{Q}^n)^* \otimes \mathbb{Q}^n \longrightarrow \mathbb{Q}$$

defined by $\text{tr}(f,v) = f(v)$. Theorem B says that $\Phi\colon \mathfrak{A}_{2g} \to \mathfrak{sl}_{2g}(\mathbb{Q})$ is an isomorphism. Abusing notation slightly, we will identify the isomorphism $\mu_1 \otimes \mu_2\colon H_{\mathbb{Z}}^{\otimes 2} \to (\mathbb{Z}^n)^* \otimes \mathbb{Z}^n$ with the corresponding isomorphism $\mu_1 \otimes \mu_2\colon H^{\otimes 2} \to (\mathbb{Q}^n)^* \otimes \mathbb{Q}^n$. By (9.1), this takes $\mathcal{Z}_g$ to $\mathfrak{sl}_{2g}(\mathbb{Q})$. This all fits into a commutative diagram

$$\begin{array}{ccc} \mathfrak{Z}_g & \xrightarrow[\cong]{M} & \mathfrak{A}_{2g} \\ \downarrow{\scriptstyle \Phi} & & {\scriptstyle \cong}\downarrow{\scriptstyle \Phi} \\ \mathcal{Z}_g & \xrightarrow[\cong]{\mu_1 \otimes \mu_2} & \mathfrak{sl}_{2g}(\mathbb{Q}). \end{array}$$

From this, we conclude that $\Phi\colon \mathfrak{Z}_g \to \mathcal{Z}_g$ is an isomorphism. □

9.3. **Consequences.** Let $\iota\colon H^{\otimes 2} \to H^{\otimes 2}$ be the involution $\iota(v_1 \otimes v_2) = v_2 \otimes v_1$. This involution induces a decomposition of $H^{\otimes 2}$ into into its $+1$ and $-1$ eigenspaces. This gives the familiar decomposition

$$H^{\otimes 2} = \text{Sym}^2(H) \oplus \wedge^2 H.$$

Recall that $\mathcal{Z}_g^a$ is the kernel of the map $\wedge^2 H \to \mathbb{Q}$ given by the symplectic form. The vector spaces $\mathcal{Z}_g$ and $\mathcal{Z}_g^a$ fit into the above decomposition as

$$\mathcal{Z}_g = \text{Sym}^2(H) \oplus \mathcal{Z}_g^a.$$

The involution $\iota$ lifts to the canonical involution $I\colon \mathfrak{Z}_g \to \mathfrak{Z}_g$ in the sense that diagram

$$\begin{array}{ccc} \mathfrak{Z}_g & \xrightarrow{I} & \mathfrak{Z}_g \\ {\scriptstyle \cong}\downarrow{\scriptstyle \Phi} & & {\scriptstyle \cong}\downarrow{\scriptstyle \Phi} \\ \mathcal{Z}_g & \xrightarrow{\iota} & \mathcal{Z}_g \end{array}$$

commute. This implies that the isomorphism $\Phi\colon \mathfrak{Z}_g \to \mathcal{Z}_g$ from Theorem 9.3 matches up the $+1$ and $-1$ eigenspaces of $I$ and $\iota$. For $\mathfrak{Z}_g$, these eigenspaces were identified by Lemma 9.2, and the following theorems from the introduction follow:

**Theorem E.** *For $g \geq 2$, the linearization map $\Phi\colon \mathfrak{Z}_g^s \to \text{Sym}^2(H)$ is an isomorphism.*

**Theorem D.** *For $g \geq 1$, the linearization map $\Phi\colon \mathfrak{Z}_g^a \to \mathcal{Z}_g^a$ is an isomorphism.*

*Remark* 9.4. One tiny issue with the above argument is that it only works for $g \geq 2$, while Theorem D is also supposed to hold for $g = 1$. However, for $g = 1$ this theorem is trivial since $\mathfrak{Z}_1^a = 0$ and $\mathcal{Z}_1^a = 0$. □

## Part 2. Improving the presentation for the symmetric kernel

We now turn to our theorems on the symmetric kernel. In this part of the paper, we enlarge its purported presentation by adding some additional generators. The key result needed to add these generators (Proposition 13.1 below) uses the proof technique from §3 that we have already used to prove Theorems A – E. See the introductory §10 for a more detailed discussion of what we will do. Our main theorems will be proved in Parts 3 and 4, again using the proof technique from §3.

To avoid having to constantly impose genus hypotheses, we make the following blanket assumption:

**Assumption 9.5.** Throughout Part 2, we assume that $g \geq 4$. □

## 10. Introduction to Part 2

Recall from §1.13 that $H = \mathbb{Q}^{2g}$ and $H_{\mathbb{Z}} = \mathbb{Z}^{2g}$ and $\omega\colon H \times H \to \mathbb{Q}$ is the standard symplectic form on $H$. We start by recalling some definitions and notation from the introduction and proving some preliminary results, and then we will outline the rest of this part.

10.1. **Quotient representation.** The symplectic form $\omega$ on $H$ identifies $H$ with its dual. Using this, we can identify alternating forms on $H$ with elements of $\wedge^2 H$. If $\{a_1, b_1, \ldots, a_g, b_g\}$ is a symplectic basis for $H$, then

$$\omega = a_1 \wedge b_1 + \cdots a_g \wedge b_g.$$

The $\mathbb{Q}$-span of $\omega$ in $\wedge^2 H$ is a copy of $\mathbb{Q}$. The quotient $(\wedge^2 H)/\mathbb{Q}$ will always mean the quotient by the $\mathbb{Q}$-span of $\omega$. Similarly, $(\wedge^2 H_{\mathbb{Z}})/\mathbb{Z}$ will always mean the quotient of $\wedge^2 H_{\mathbb{Z}}$ by the $\mathbb{Z}$-span of $\omega$.

10.2. **Symmetric contraction.** As we discussed in the introduction, the symmetric contraction is the bilinear map

$$\mathfrak{c}\colon ((\wedge^2 H)/\mathbb{Q}) \times ((\wedge^2 H)/\mathbb{Q}) \longrightarrow \mathrm{Sym}^2(H)$$

defined by the formula

$$\mathfrak{c}(x \wedge y, z \wedge w) = \omega(x,z) y{\cdot}w - \omega(x,w) y{\cdot}z - \omega(y,z) x{\cdot}w + \omega(y,w) x{\cdot}z \text{ for } x,y,z,w \in H.$$

The bilinear form $\mathfrak{c}$ is alternating:

$$\mathfrak{c}(\kappa_2, \kappa_1) = -\mathfrak{c}(\kappa_1, \kappa_2) \text{ for all } \kappa_1, \kappa_2 \in (\wedge^2 H)/\mathbb{Q}.$$

It induces a map

$$((\wedge^2 H)/\mathbb{Q})^{\otimes 2} \longrightarrow \mathrm{Sym}^2(H)$$

whose kernel $\mathcal{K}_g$ is the *symmetric kernel*. We say that $\kappa_1, \kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ are *sym-orthogonal* if

$$\mathfrak{c}(\kappa_1, \kappa_2) = -\mathfrak{c}(\kappa_2, \kappa_1) = 0,$$

or equivalently if $\kappa_1 \otimes \kappa_2$ and $\kappa_2 \otimes \kappa_1$ lie in $\mathcal{K}_g$. For $\kappa \in (\wedge^2 H)/\mathbb{Q}$, the *symmetric orthogonal complement* of $\kappa$, denoted $\kappa^{\perp}$, is the subspace of all $\kappa' \in (\wedge^2 H)/\mathbb{Q}$ that are sym-orthogonal to $\kappa$.

10.3. **Symplectic pairs.** A *symplectic pair* is an element of $(\wedge^2 H_{\mathbb{Z}})/\mathbb{Z}$ of the form $a \wedge b$, where $a, b \in H_{\mathbb{Z}}$ are such that $\omega(a,b) = 1$. Equivalently, there exists a symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $H_{\mathbb{Z}}$ with $a_1 = a$ and $b_1 = b$. For $X \subset \wedge^2 H$, let $\overline{X}$ be its image in $(\wedge^2 H)/\mathbb{Q}$. Also, for $V \subset H_{\mathbb{Z}}$ let $V_{\mathbb{Q}} = V \otimes \mathbb{Q} \subset H$. We have:

**Lemma 10.1.** *Let $a \wedge b$ be a symplectic pair and let $V = \langle a, b \rangle$. Then $(a \wedge b)^{\perp} = \overline{\wedge^2 V_{\mathbb{Q}}^{\perp}}$.*

*Proof.* Let $\{a_1, b_1, \ldots, a_g, b_g\}$ be a symplectic basis for $H$ with $a_1 = a$ and $b_1 = b$, so $V_{\mathbb{Q}}^{\perp} = \langle a_2, b_2, \ldots, a_g, b_g \rangle$. It is immediate from the formula for $\mathfrak{c}$ in §10.2 that $\mathfrak{c}(a_1 \wedge b_1, \kappa) = 0$ for $\kappa \in \overline{\wedge^2 V_{\mathbb{Q}}^{\perp}}$, so $\overline{\wedge^2 V_{\mathbb{Q}}^{\perp}} \subset (a_1 \wedge b_1)^{\perp}$. We must show the other inclusion. Via the decomposition

$$\wedge^2 H = \wedge^2 (V_{\mathbb{Q}} \oplus V_{\mathbb{Q}}^{\perp}) = \left(\wedge^2 V_{\mathbb{Q}}\right) \oplus \left(\wedge^2 V_{\mathbb{Q}}^{\perp}\right) \oplus \left(V_{\mathbb{Q}} \wedge V_{\mathbb{Q}}^{\perp}\right),$$

this is equivalent to showing that the intersection of $(a_1 \wedge b_1)^{\perp}$ and

$$\overline{(\wedge^2 V_{\mathbb{Q}}) \oplus \left(V_{\mathbb{Q}} \wedge V_{\mathbb{Q}}^{\perp}\right)} = \overline{\langle a_1 \wedge b_1 \rangle \oplus \left(V_{\mathbb{Q}} \wedge V_{\mathbb{Q}}^{\perp}\right)} \tag{10.1}$$

is contained in $\overline{\wedge^2 V_{\mathbb{Q}}^{\perp}}$.

Note that (10.1) is spanned by $a_1 \wedge b_1$ and $a_1 \wedge z$ and $b_1 \wedge z$ as $z$ ranges over $\{a_2, b_2, \ldots, a_g, b_g\}$. For such $z$, we have

$$\begin{aligned} \mathfrak{c}(a_1 \wedge b_1, a_1 \wedge z) &= -a_1 \cdot z, \\ \mathfrak{c}(a_1 \wedge b_1, b_1 \wedge z) &= b_1 \cdot z, \\ \mathfrak{c}(a_1 \wedge b_1, a_1 \wedge b_1) &= -b_1 \cdot a_1 + a_1 \cdot b_1 = 0. \end{aligned}$$

Other than 0, the elements of $\mathrm{Sym}^2(H)$ appearing on the right hand side of this equation as $z$ ranges over $\{a_2, b_2, \ldots, a_g, b_g\}$ are linearly independent. It follows that the intersection of $(a_1 \wedge b_1)^{\perp}$ with (10.1) is spanned by

$$a_1 \wedge b_1 = -(a_2 \wedge b_2 + \cdots + a_g \wedge b_g) \in \overline{\wedge^2 V_{\mathbb{Q}}^{\perp}},$$

as desired. Note that here we are using the fact that we are working in $(\wedge^2 H)/\mathbb{Q}$, so $\omega \in \wedge^2 H$ equals 0. □

10.4. **Isotropic pairs.** An *isotropic pair* is an element of $(\wedge^2 H_{\mathbb{Z}})/\mathbb{Z}$ of the form $a \wedge a'$, where $a, a' \in H_{\mathbb{Z}}$ are linearly independent elements such that $\omega(a, a') = 0$. The following is the analogue for isotropic pairs of Lemma 10.1:

**Lemma 10.2.** *Let $a \wedge a'$ be an isotropic pair and let $I = \langle a, a' \rangle$. Then $(a \wedge a')^{\perp} = \overline{\wedge^2 I_{\mathbb{Q}}^{\perp}}$.*

*Proof.* We can find a symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $H$ with $a_1 = a$ and $a_2 = a'$, so $I_{\mathbb{Q}}^{\perp} = \langle a_1, a_2, a_3, b_3, \ldots, a_g, b_g \rangle$. Note that we might not be able to find such a basis for $H_{\mathbb{Z}}$ since $\{a, a'\}$ might not span a direct summand of $H_{\mathbb{Z}}$ (see §10.5 below). It is immediate from the formula for $\mathfrak{c}$ in §10.2 that $\mathfrak{c}(a_1 \wedge a_2, \kappa) = 0$ for $\kappa \in \overline{\wedge^2 I_{\mathbb{Q}}^{\perp}}$, so $\overline{\wedge^2 I_{\mathbb{Q}}^{\perp}} \subset (a_1 \wedge a_2)^{\perp}$. We must show the other inclusion. Via the decomposition

$$\wedge^2 H = \wedge^2 (I_{\mathbb{Q}}^{\perp} \oplus \langle b_1, b_2 \rangle) = \left(\wedge^2 I_{\mathbb{Q}}^{\perp}\right) \oplus \left(b_1 \wedge I_{\mathbb{Q}}^{\perp}\right) \oplus \left(b_2 \wedge I_{\mathbb{Q}}^{\perp}\right) \oplus \langle b_1 \wedge b_2 \rangle,$$

this is equivalent to showing that the intersection of $(a_1 \wedge a_2)^{\perp}$ and

$$\overline{\left(b_1 \wedge I_{\mathbb{Q}}^{\perp}\right) \oplus \left(b_2 \wedge I_{\mathbb{Q}}^{\perp}\right) \oplus \langle b_1 \wedge b_2 \rangle}. \tag{10.2}$$

is contained in $\overline{\wedge^2 I_{\mathbb{Q}}^{\perp}}$. For $z \in \{a_1, a_2, a_3, b_3, \ldots, a_g, b_g\}$, we have

$$\begin{aligned} \mathfrak{c}(a_1 \wedge a_2, b_1 \wedge z) &= a_2 \cdot z, \\ \mathfrak{c}(a_1 \wedge a_2, b_2 \wedge z) &= -a_1 \cdot z, \\ \mathfrak{c}(a_1 \wedge a_2, b_1 \wedge b_2) &= a_2 \cdot b_2 + a_1 \cdot b_1. \end{aligned}$$

The only linear dependence among the elements of $\mathrm{Sym}^2(H)$ appearing on the right hand side of this equation as $z$ ranges over $\{a_1, a_2, a_3, b_3, \ldots, a_g, b_g\}$ is

$$\mathfrak{c}(a_1 \wedge a_2, b_1 \wedge a_1) + \mathfrak{c}(a_1 \wedge a_2, b_2 \wedge a_2) = a_2 \cdot a_1 - a_1 \cdot a_2 = 0.$$

It follows that the intersection of $(a_1 \wedge a_2)^{\perp}$ with (10.2) is spanned by

$$b_1 \wedge a_1 + b_2 \wedge a_2 = -(a_1 \wedge b_1 + a_2 \wedge b_2) = a_3 \wedge b_3 + \cdots + a_g \wedge b_g \in \overline{\wedge^2 I_{\mathbb{Q}}^{\perp}},$$

as desired. □

10.5. **Strong isotropic pairs.** A *strong isotropic pair* is an isotropic pair $a \wedge a'$ such that $\{a, a'\}$ forms a basis for a rank-2 direct summand of $H_{\mathbb{Z}}$. Equivalently, there exists a symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $H_{\mathbb{Z}}$ with $a_1 = a$ and $a_2 = a'$. We will prove that every isotropic pair is a multiple of a strong isotropic pair. This requires the following lemma:

**Lemma 10.3.** *Let $V$ be a subspace of $\mathbb{Q}^n$. Then $V_{\mathbb{Z}} = V \cap \mathbb{Z}^n$ is a direct summand of $\mathbb{Z}^n$.*

*Proof.* The short exact sequence

$$0 \longrightarrow V \longrightarrow \mathbb{Q}^n \xrightarrow{\pi} \mathbb{Q}^n/V \longrightarrow 0$$

restricts to a short exact sequence

$$0 \longrightarrow V_{\mathbb{Z}} \longrightarrow \mathbb{Z}^n \xrightarrow{\pi} \pi(\mathbb{Z}^n) \longrightarrow 0. \qquad (10.3)$$

The subgroup $\pi(\mathbb{Z}^n)$ of $\mathbb{Q}^n/V \cong \mathbb{Q}^{n-\dim(V)}$ is finitely generated and torsion-free, and hence free abelian. The short exact sequence (10.3) thus splits, so $V_{\mathbb{Z}}$ is a direct summand of $\mathbb{Z}^n$. □

For a subspace $V$ of $H_{\mathbb{Z}}$, the *saturation* of $V$ in $H_{\mathbb{Z}}$ is $V_{\mathbb{Q}} \cap H_{\mathbb{Z}}$. By Lemma 10.3, this is a direct summand of $H_{\mathbb{Z}}$. We have:

**Lemma 10.4.** *Let $a \wedge a'$ be an isotropic pair. Then there exists a strong isotropic pair $a_0 \wedge a_0'$ and $n \in \mathbb{Z}$ such that $a \wedge a' = n a_0 \wedge a_0'$. Moreover, $\langle a_0, a_0' \rangle$ is the saturation in $H_{\mathbb{Z}}$ of $\langle a, a' \rangle$.*

*Proof.* Set $I = \langle a, a' \rangle$ and let $\overline{I}$ be the saturation of $I$ in $H_{\mathbb{Z}}$. Let $\{a_0, a_0'\}$ be a basis for $\overline{I}$. Regarding $a \wedge a'$ and $a_0 \wedge a_0'$ as elements of $\wedge^2 H$, they correspond to the same 2-dimensional subspace of $H$, namely $I_{\mathbb{Q}} = \overline{I}_{\mathbb{Q}}$. It follows that there exists some $n \in \mathbb{Q}$ such that $a \wedge a' = n a_0 \wedge a_0'$. Since $a \wedge a' \in \wedge^2 H_{\mathbb{Z}}$ and $a_0 \wedge a_0'$ is a primitive element of $\wedge^2 H_{\mathbb{Z}}$, we must have $n \in \mathbb{Z}$, as desired. □

10.6. **Symmetric kernel presentation.** We defined $\mathfrak{Z}_g^s$ and $\mathfrak{Z}_g^a$ in Definitions 1.17 and 1.15. Just like we did for $\mathfrak{Z}_g^s$ and $\mathfrak{Z}_g^a$ in §9.1, we now define a version of them that does not include their symmetric/anti-symmetric relations:

**Definition 10.5.** Define $\mathfrak{K}_g$ to be the $\mathbb{Q}$-vector space with the following presentation:

- **Generators**. A generator $[\![\kappa_1, \kappa_2]\!]$ for all sym-orthogonal $\kappa_1, \kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ such that either $\kappa_1$ or $\kappa_2$ (or both) is a symplectic pair in $(\wedge^2 H_{\mathbb{Z}})/\mathbb{Z}$.
- **Relations**. For all symplectic pairs $a \wedge b \in (\wedge^2 H_{\mathbb{Z}})/\mathbb{Z}$ and all $\kappa_1, \kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ that are sym-orthogonal to $a \wedge b$ and all $\lambda_1, \lambda_2 \in \mathbb{Q}$, the relations

$$[\![a \wedge b, \lambda_1\kappa_1 + \lambda_2\kappa_2]\!] = \lambda_1[\![a \wedge b, \kappa_1]\!] + \lambda_2[\![a \wedge b, \kappa_2]\!] \quad \text{and}$$
$$[\![\lambda_1\kappa_1 + \lambda_2\kappa_2, a \wedge b]\!] = \lambda_1[\![\kappa_1, a \wedge b]\!] + \lambda_2[\![\kappa_2, a \wedge b]\!].$$ □

There is an involution $I\colon \mathfrak{K}_g \to \mathfrak{K}_g$ defined by $I([\![\kappa_1, \kappa_2]\!]) = [\![\kappa_2, \kappa_1]\!]$ that we will call the *canonical involution*. We have:

**Lemma 10.6.** *We have $\mathfrak{K}_g = \mathfrak{K}_g^s \oplus \mathfrak{K}_g^a$, where $\mathfrak{K}_g^s$ and $\mathfrak{K}_g^a$ are identified with the $+1$ and $-1$ eigenspaces of the canonical involution.*

*Proof.* Identical to the proof of Lemma 9.2. □

There is a linearization map $\Phi\colon \mathfrak{K}_g \to ((\wedge^2 H)/\mathbb{Q})^{\otimes 2}$ defined by $\Phi([\![\kappa_1, \kappa_2]\!]) = \kappa_1 \otimes \kappa_2$. This takes relations to relations, and thus gives a well-defined map. Since in the generator $[\![\kappa_1, \kappa_2]\!]$ the elements $\kappa_1$ and $\kappa_2$ are sym-orthogonal, the image of $\Phi$ lies in $\mathcal{K}_g$. In light of Lemma 10.6, Theorems F and G are equivalent to:

**Theorem 10.7.** *For $g \geq 4$, the linearization map $\Phi\colon \mathfrak{K}_g \to \mathcal{K}_g$ is an isomorphism.*

10.7. **Goal of Part 2.** Our goal in the rest of this paper is to prove Theorem 10.7. Actually, it will turn out that it is more convenient to prove Theorems F and G separately. We introduced the representation $\mathfrak{K}_g$ from Theorem 10.7 for the sake of the calculations in this part of the paper, which later will give results about $\mathfrak{K}_g^a$ and $\mathfrak{K}_g^s$ that will be needed for the proofs of Theorems F and G.

Our goal in this part is to enhance $\mathfrak{K}_g$ by showing that in the above presentation we can add generators $[\![\kappa_1, \kappa_2]\!]$ such that $\kappa_1, \kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ are sym-orthogonal elements with either $\kappa_1$ or $\kappa_2$ (or both) a symplectic pair or an isotropic pair.[11] Lemma 10.6 will then imply a corresponding result about $\mathfrak{K}_g^s$ and $\mathfrak{K}_g^a$. We accomplish this in §15. This is preceded by a series of preliminary results in §11 – §14.

## 11. Isotropic pairs I: setup

This section contains the basic framework for constructing our new generators.

11.1. **Generation by symplectic pairs.** We start with a technical lemma. Let $X$ be a direct summand of $H_{\mathbb{Z}}$. Define $\ker(X)$ to be the subspace of all $x_0 \in X$ such that $\omega(x_0, x) = 0$ for all $x \in X$. The rank of $\ker(X)$ is the *kernel rank* of $X$.

The restriction of $\omega$ to $X$ induces an alternating bilinear form $\iota$ on $X/\ker(X)$, and we say that $X$ is a *near symplectic summand* of $H_{\mathbb{Z}}$ if $\iota$ is a symplectic form. This is equivalent to requiring that there be a symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $H_{\mathbb{Z}}$ such that $X = \langle a_1, b_1, \ldots, a_h, b_h, a_{h+1}, \ldots, a_{h+k}\rangle$ for some $h \leq g$ and $k \leq g - h$. The integer $k$ is the kernel rank of $X$, and we call $h$ the *genus* of $X$. Here is an example of this:

**Lemma 11.1.** *Let $I$ be a rank-$k$ subgroup of $H_{\mathbb{Z}}$ on which $\omega$ vanishes identically. Then $I^{\perp}$ is a near symplectic summand of genus $g - k$ and kernel rank $k$.*

*Proof.* Let $\overline{I}$ be the saturation of $I$ in $H_{\mathbb{Z}}$. Since $I_{\mathbb{Q}} = \overline{I}_{\mathbb{Q}}$ we have that $I^{\perp} = \overline{I}^{\perp}$. Lemma 10.3 implies that $\overline{I}$ is a direct summand of $H_{\mathbb{Z}}$. We can therefore find a symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $H_{\mathbb{Z}}$ such that $\overline{I} = \langle a_1, a_2, \ldots, a_k\rangle$. It follows that

$$I^{\perp} = \overline{I}^{\perp} = \langle a_1, a_2, \ldots, a_k, a_{k+1}, b_{k+1}, \ldots, a_g, b_g\rangle.$$

The lemma follows. □

Our main result about near symplectic summands is:

**Lemma 11.2.** *Let $X$ be a near symplectic summand of $H_{\mathbb{Z}}$ of genus $h \geq 1$. Then $\overline{\wedge^2 X_{\mathbb{Q}}}$ is spanned by symplectic pairs $\sigma$ with $\sigma \in \overline{\wedge^2 X_{\mathbb{Q}}}$.*

*Proof.* Let $k$ be the kernel rank of $X$ and let $\{a_1, b_1, \ldots, a_g, b_g\}$ be a symplectic basis for $H_{\mathbb{Z}}$ such that $X = \langle a_1, b_1, \ldots, a_h, b_h, a_{h+1}, \ldots, a_{h+k}\rangle$. The vector space $\overline{\wedge^2 X_{\mathbb{Q}}}$ is spanned by elements of the form $x \wedge y$ with $x, y \in \{a_1, b_1, \ldots, a_h, b_h, a_{h+1}, \ldots, a_{h+k}\}$ distinct. We must write each of these as a linear combination of symplectic pairs $\sigma$ with $\sigma \in \overline{\wedge^2 X_{\mathbb{Q}}}$.

Up to flipping $x$ and $y$, there are several cases. In each of them, we will use blue to denote symplectic pairs $\sigma$ with $\sigma \in \overline{\wedge^2 X_{\mathbb{Q}}}$.

- If $\omega(x, y) = 1$, then we have $x = a_i$ and $y = b_i$ for some $1 \leq i \leq h$ and $x \wedge y$ is already of the desired form.
- If $\omega(x, y) = 0$ and $x \in \{a_1, b_1, \ldots, a_h, b_h\}$ and $y \in \{a_1, b_1, \ldots, a_h, b_h, a_{h+1}, \ldots, a_{h+k}\}$, then for $1 \leq i \leq h$ we have either:

$$x \wedge y = a_i \wedge y = a_i \wedge (b_i + y) - a_i \wedge b_i, \text{ or}$$
$$x \wedge y = b_i \wedge y = -(a_i + y) \wedge b_i + a_i \wedge b_i.$$

[11] We will actually prove something slightly more general.

- If $x, y \in \{a_{h+1}, \ldots, a_{h+k}\}$, then we have
$$x \wedge y = (a_1 + x) \wedge (b_1 + y) - a_1 \wedge (b_1 + y) - (a_1 + x) \wedge b_1 + a_1 \wedge b_1. \qquad \square$$

11.2. **Right compatible subspaces.** Let $a \wedge a'$ be an isotropic pair. Define $\mathfrak{K}_g[-, a \wedge a']$ to be the subspace of $\mathfrak{K}_g$ spanned by elements of the form $[\![\sigma, a \wedge a']\!]$ with $\sigma$ a symplectic pair that is sym-orthogonal to $a \wedge a'$. Let $\Phi\colon \mathfrak{K}_g \to ((\wedge^2 H)/\mathbb{Q})^{\otimes 2}$ be the linearization map. Our main technical result will be that $\Phi$ takes $\mathfrak{K}_g[-, a \wedge a']$ isomorphically onto $(a \wedge a')^{\perp} \otimes (a \wedge a')$. The proof of this is spread over §11 – §13, with the result being Proposition 13.1. In §14, we use this to construct our new generators.

*Remark* 11.3. We could also define $\mathfrak{K}_g[a \wedge a', -]$ similarly, and all of our results would have analogues for $\mathfrak{K}_g[a \wedge a', -]$. To avoid repetition, we will focus on $\mathfrak{K}_g[-, a \wedge a']$ and then at the very end formally derive these analogues; see §14.2. $\square$

11.3. **Calculating the image.** We start by proving:

**Lemma 11.4.** *Let $a \wedge a'$ be an isotropic pair and let $\Phi\colon \mathfrak{K}_g \to ((\wedge^2 H)/\mathbb{Q})^{\otimes 2}$ be the linearization map. Then $\Phi$ takes $\mathfrak{K}_g[-, a \wedge a']$ onto $(a \wedge a')^{\perp} \otimes (a \wedge a')$.*

*Proof.* By definition, $\mathfrak{K}_g[-, a \wedge a']$ is spanned by elements of the form $[\![\sigma, a \wedge a']\!]$ with $\sigma$ a symplectic pair such that $\sigma \in (a \wedge a')^{\perp}$. Since $\Phi([\![\sigma, a \wedge a']\!]) = \sigma \otimes (a \wedge a')$, this implies that
$$\Phi(\mathfrak{K}_g[-, a \wedge a']) \subset (a \wedge a')^{\perp} \otimes (a \wedge a').$$
To see that this is an equality, let $I = \langle a, a' \rangle$. Lemma 10.2 says that $(a \wedge a')^{\perp} = \overline{\wedge^2 I^{\perp}_{\mathbb{Q}}}$. Lemma 11.1 says that $I^{\perp}$ is a near symplectic summand of genus[12] $g - 2 \geq 1$. Lemma 11.2 therefore implies that $\overline{\wedge^2 I^{\perp}_{\mathbb{Q}}}$ is spanned by symplectic pairs $\sigma$ such that $\sigma \in \overline{\wedge^2 I^{\perp}_{\mathbb{Q}}}$. The desired equality follows. $\square$

## 12. Isotropic pairs II: lifting orthogonal elements

Let $a \wedge a'$ be an isotropic pair and let $\kappa \in (a \wedge a')^{\perp}$. In this section, for certain $\kappa$ we show how to find specific elements of $\mathfrak{K}_g[-, a \wedge a']$ projecting to $\kappa \otimes (a \wedge a')$.

12.1. **Separating classes.** A subgroup $X$ of $H_{\mathbb{Z}}$ is said to *separate* $\kappa$ from $a \wedge a'$ if:

- $X \subset \langle a, a' \rangle^{\perp}$; and
- $\kappa \in \overline{\wedge^2 X_{\mathbb{Q}}}$; and
- $X$ is a near symplectic summand of $H_{\mathbb{Z}}$ of positive genus. This implies in particular that $X$ is a direct summand of $H_{\mathbb{Z}}$.

Let $X$ be a direct summand of $H_{\mathbb{Z}}$ separating $\kappa$ from $a \wedge a'$. Use Lemma 11.2 to write
$$\kappa = \sum_{i=1}^{n} \lambda_i \sigma_i \quad \text{with } \lambda_i \in \mathbb{Q} \text{ and } \sigma_i \text{ a symplectic pair with } \sigma_i \in \overline{\wedge^2 X_{\mathbb{Q}}}. \tag{12.1}$$

12.2. **Constructing the lift.** We would like to define
$$[\![(\kappa; X), a \wedge a']\!] = \sum_{i=1}^{n} \lambda_i [\![\sigma_i, a \wedge a']\!] \in \mathfrak{K}_g.$$
This is in orange to emphasize that it is not one of our generators. It appears to depend on the expression (12.1), but below we will prove that under favorable circumstances it does not depend on this expression.

[12] Here we are using our standing assumption that $g \geq 4$; see Assumption 9.5.

To state our result, recall that a *Lagrangian* in $H_{\mathbb{Z}}$ is a direct summand $L$ with $L^{\perp} = L$. Equivalently, we can find a symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $H_{\mathbb{Z}}$ with $L = \langle a_1, \ldots, a_g \rangle$. We say that $X$ is *Lagrangian-free* if $X$ does not contain a Lagrangian of $H_{\mathbb{Z}}$. Then:

**Lemma 12.1.** *Let the notation be as above, and assume that $X$ is Lagrangian-free. Then $[\![(\kappa; X), a \wedge a']\!]$ does not depend on (12.1).*

*Proof.* Let

$$\kappa = \sum_{j=1}^{m} \lambda'_j \sigma'_j \quad \text{with } \lambda'_j \in \mathbb{Q} \text{ and } \sigma'_j \text{ a symplectic pair with } \sigma'_j \in \overline{\wedge^2 X_{\mathbb{Q}}}$$

be another expression. We must prove that

$$\sum_{i=1}^{n} \lambda_i [\![\sigma_i, a \wedge a']\!] = \sum_{j=1}^{m} \lambda'_j [\![\sigma'_j, a \wedge a']\!]. \tag{12.2}$$

Let $h \geq 1$ be the genus of $X$ and let $k$ be the kernel rank of $X$. Pick a symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $H_{\mathbb{Z}}$ with $X = \langle a_1, b_1, \ldots, a_h, b_h, a_{h+1}, \ldots, a_{h+k} \rangle$. We then have

$$X^{\perp} = \langle a_{h+1}, \ldots, a_{h+k}, a_{h+k+1}, b_{h+k+1}, \ldots, a_g, b_g \rangle.$$

Since $X$ is Lagrangian-free, we have $h + k < g$. It follows that $X^{\perp}$ is a near symplectic summand of $H_{\mathbb{Z}}$ of positive genus. By assumption, $a, a' \in X^{\perp}$. Using Lemma 11.2, we can write

$$a \wedge a' = \sum_{\ell=1}^{p} c_\ell s_\ell \quad \text{with } c_\ell \in \mathbb{Q} \text{ and } s_\ell \text{ a symplectic pair with } s_\ell \in \overline{\wedge^2 X_{\mathbb{Q}}^{\perp}}.$$

We have the following relation in $\mathfrak{K}_g$:

$$[\![\sigma_i, a \wedge a']\!] = \sum_{\ell=1}^{p} c_\ell [\![\sigma_i, s_\ell]\!].$$

Using the relations in $\mathfrak{K}_h$ again, it follows that

$$\sum_{i=1}^{n} \lambda_i [\![\sigma_i, a \wedge a']\!] = \sum_{\ell=1}^{p} \left( c_\ell \sum_{i=1}^{n} \lambda_i [\![\sigma_i, s_\ell]\!] \right) = \sum_{\ell=1}^{p} \left( c_\ell [\![\sum_{i=1}^{n} \lambda_i \sigma_i, s_\ell]\!] \right) = \sum_{\ell=1}^{p} c_\ell [\![\kappa, s_\ell]\!].$$

Similarly, we have

$$\sum_{j=1}^{m} \lambda'_j [\![\sigma'_j, a \wedge a']\!] = \sum_{\ell=1}^{p} c_\ell [\![\kappa, s_\ell]\!].$$

The equality (12.2) follows. □

12.3. **Properties of the lift.** We now give three properties of our lifts. The first is linearity:

**Lemma 12.2** (Linearity of the lifts)**.** *Let $a \wedge a'$ be an isotropic pair, let $\kappa_1, \kappa_2 \in (a \wedge a')^{\perp}$, and let $\lambda_1, \lambda_2 \in \mathbb{Q}$. Let $X$ be a direct summand of $H_{\mathbb{Z}}$ that is Lagrangian-free and separates both $\kappa_1$ and $\kappa_2$ from $a \wedge a'$. Then*

$$[\![(\lambda_1 \kappa_1 + \lambda_2 \kappa_2; X), a \wedge a']\!] = \lambda_1 [\![(\kappa_1; X), a \wedge a']\!] + \lambda_2 [\![(\kappa_2; X), a \wedge a']\!].$$

*Proof.* By taking the corresponding linear combination of the expressions in $\mathfrak{K}_g[-, a \wedge a']$ that we used to write $[\![(\kappa_1; X); a \wedge a']\!]$ and $[\![(\kappa_2; X), a \wedge a']\!]$, we obtain an expression that can be used to write $[\![(\lambda_1 \kappa_1 + \lambda_2 \kappa_2; X), a \wedge a']\!]$. The lemma follows. □

The second is equivariance. For a strong isotropic pair $a \wedge a'$ and $f \in \mathrm{Sp}_{2g}(\mathbb{Z})$, note that $f(a) \wedge f(a')$ is another strong isotropic pair. The group $\mathrm{Sp}_{2g}(\mathbb{Z})$ also acts on $\mathfrak{K}_g$, and $f$ takes $\mathfrak{K}_g[-, a \wedge a']$ to $\mathfrak{K}_g[-, f(a) \wedge f(a')]$. We have:

**Lemma 12.3** (Equivariance of the lifts)**.** *Let $a \wedge a'$ be an isotropic pair and let $\kappa \in (a \wedge a')^{\perp}$. Let $X$ be a direct summand of $H_{\mathbb{Z}}$ that is Lagrangian-free and separates $\kappa$ from $a \wedge a'$. Then for all $f \in \mathrm{Sp}_{2g}(\mathbb{Z})$ we have $f([\![(\kappa; X), a \wedge a']\!]) = [\![(f(\kappa); f(X)), f(a) \wedge f(a')]\!]$.*

*Proof.* The map $f$ takes the expression in $\mathfrak{K}_g[-, a \wedge a']$ we used to write $[\![(\kappa; X), a \wedge a']\!]$ to one that can be used to write $[\![(f(\kappa); f(X)), f(a) \wedge f(a')]\!]$. The lemma follows. □

Our final lemma lets us change $X$:

**Lemma 12.4** (Changing the separator in the lifts)**.** *Let $a \wedge a'$ be an isotropic pair and let $\kappa \in (a \wedge a')^{\perp}$. Let $X$ and $X'$ be direct summands of $H_{\mathbb{Z}}$ that are Lagrangian-free and separate $\kappa$ from $a \wedge a'$. Assume that $X \subset X'$. Then $[\![(\kappa; X), a \wedge a']\!] = [\![(\kappa; X'), a \wedge a']\!]$.*

*Proof.* Since $X \subset X'$, the expression in $\mathfrak{K}_g[-, a \wedge a']$ we used to write $[\![(\kappa; X), a \wedge a']\!]$ can also be used to write $[\![(\kappa; X'), a \wedge a']\!]$. The lemma follows. □

12.4. **Symplectic automorphism group.** We pause now to prove a lemma about the symplectic group. Let $I$ be a rank-$k$ direct summand of $H_{\mathbb{Z}}$ on which $\omega$ vanishes identically. Let $\mathrm{Sp}_{2g}(\mathbb{Z}, I)$ be the subgroup of all $f \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that $f$ fixes $I$ pointwise. The group $\mathrm{Sp}_{2g}(\mathbb{Z}, I)$ acts on $I^{\perp}$. Let $\mathrm{Sp}_{2g}(\mathbb{Z}, I)|_{I^{\perp}}$ be the image of $\mathrm{Sp}_{2g}(\mathbb{Z}, I)$ in $\mathrm{Aut}(I^{\perp})$. Lemma 11.1 says that $I^{\perp}$ is a near symplectic summand of $H_{\mathbb{Z}}$ of genus $g-k$, so we can find a symplectic summand $X$ of $H_{\mathbb{Z}}$ of genus $g-k$ such that $I^{\perp} = X \oplus I$. We have:

**Lemma 12.5.** *Let $X$ and $I$ be as above. We then have a semidirect product decomposition*

$$\mathrm{Sp}_{2g}(\mathbb{Z}, I)|_{I^{\perp}} = \mathrm{Hom}(X, I) \ltimes \mathrm{Sp}(X),$$

*where for $\lambda \in \mathrm{Hom}(X, I)$ the associated element $f \in \mathrm{Sp}_{2g}(\mathbb{Z}, I)|_{I^{\perp}}$ satisfies $f(x) = x + \lambda(x)$ for all $x \in X$.*

*Proof.* Set $\Gamma = \mathrm{Sp}_{2g}(\mathbb{Z}, I)|_{I^{\perp}}$. The action of $\Gamma$ on $I^{\perp}$ descends to an action on $I^{\perp}/I$. The symplectic form on $H_{\mathbb{Z}}$ induces a symplectic form on $I^{\perp}/I$. We thus get a homomorphism

$$\rho \colon \Gamma \longrightarrow \mathrm{Sp}(I^{\perp}/I).$$

The map $I^{\perp} \to I^{\perp}/I$ restricts to an isomorphism $X \cong I^{\perp}/I$. Identifying $\mathrm{Sp}(X)$ with the subgroup of $\Gamma$ consisting of automorphisms that act trivially on $X^{\perp}$, the homomorphism $\rho$ splits via the map

$$\mathrm{Sp}(I^{\perp}/I) \cong \mathrm{Sp}(X) \longleftrightarrow \Gamma.$$

We therefore get a semidirect product decomposition

$$\Gamma = \ker(\rho) \ltimes \mathrm{Sp}(X).$$

To identify $\ker(\rho)$ with $\mathrm{Hom}(X, I)$, consider $f \in \ker(\rho)$. By definition, for $x \in X$ we have $f(x) - x \in I$. We can therefore define a homomorphism $\lambda_f \colon X \to I$ via the formula $\lambda_f(x) = f(x) - x$. If $\lambda_f = 0$, then $f$ fixes both $X$ and $I$, so it fixes $I^{\perp} = X \oplus I$ and is the identity.

The map $f \mapsto \lambda_f$ is thus an injective homomorphism from $\ker(\rho)$ to $\mathrm{Hom}(X, I)$. We remark that the fact that is a homomorphism uses the fact that $f(y) = y$ for all $y \in I$. To see that it is a surjection and thus an isomorphism, consider $\lambda \in \mathrm{Hom}(X, I)$. Let $\{a_1, b_1, \ldots, a_k, b_k\}$ be a symplectic basis for $X^{\perp}$ such that $I = \langle a_1, a_2, \ldots, a_k \rangle$. Set $J = \langle b_1, b_2, \ldots, b_k \rangle$. Since $\omega$

restricts to a symplectic form on $X$, it identifies $X$ with $\text{Hom}(X, \mathbb{Z})$. There is thus a unique homomorphism $\delta\colon J \to X$ such that

$$\omega(x, \delta(z)) = -\omega(\lambda(x), z) \quad \text{for all } x \in X \text{ and } z \in J.$$

Since $X$ and $J$ are orthogonal to each other, for $x \in X$ and $z \in J$ we have

$$\omega(x + \lambda(x), z + \delta(z)) = \omega(\lambda(x), z) + \omega(x, \delta(z)) = 0.$$

In other words, the map $f\colon H_{\mathbb{Z}} \to H_{\mathbb{Z}}$ defined by $f(x) = x + \lambda(x)$ for $x \in X$ and

$$f(a_i) = a_i \quad \text{and} \quad f(b_i) = b_i + \delta(b_i)$$

for $1 \leq i \leq k$ is an element of $\text{Sp}_{2g}(\mathbb{Z}, I)$ whose restriction to $I^{\perp}$ satisfies $\lambda = \lambda_f$. The lemma follows. □

12.5. **Fixed lift.** Let $a \wedge a'$ be a strong isotropic pair, so $I = \langle a, a' \rangle$ is a direct summand of $H_{\mathbb{Z}}$. There is a special element $(a \wedge a') \otimes (a \wedge a')$ in $(a \wedge a')^{\perp} \otimes (a \wedge a')$ that is fixed by $\text{Sp}_{2g}(\mathbb{Z}, I)$. We close this section by showing that we can lift this to an element of $\mathfrak{K}_g[-, a \wedge a']$ that is fixed by $\text{Sp}_{2g}(\mathbb{Z}, I)$. To state our result, let $\{a_1, b_1, \ldots, a_g, b_g\}$ be a symplectic basis for $H_{\mathbb{Z}}$ with[13] $a_{g-1} = a$ and $a_g = a'$. For $1 \leq i \leq g-2$, let $W_i = \langle a_i, b_i, a_{g-1}, a_g \rangle$. We then have:

**Lemma 12.6.** *Let $a_{g-1}$ and $a_g$ and $W_i$ be as above. The following then hold:*

- *For $1 \leq i, j \leq g-2$ we have*
$$[\![(a_{g-1} \wedge a_g; W_i), a_{g-1} \wedge a_g]\!] = [\![(a_{g-1} \wedge a_g; W_j), a_{g-1} \wedge a_g]\!].$$
- *For $1 \leq i \leq g-2$, the group $\text{Sp}_{2g}(\mathbb{Z}, I)$ fixes $[\![(a_{g-1} \wedge a_g; W_i), a_{g-1} \wedge a_g]\!]$.*

*Proof.* For $\kappa \in (a \wedge a')^{\perp}$ and a summand $X$ of $H_{\mathbb{Z}}$ that is Lagrangian-free and separates $\kappa$ from $a_{g-1} \wedge a_g$, we will drop the $a_{g-1} \wedge a_g$ from our notation and write $[\![\kappa; X]\!]$ instead of $[\![(\kappa; X), a_{g-1} \wedge a_g]\!]$. We encourage the reader to verify that all the summands appearing in our calculations are Lagrangian-free and separate the appropriate elements of $(a \wedge a')^{\perp}$ from $a_{g-1} \wedge a_g$. In particular, they all have positive genus. We have:

**Claim.** *For $1 \leq i, j \leq g-2$, we have $[\![a_{g-1} \wedge a_g; W_i]\!] = [\![a_{g-1} \wedge a_g; W_j]\!]$.*

Using Lemma 12.2 (linearity of the lifts), we have

$$[\![a_{g-1} \wedge a_g; W_i]\!] = [\![(a_i + a_{g-1}) \wedge a_g; W_i]\!] - [\![a_i \wedge a_g; W_i]\!],$$
$$[\![a_{g-1} \wedge a_g; W_j]\!] = [\![(a_j + a_{g-1}) \wedge a_g; W_j]\!] - [\![a_j \wedge a_g; W_j]\!].$$

To prove that these are equal, we must show that

$$[\![(a_i + a_{g-1}) \wedge a_g; W_i]\!] + [\![a_j \wedge a_g; W_j]\!] = [\![(a_j + a_{g-1}) \wedge a_g; W_j]\!] + [\![a_i \wedge a_g; W_i]\!]. \tag{12.3}$$

Using Lemma 12.4 (changing the separator in the lifts), we have

$$\begin{aligned}
[\![(a_i + a_{g-1}) \wedge a_g; W_i]\!] &= [\![(a_i + a_{g-1}) \wedge a_g; \langle a_i + a_{g-1}, b_i, a_g \rangle]\!] \\
&= [\![(a_i + a_{g-1}) \wedge a_g; \langle a_i + a_{g-1}, b_i, a_j, b_j, a_g \rangle]\!], \\
[\![a_j \wedge a_g; W_j]\!] &= [\![a_j \wedge a_g; \langle a_j, b_j, a_g \rangle]\!] \\
&= [\![a_j \wedge a_g; \langle a_i + a_{g-1}, b_i, a_j, b_j, a_g \rangle]\!].
\end{aligned}$$

[13]Indexing it like this rather than $a_1 = a$ and $a_2 = a'$ will simplify our notation later.

Adding these and using Lemmas 12.2 (linearity of the lifts) and 12.4 (changing the separator in the lifts), we see that[14]

$$\begin{aligned}[\![(a_i+a_{g-1})\wedge a_g;W_i]\!]+[\![a_j\wedge a_g;W_j]\!]&=[\![(a_i+a_j+a_{g-1})\wedge a_g;\langle a_i+a_{g-1},b_i,a_j,b_j,a_g\rangle]\!]\\&=[\![(a_i+a_j+a_{g-1})\wedge a_g;\langle a_i+a_j+a_{g-1},b_i,b_j,a_g\rangle]\!].\end{aligned}$$

Similarly, we have

$$[\![(a_j+a_{g-1})\wedge a_g;W_j]\!]+[\![a_i\wedge a_g;W_i]\!]=[\![(a_i+a_j+a_{g-1})\wedge a_g;\langle a_i+a_j+a_{g-1},b_i,b_j,a_g\rangle]\!].$$

The identity (12.3) follows.

**Claim.** *For $1\leq i\leq g-2$, the group $\mathrm{Sp}_{2g}(\mathbb{Z},I)$ fixes $[\![a_{g-1}\wedge a_g;W_i]\!]$.*

As we noted in §12.4, the action of $\mathrm{Sp}_{2g}(\mathbb{Z},I)$ on our lifts factors through $\Gamma=\mathrm{Sp}_{2g}(\mathbb{Z},I)|_{I^\perp}$. Lemma 12.5 says that

$$\Gamma=\mathrm{Hom}(\mathbb{Z}^{2(g-2)},I)\ltimes\mathrm{Sp}_{2(g-2)}(\mathbb{Z}).$$

We must prove that the subgroups $\mathrm{Hom}(\mathbb{Z}^{2(g-2)},I)$ and $\mathrm{Sp}_{2(g-2)}(\mathbb{Z})$ both fix $[\![a_{g-1}\wedge a_g;W_i]\!]$.

We start with $\mathrm{Sp}_{2(g-2)}(\mathbb{Z})$. It is classical that $\mathrm{Sp}_{2(g-2)}(\mathbb{Z})$ is generated by the stabilizer of $a_1$ and the stabilizer of $a_2$. For instance,[15] the mapping class group $\mathrm{Mod}_{g-2}$ surjects onto $\mathrm{Sp}_{2(g-2)}(\mathbb{Z})$, and choosing a basis for $\mathrm{H}_1(\Sigma_{g-2})$ appropriately the usual Dehn twist generators for $\mathrm{Mod}_{g-2}$ from [1, Theorem 4.13] each fix either $a_1$ or $a_2$. It is thus enough to prove that both of these stabilizers fix $[\![a_{g-1}\wedge a_g;W_i]\!]$.

The proofs for both are similar, so we will give the details for the stabilizer of $a_2$ and leave the other case to the reader. Consider $f\in\mathrm{Sp}_{2(g-2)}(\mathbb{Z})$ with $f(a_2)=a_2$. By the previous claim, it is enough to prove that $f$ fixes $[\![a_{g-1}\wedge a_g;W_2]\!]$. Using Lemma 12.2 (linearity of the lifts), we have

$$[\![a_{g-1}\wedge a_g;W_2]\!]=[\![(a_2+a_{g-1})\wedge a_g;W_2]\!]-[\![a_2\wedge a_g;W_2]\!].$$

We will prove that $f$ fixes $[\![(a_2+a_{g-1})\wedge a_g;W_2]\!]$ and $[\![a_2\wedge a_g;W_2]\!]$.

For the first, Lemma 12.4 (changing the separator in the lifts) says that

$$\begin{aligned}[\![(a_2+a_{g-1})\wedge a_g;W_2]\!]&=[\![(a_2+a_{g-1})\wedge a_g;\langle a_2+a_{g-1},b_2,a_g\rangle]\!]\\&=[\![(a_2+a_{g-1})\wedge a_g;\langle a_1,b_1,a_2+a_{g-1},b_2,a_g\rangle]\!].\end{aligned}$$

Lemma 12.3 (equivariance of the lifts) says that $f$ takes this to

$$\begin{aligned}&[\![(f(a_2)+f(a_{g-1}))\wedge f(a_g);f(\langle a_1,b_1,a_2+a_{g-1},b_2,a_g\rangle)]\!]\\&\qquad=[\![(a_2+a_{g-1})\wedge a_g;\langle a_1,b_1,a_2+a_{g-1},b_2,a_g\rangle]\!],\end{aligned}$$

as desired.

For the second, Lemma 12.4 (changing the separator in the lifts) says that

$$\begin{aligned}[\![a_2\wedge a_g;W_2]\!]&=[\![a_2\wedge a_g;\langle a_2,b_2,a_g\rangle]\!]\\&=[\![a_2\wedge a_g;\langle a_1,b_1,a_2,b_2,a_g\rangle]\!].\end{aligned}$$

Lemma 12.3 (equivariance of the lifts) says that $f$ takes this to

$$[\![f(a_2)\wedge f(a_g);f(\langle a_1,b_1,a_2,b_2,a_g\rangle)]\!]=[\![a_2\wedge a_g;\langle a_1,b_1,a_2,b_2,a_g\rangle]\!],$$

as desired.

It remains to prove that the subgroup $\mathrm{Hom}(\mathbb{Z}^{2(g-2)},I)$ of $\Gamma$ fixes our lift. Observe that $\mathrm{Hom}(\mathbb{Z}^{2(g-2)},I)$ is generated by elements that fix all but one element of the basis

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[14]The term $\langle a_i+a_j+a_{g-1},b_i,b_j,a_g\rangle$ appearing here is a near symplectic summand even though the given basis does not reflect this.

[15]This could also be deduced from the generating set of Hua–Reiner [3] discussed in §7, but be warned that their generating set does not consist of elements that fix either $a_1$ or $a_2$.

$\{a_1, b_1, \ldots, a_{g-2}, b_{g-2}\}$. It is enough to prove that such elements fix our lift. Consider $f \in \text{Hom}(\mathbb{Z}^{2(g-2)}, I)$ that fixes all elements of $\{a_1, b_1, \ldots, a_{g-2}, b_{g-2}\}$ except for $x \in \{a_j, b_j\}$. Letting $1 \leq i \leq g-2$ be such that $i \neq j$, it is enough to prove that $f$ fixes $[\![a_{g-1} \wedge a_g; W_i]\!]$. But this is immediate from the fact that $f$ fixes $a_{g-1}$ and $a_g$ and $W_i = \langle a_i, b_i, a_{g-1}, a_g\rangle$. □

## 13. Isotropic pairs III: isomorphism theorem

We now prove the following theorem using the proof outline from §3.

**Proposition 13.1.** *Let $a \wedge a'$ be an isotropic pair and let $\Phi\colon \mathfrak{K}_g \to ((\wedge^2 H)/\mathbb{Q})^{\otimes 2}$ be the linearization map. Then $\Phi$ takes $\mathfrak{K}_g[-, a \wedge a']$ isomorphically to $(a \wedge a')^{\perp} \otimes (a \wedge a')$.*

*Proof.* By Lemma 10.4, there exists a strong isotropic pair $a_0 \wedge a_0'$ and $n \in \mathbb{Z}$ such that $a \wedge a' = n a_0 \wedge a_0'$. Moreover, $\langle a_0, a_0'\rangle_{\mathbb{Q}} = \langle a, a'\rangle_{\mathbb{Q}}$, so by Lemma 10.2 we have $(a \wedge a')^{\perp} = (a_0 \wedge a_0')^{\perp}$. Using the linearity relations in $\mathfrak{K}_g$, multiplication by $n$ gives an isomorphism $\mathfrak{K}_g[-, a_0 \wedge a_0'] \cong \mathfrak{K}_g[-, a \wedge a']$ taking a generator $[\![\sigma, a_0 \wedge a_0']\!]$ with $\sigma$ a symplectic pair in $(a_0 \wedge a_0')^{\perp}$ to a generator $[\![\sigma, a \wedge a']\!]$. It is thus enough to prove the proposition for $a_0 \wedge a_0'$. Replacing $a \wedge a'$ with $a_0 \wedge a_0'$, we can therefore assume that $a \wedge a'$ is a strong isotropic pair.

To simplify our notation, we will drop $a \wedge a'$ from our notation in two places:

- For $\kappa \in (a \wedge a')^{\perp}$ and $X$ a Lagrangian-free direct summand of $H_{\mathbb{Z}}$ that separates $\kappa$ from $a \wedge a'$, we will drop the $a \wedge a'$ from our notation and write $[\![\kappa; X]\!]$ instead of $[\![(\kappa; X), a \wedge a']\!]$.
- We will also drop the $a \wedge a'$ from our notation for the codomain of the restriction of $\Phi$ to $\mathfrak{K}_g[-, a \wedge a']$. Thus for $[\![\kappa; X]\!]$ as in the previous bullet point, we will write $\Phi([\![\kappa; X]\!]) = \kappa$ rather than $\Phi([\![\kappa; X]\!]) = \kappa \otimes (a \wedge a')$.

The proof has three steps.

**Step 1.** *We construct a set $S \subset \mathfrak{K}_g[-, a \wedge a']$ such that the restriction of $\Phi$ to $\langle S\rangle$ is an isomorphism to $(a \wedge a')^{\perp} \otimes (a \wedge a')$.*

Let $\mathcal{B} = \{a_1, b_1, \ldots, a_g, b_g\}$ be a symplectic basis for $H_{\mathbb{Z}}$ with $a_{g-1} = a$ and $a_g = a'$. Letting $I = \langle a, a'\rangle$, we have

$$I_{\mathbb{Q}}^{\perp} = \langle a_1, b_1, \ldots, a_{g-2}, b_{g-2}, a_{g-1}, a_g\rangle.$$

Set

$$\mathcal{B}' = \{a_1, b_1, \ldots, a_{g-2}, b_{g-2}\},$$

with the total order $\prec$ as indicated in this list. Lemma 10.2 says that

$$(a \wedge a')^{\perp} = \overline{\wedge^2 I_{\mathbb{Q}}^{\perp}}.$$

This vector space has the basis[16]

$$T = \{x \wedge y \mid x, y \in \mathcal{B}',\ x \prec y\} \cup \{x \wedge a_{g-1},\ x \wedge a_g \mid x \in \mathcal{B}'\} \cup \{a_{g-1} \wedge a_g\}.$$

Define

$$X = \langle \mathcal{B}'\rangle \quad \text{and} \quad Y = \langle \mathcal{B}', a_{g-1}\rangle \quad \text{and} \quad Z = \langle \mathcal{B}', a_g\rangle$$

and

$$W_i = \langle a_i, b_i, a_{g-1}, a_g\rangle \quad \text{for } 1 \leq i \leq g-2.$$

[16]This clearly forms a basis for $\wedge^2 I_{\mathbb{Q}}^{\perp}$, and since the restriction of the map $\wedge^2 H \to (\wedge^2 H)/\mathbb{Q}$ to $\wedge^2 I_{\mathbb{Q}}^{\perp}$ is an injection the image of $T$ in $(\wedge^2 H)/\mathbb{Q}$ also forms a basis for $\overline{\wedge^2 I_{\mathbb{Q}}^{\perp}}$.

These are all Lagrangian-free near symplectic summands of $H_{\mathbb{Z}}$ of positive genus. Finally, define

$$S = \{ [\![x \wedge y; X]\!] \mid x, y \in \mathcal{B}',\, x \prec y \} \\ \cup \{ [\![x \wedge a_{g-1}; Y]\!],\, [\![x \wedge a_g; Z]\!] \mid x \in \mathcal{B}' \} \cup \{ [\![a_{g-1} \wedge a_g; W_1]\!] \}.$$

By construction, $\Phi$ takes $S$ bijectively to $T$. Since $T$ is a basis for $(a \wedge a')^{\perp}$, it follows that the restriction of $\Phi$ to $\langle S \rangle$ is an isomorphism.

**Step 2.** *We prove that the* $\mathrm{Sp}_{2g}(\mathbb{Z})$*-orbit of* $S$ *spans* $\mathfrak{K}_g[-, a_{g-1} \wedge a_g]$.

By definition, $\mathfrak{K}_g[-, a_{g-1} \wedge a_g]$ is spanned by elements of the form $[\![\sigma, a_{g-1} \wedge a_g]\!]$, where $\sigma$ is a symplectic pair with $\sigma \in (a \wedge a')^{\perp}$. The image of $\Phi$ contains some elements of this form; for instance, it contains

$$[\![a_1 \wedge b_1; X]\!] = [\![a_1 \wedge b_1, a_{g-1} \wedge a_g]\!].$$

Since $\mathrm{Sp}_{2g}(\mathbb{Z}, I)$ acts transitively on symplectic pairs lying in $(a \wedge a')^{\perp} = \overline{\wedge^2 I_{\mathbb{Q}}^{\perp}}$, it follows that the $\mathrm{Sp}_{2g}(\mathbb{Z})$-orbit of $S$ spans $\mathfrak{K}_g[-, a_{g-1} \wedge a_g]$..

**Step 3.** *We prove that* $\mathrm{Sp}_{2g}(\mathbb{Z})$ *takes* $\langle S \rangle$ *to itself. By Step 2 this will imply that* $\langle S \rangle = \mathfrak{K}_g[-, a_{g-1} \wedge a_g]$*, and thus by Step 1 that* $\Phi$ *is an isomorphism.*

The action of $\mathrm{Sp}_{2g}(\mathbb{Z}, I)$ on $\mathfrak{K}_g[-, a_{g-1} \wedge a_g]$ factors through $\Gamma = \mathrm{Sp}_{2g}(\mathbb{Z}, I)|_{I^{\perp}}$, and by Lemma 12.5 we have

$$\Gamma = \mathrm{Hom}(X, I) \ltimes \mathrm{Sp}_{2(g-2)}(\mathbb{Z}).$$

We must prove that $\mathrm{Hom}(X, I)$ and $\mathrm{Sp}_{2(g-2)}(\mathbb{Z})$ both take $\langle S \rangle$ to itself. We divide this into two claims:

**Claim 3.1.** *The action of* $\mathrm{Sp}_{2(g-2)}(\mathbb{Z})$ *on* $\mathfrak{K}_g[-, a_{g-1} \wedge a_g]$ *takes* $\langle S \rangle$ *to itself.*

Consider $f \in \mathrm{Sp}_{2(g-2)}(\mathbb{Z})$ and $s \in S$. We must prove that $f(s)$ is a linear combination of elements of $S$. This is trivial for $s = [\![a_{g-1} \wedge a_g; W_1]\!]$ since Lemma 12.6 implies that $f$ fixes $s$. The other $s$ fall into three cases.

The first is $s = [\![x \wedge y; X]\!]$ with

$$x, y \in \mathcal{B}' = \{a_1, b_1, \ldots, a_{g-2}, b_{g-2}\} \text{ such that } x \prec y.$$

Since $f(X) = X$, Lemma 12.3 (equivariance of the lifts) implies that

$$f([\![x \wedge y; X]\!]) = [\![f(x) \wedge f(y); f(X)]\!] = [\![f(x) \wedge f(y); X]\!].$$

The element $f(x) \wedge f(y) \in \overline{\wedge^2 X_{\mathbb{Q}}}$ is a linear combination of terms of the form $x' \wedge y'$ with $x', y' \in \mathcal{B}'$ such that $x' \prec y'$, and by Lemma 12.2 (linearity of the lifts) the element $[\![f(x) \wedge f(y); X]\!]$ equals the corresponding linear combination of elements of the form $[\![x' \wedge y'; X]\!] \in S$, as desired.

The second is $s = [\![x \wedge a_{g-1}; Y]\!]$ with $x \in \mathcal{B}'$. Since $f(a_{g-1}) = a_{g-1}$ and $f(Y) = Y$, Lemma 12.3 (equivariance of the lifts) implies that

$$f([\![x \wedge a_{g-1}; Y]\!]) = [\![f(x) \wedge f(a_{g-1}); f(Y)]\!] = [\![f(x) \wedge a_{g-1}; Y]\!].$$

The element $f(x) \wedge a_{g-1} \in \overline{\wedge^2 Y_{\mathbb{Q}}}$ is a linear combination of terms of the form $x' \wedge a_{g-1}$ with $x' \in \mathcal{B}'$, and by Lemma 12.2 (linearity of the lifts) the element $[\![f(x) \wedge a_{g-1}; Y]\!]$ equals the corresponding linear combination of elements of the form $[\![x' \wedge a_{g-1}; Y]\!] \in S$, as desired.

The third is $s = [\![x \wedge a_g; Z]\!]$ with $x \in \mathcal{B}'$. This is handled in the same way as the previous case, so we omit the details.

**Claim 3.2.** *The action of* $\mathrm{Hom}(X, I)$ *on* $\mathfrak{K}_g[-, a_{g-1} \wedge a_g]$ *takes* $\langle S \rangle$ *to itself.*

Recall that $X = \langle a_1, b_1, \ldots, a_{g-2}, b_{g-2}\rangle$ and $I = \{a_{g-1}, a_g\}$. The group $\text{Hom}(X, I)$ is generated by $\text{Hom}(X, \langle a_{g-1}\rangle)$ and $\text{Hom}(X, \langle a_g\rangle)$. It is enough to check that all $\lambda$ lying in one of these two subgroups take $\langle S\rangle$ to itself. For concreteness, we will explain how to do this for $\lambda \in \text{Hom}(X, \langle a_{g-1}\rangle)$. The other case is similar. The corresponding $f \in \text{Sp}_{2g}(\mathbb{Z}, I)|_{I^\perp}$ satisfies

$$f(a_{g-1}) = a_{g-1} \quad \text{and} \quad f(a_g) = a_g \quad \text{and} \quad f(x) = x + \lambda(x) \text{ for all } x \in X.$$

Consider $s \in S$. We must prove that $f(s)$ is a linear combination of elements of $S$. This is trivial for $s = [\![a_{g-1} \wedge a_g; W_1]\!]$ since in this case Lemma 12.6 implies that $f$ fixes $s$. The other $s$ fall into three cases.

The first is $s = [\![x \wedge y; X]\!]$ with $x, y \in \mathcal{B}'$ such that $x \prec y$. By Lemma 12.4 (changing the separator in the lifts), this equals $[\![x \wedge y; Y]\!]$. The reason for doing this is that $f(Y) = Y$. Write $f(x) = x + ca_{g-1}$ and $f(y) = y + da_{g-1}$ with $c, d \in \mathbb{Z}$. Using all three properties of our lifts from §12.3, we have

$$\begin{aligned} f([\![x \wedge y; Y]\!]) &= [\![(x + ca_{g-1}) \wedge (y + da_{g-1}); Y]\!] \\ &= [\![x \wedge y; Y]\!] + d[\![x \wedge a_{g-1}; Y]\!] - c[\![y \wedge a_{g-1}; Y]\!] \\ &= [\![x \wedge y; X]\!] + d[\![x \wedge a_{g-1}; Y]\!] - c[\![y \wedge a_{g-1}; Y]\!]. \end{aligned}$$

This is a linear combination of elements of $S$, as desired.

The second is $s = [\![x \wedge a_{g-1}; Y]\!]$ with $x \in \mathcal{B}'$. Write $f(x) = x + ca_{g-1}$ with $c \in \mathbb{Z}$. We then have

$$f(x \wedge a_{g-1}) = (x + ca_{g-1}) \wedge a_{g-1} = x \wedge a_{g-1}.$$

Since $f$ fixes $x \wedge a_{g-1}$ and $Y$, by Lemma 12.3 (equivariance of the lifts) the map $f$ also fixes $[\![x \wedge a_{g-1}; Y]\!]$ and there is nothing to prove.

The third is $s = [\![x \wedge a_g; Z]\!]$ with $x \in \mathcal{B}'$. Write $f(x) = x + ca_{g-1}$ with $c \in \mathbb{Z}$. Pick $i$ such that $x \in \{a_i, b_i\}$. By Lemma 12.4 (changing the separator in the lifts), we have

$$[\![x \wedge a_g; Z]\!] = [\![x \wedge a_g; \langle a_i, b_i, a_g\rangle]\!] = [\![x \wedge a_g; W_i]\!].$$

The reason for doing is that $f(W_i) = W_i$. Using all three properties of our lifts from §12.3 along with Lemma 12.6, we have

$$\begin{aligned} f([\![x \wedge a_g; W_i]\!]) &= [\![(x + ca_{g-1}) \wedge a_g; W_i]\!] = [\![x \wedge a_g; W_i]\!] + c[\![a_{g-1} \wedge a_g; W_i]\!] \\ &= [\![x \wedge a_g; Z]\!] + c[\![a_{g-1} \wedge a_g; W_1]\!]. \end{aligned}$$

This is a linear combination of elements of $S$, as desired. □

## 14. Isotropic pairs IV: refining the presentation I

We now bring all our work together to add new generators to $\mathfrak{K}_g$ involving isotropic pairs. Let $a \wedge a'$ be an isotropic pair and let $\Phi\colon \mathfrak{K}_g \to ((\wedge^2 H)/\mathbb{Q})^{\otimes 2}$ be the linearization map.

### 14.1. Right elements.

Proposition 13.1 says that for all $\kappa \in (a \wedge a')^\perp$, there is a unique element $[\![\kappa, a \wedge a']\!]_R \in \mathfrak{K}_g[-, a \wedge a']$ satisfying

$$\Phi([\![\kappa, a \wedge a']\!]_R) = \kappa \otimes (a \wedge a').$$

For $\lambda_1, \lambda_2 \in \mathbb{Q}$ and $\kappa_1, \kappa_2 \in (a \wedge a')^\perp$, Proposition 13.1 implies that

$$[\![\lambda_1\kappa_1 + \lambda_2\kappa_2, a \wedge a']\!]_R = \lambda_1[\![\kappa_1, a \wedge a']\!]_R + \lambda_2[\![\kappa_2, a \wedge a']\!]_R.$$

14.2. **Left elements.** Define $\mathfrak{K}_g[a \wedge a', -]$ to be the subspace of $\mathfrak{K}_g$ spanned by elements of the form $[\![a \wedge a', \sigma]\!]$ with $\sigma$ a symplectic pair such that $\sigma \in (a \wedge a')^{\perp}$. There is an involution $\iota\colon \mathfrak{K}_g \to \mathfrak{K}_g$ taking a generator $[\![\kappa_1, \kappa_2]\!]$ to $[\![\kappa_2, \kappa_1]\!]$, and $\iota$ takes $\mathfrak{K}_g[a \wedge a', -]$ isomorphically to $\mathfrak{K}_g[-, a \wedge a']$. For $\kappa \in (a \wedge a')^{\perp}$, define

$$[\![a \wedge a', \kappa]\!]_L = \iota([\![\kappa, a \wedge a']\!]_R).$$

The element $[\![a \wedge a', \kappa]\!]_L$ is then the unique element of $\mathfrak{K}_g[a \wedge a', -]$ satisfying

$$\Phi([\![a \wedge a', \kappa]\!]_L) = (a \wedge a') \otimes \kappa.$$

For $\lambda_1, \lambda_2 \in \mathbb{Q}$ and $\kappa_1, \kappa_2 \in (a \wedge a')^{\perp}$, we have

$$[\![a \wedge a', \lambda_1\kappa_1 + \lambda_2\kappa_2]\!]_L = \lambda_1[\![a \wedge a', \kappa_1]\!]_L + \lambda_2[\![a \wedge a', \kappa_2]\!]_L.$$

14.3. **Ambiguity.** We would like to drop the $L$ and $R$ from $[\![a \wedge a', \kappa]\!]_L$ and $[\![\kappa, a \wedge a']\!]_R$. To do this, we must first show that this does not introduce ambiguity into our notation. The issue is that there exist isotropic pairs $a_1 \wedge a_1'$ and $a_2 \wedge a_2'$ that are sym-orthogonal to each other. In this case, we have elements $[\![a_1 \wedge a_1', a_2 \wedge a_2']\!]_L$ and $[\![a_1 \wedge a_1', a_2 \wedge a_2']\!]_R$, and we need to prove that they are equal:

**Lemma 14.1.** *Let $a_1 \wedge a_1'$ and $a_2 \wedge a_2'$ be isotropic pairs that are sym-orthogonal. Then $[\![a_1 \wedge a_1', a_2 \wedge a_2']\!]_L = [\![a_1 \wedge a_1', a_2 \wedge a_2']\!]_R$.*

*Proof.* The proof uses the same idea as the proof of Lemma 12.1. Set $I_1 = \langle a_1, a_1' \rangle$ and $I_2 = \langle a_2, a_2' \rangle$. By Lemma 10.2, we have $a_1 \wedge a_1' \in \overline{\wedge^2(I_2)^{\perp}_{\mathbb{Q}}}$ and $a_2 \wedge a_2' \in \overline{\wedge^2(I_1)^{\perp}_{\mathbb{Q}}}$. This implies that $I_1 \subset I_2^{\perp}$ and $I_2 \subset I_1^{\perp}$. Recall that from §12.2 that a Lagrangian in $H_{\mathbb{Z}}$ is a direct summand $L$ of $H_{\mathbb{Z}}$ with $L^{\perp} = L$. We start with:

**Claim.** *There exists a Lagrangian $L$ in $H_{\mathbb{Z}}$ such that $I_1, I_2 \subset L$.*

*Proof of claim.* Recall that a subspace $J$ of $H$ is isotropic if $J \subset J^{\perp}$ and is a Lagrangian if $J = J^{\perp}$. It is standard that $J$ being a Lagrangian is equivalent to $J$ being isotropic and $g$-dimensional, and also that every isotropic subspace is contained in a Lagrangian. The subspace $\langle (I_1)_{\mathbb{Q}}, (I_2)_{\mathbb{Q}} \rangle$ of $H$ is isotropic, so it is contained in a Lagrangian $L_{\mathbb{Q}}$. Define $L = L_{\mathbb{Q}} \cap H_{\mathbb{Z}}$. Lemma 10.3 implies that $L$ is a direct summand of $H_{\mathbb{Z}}$, and by construction it is a Lagrangian containing $I_1$ and $I_2$. □

Using this, we will prove:

**Claim.** *There exists a Lagrangian-free near symplectic summand $X$ of $H_{\mathbb{Z}}$ of genus 1 such that $I_1 \subset X \subset I_2^{\perp}$.*

*Proof of claim.* By the previous claim, we can find a Lagrangian $L$ in $H_{\mathbb{Z}}$ with $I_1, I_2 \subset L$. Since $I_2 \cong \mathbb{Z}^2$ is a subspace of $L \cong \mathbb{Z}^g$ and $g \geq 4$ (see Assumption 9.5), the quotient $L/I_2$ cannot consist entirely of torsion. It follows that there exists a surjection $\pi\colon L \to \mathbb{Z}$ with $I_2 \subset \ker(\pi)$. Since the symplectic form $\omega$ identifies $H_{\mathbb{Z}}$ with its dual, we can find $y_1 \in H_{\mathbb{Z}}$ with

$$\omega(z, y_1) = \pi(z) \text{ for all } z \in L.$$

In particular, $y_1 \in (I_2)^{\perp}$. Pick $x_1 \in L$ with $\omega(x_1, y_1) = 1$. Define $J = \langle (I_1)_{\mathbb{Q}}, \langle x_1 \rangle_{\mathbb{Q}} \rangle \cap L$, so by Lemma 10.3 the subgroup $J$ is a direct summand of $L$ with $I_1 \subset J$ and $x_1 \in J$. Let $r$ be the rank of $J$. Since the rank of $I_1$ is 2, we have $2 \leq r \leq 3$. We can now extend $x_1$ to a basis $\{x_1, \ldots, x_r\}$ for $J$ such that $\omega(x_i, y_1) = 0$ for $2 \leq i \leq r$. Set $X = \langle x_1, y_1, x_2, \ldots, x_r \rangle$. By construction, $X$ is a near-symplectic summand of $H_{\mathbb{Z}}$ of genus 1 such that $I_1 \subset X \subset I_2^{\perp}$. Since $r \leq 3$, our standing assumption that $g \geq 4$ (see Assumption 9.5) implies that $X$ is Lagrangian-free. □

Let $X$ be as in the previous claim. We have:

**Claim.** *The subspace $X^{\perp}$ of $H_{\mathbb{Z}}$ is a near symplectic summand of positive genus.*

*Proof of claim.* Since $X$ is a near-symplectic summand of genus 1, we can find a symplectic basis $\{x_1, y_1, \ldots, x_g, y_g\}$ for $H_{\mathbb{Z}}$ such that $X = \langle x_1, y_1, x_2, \ldots, x_r \rangle$. Since $X$ is Lagrangian-free, we have $r < g$. We have $X^{\perp} = \langle x_2, \ldots, x_r, x_{r+1}, y_{r+1}, \ldots, x_g, y_g \rangle$. This is a near symplectic summand, and since $r < g$ its genus is positive. □

Since $I_1 \subset X$ and $I_2 \subset X^{\perp}$, we can use Lemma 11.2 to write

$$a_1 \wedge a_1' = \sum_{i=1}^{n} \lambda_i \sigma_i \quad \text{and} \quad a_2 \wedge a_2' = \sum_{j=1}^{m} c_j s_j$$

with $\lambda_i, c_j \in \mathbb{Q}$ and with each $\sigma_i$ and $s_j$ a symplectic pair with $\sigma_i \in \overline{\wedge^2 X_{\mathbb{Q}}}$ and $s_j \in \overline{\wedge^2 X_{\mathbb{Q}}^{\perp}}$, respectively. The element $[\![a_1 \wedge a_1', a_2 \wedge a_2']\!]_R$ then equals

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i [\![\sigma_i, a_2 \wedge a_2']\!] &= \sum_{i=1}^{n} \lambda_i \left( \sum_{j=1}^{m} c_j [\![\sigma_i, s_j]\!] \right) = \sum_{j=1}^{m} c_j \left( \sum_{i=1}^{n} \lambda_i [\![\sigma_i, s_j]\!] \right) \\ &= \sum_{j=1}^{m} c_j [\![a_1 \wedge a_1', s_j]\!] = [\![a_1 \wedge a_1', a_2 \wedge a_2']\!]_L. \end{aligned}$$

□

14.4. **New generators.** Let $a \wedge a'$ be an isotropic pair and let $\kappa \in (a \wedge a')^{\perp}$. We then have elements $[\![\kappa, a \wedge a']\!]_R$ and $[\![a \wedge a', \kappa]\!]_L$ of $\mathfrak{K}_g$. By Lemma 14.1, we can unambiguously drop the "L" and "R" from our notation. Since everything is now canonical, we will also stop writing our elements in orange and define

$$[\![\kappa, a \wedge a']\!] = [\![\kappa, a \wedge a']\!]_R \quad \text{and} \quad [\![a \wedge a', \kappa]\!] = [\![a \wedge a', \kappa]\!]_L.$$

14.5. **Summary.** The following summarizes what we have accomplished in Proposition 13.1 and Lemma 14.1:

**Theorem 14.2.** *The vector space $\mathfrak{K}_g$ has the following presentation:*

- **Generators**. *A generator $[\![\kappa_1, \kappa_2]\!]$ for all sym-orthogonal $\kappa_1, \kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ such that either $\kappa_1$ or $\kappa_2$ (or both) is a symplectic pair or an isotropic pair.*
- **Relations**. *For all $\zeta \in (\wedge^2 H)/\mathbb{Q}$ that are symplectic pairs or strong isotropic pairs and all $\kappa_1, \kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ that are sym-orthogonal to $\zeta$ and all $\lambda_1, \lambda_2 \in \mathbb{Q}$, the relations*

$$[\![\zeta, \lambda_1 \kappa_1 + \lambda_2 \kappa_2]\!] = \lambda_1 [\![\zeta, \kappa_1]\!] + \lambda_2 [\![\zeta, \kappa_2]\!] \quad \text{and}$$
$$[\![\lambda_1 \kappa_1 + \lambda_2 \kappa_2, \zeta]\!] = \lambda_1 [\![\kappa_1, \zeta]\!] + \lambda_2 [\![\kappa_2, \zeta]\!].$$

## 15. Isotropic pairs V: refining the presentation II

In Theorem 14.2, we added generators involving isotropic pairs. In this section, we add a few more generators and verify some additional relations. A *special pair* is an element $x \wedge y$ of $(\wedge^2 H_{\mathbb{Z}})/\mathbb{Z}$ such that $\omega(x, y) \in \{-1, 0, 1\}$. These fall into four classes:

- $\omega(x, y) = 1$, so $x \wedge y$ is a symplectic pair; and
- $\omega(x, y) = 0$ with $x$ and $y$ linearly independent, so $x \wedge y$ is an isotropic pair;[17] and
- $\omega(x, y) = 0$ with $x$ and $y$ linearly dependent, so $x \wedge y = 0$; and
- $\omega(x, y) = -1$, so $y \wedge x = -x \wedge y$ is a symplectic pair.

Our main result is:

[17] They are strong isotropic pairs if $x$ and $y$ also span a direct summand of $H_{\mathbb{Z}}$; see §10.4.

**Theorem 15.1.** *For*[18] $g \geq 4$, *the vector space* $\mathfrak{K}_g$ *has the following presentation:*

- **Generators**. *A generator* $[\![\kappa_1, \kappa_2]\!]$ *for all sym-orthogonal* $\kappa_1, \kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ *such that either* $\kappa_1$ *or* $\kappa_2$ *(or both) is a special pair.*
- **Relations**. *The following two families of relations:*
  - *For all special pairs* $\zeta \in (\wedge^2 H)/\mathbb{Q}$ *and all* $\kappa_1, \kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ *that are sym-orthogonal to* $\zeta$ *and all* $\lambda_1, \lambda_2 \in \mathbb{Q}$, *the linearity relations*

$$[\![\zeta, \lambda_1\kappa_1 + \lambda_2\kappa_2]\!] = \lambda_1[\![\zeta, \kappa_1]\!] + \lambda_2[\![\zeta, \kappa_2]\!] \quad \textit{and}$$
$$[\![\lambda_1\kappa_1 + \lambda_2\kappa_2, \zeta]\!] = \lambda_1[\![\kappa_1, \zeta]\!] + \lambda_2[\![\kappa_2, \zeta]\!].$$

  - *For all special pairs* $\zeta \in (\wedge^2 H)/\mathbb{Q}$ *and all* $\kappa \in (\wedge^2 H)/\mathbb{Q}$ *that are sym-orthogonal to* $\zeta$ *and all* $n \in \mathbb{Z}$ *such that* $n\zeta$ *is a special pair,*[19] *the relations*

$$[\![n\zeta, \kappa]\!] = n[\![\zeta, \kappa]\!] \quad \textit{and}$$
$$[\![\kappa, n\zeta]\!] = n[\![\kappa, \zeta]\!].$$

*Proof.* We divide the proof into three steps: we first define our new generators in terms of the generators given by Theorem 14.2, and then we check the two families of relations.

**Step 1.** *We define our new generators.*

Consider a special pair $x \wedge y \in (\wedge^2 H)/\mathbb{Q}$ and $\kappa \in (\wedge^2 H)/\mathbb{Q}$ that is sym-orthogonal to $x \wedge y$. We will express $[\![x \wedge y, \kappa]\!]$ and $[\![\kappa, x \wedge y]\!]$ in terms of the generators given by Theorem 14.2. By definition, we have $\omega(x, y) \in \{-1, 0, 1\}$. There are four cases:

- If $\omega(x, y) = 1$, then $x \wedge y$ is a symplectic pair and $[\![x \wedge y, \kappa]\!]$ and $[\![\kappa, x \wedge y]\!]$ are already defined.
- If $\omega(x, y) = 0$ and $x$ and $y$ are linearly independent, then $x \wedge y$ is an isotropic pair and $[\![x \wedge y, \kappa]\!]$ and $[\![\kappa, x \wedge y]\!]$ are already defined.
- If $\omega(x, y) = 0$ and $x$ and $y$ are linearly dependent, then $x \wedge y = 0$. We define $[\![0, \kappa]\!] = 0$ and $[\![\kappa, 0]\!] = 0$.
- If $\omega(x, y) = -1$, then $x \wedge y = -y \wedge x$ and $y \wedge x$ is a symplectic pair. We define $[\![x \wedge y, \kappa]\!] = -[\![y \wedge x, \kappa]\!]$ and $[\![\kappa, x \wedge y]\!] = -[\![\kappa, y \wedge x]\!]$.

The only issue with this definition is that if $\kappa = z \wedge w$ is also a special pair, then in a few cases we have two different definitions of $[\![x \wedge y, z \wedge w]\!]$. We must check that they give the same element of $\mathfrak{K}_g$. If both $x \wedge y$ and $z \wedge w$ are symplectic or isotropic pairs, then there is no ambiguity. Also, if one of them is 0, then both definitions give 0. The only potential issue is therefore when either $\omega(x, y)$ or $\omega(z, w)$ (or both) is $-1$.

There are several cases. All are handled the same way, so we will give the details for when $\omega(x, y) = \omega(z, w) = -1$, which is slightly harder. Our two definitions are

$$[\![x \wedge y, z \wedge w]\!] = -[\![y \wedge x, z \wedge w]\!] \quad \text{and} \quad [\![x \wedge y, z \wedge w]\!] = -[\![x \wedge y, w \wedge z]\!].$$

We appeal to the linearity relations from Theorem 14.2 to see these are equal:

$$-[\![y \wedge x, z \wedge w]\!] = -[\![y \wedge x, -w \wedge z]\!] = [\![y \wedge x, w \wedge z]\!] = [\![-x \wedge y, w \wedge z]\!] = -[\![x \wedge y, w \wedge z]\!].$$

**Step 2.** *Let* $\zeta \in (\wedge^2 H)/\mathbb{Q}$ *be a special pair, let* $\kappa_1, \kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ *be sym-orthogonal to* $\zeta$, *and let* $\lambda_1, \lambda_2 \in \mathbb{Q}$. *Then:*

$$[\![\zeta, \lambda_1\kappa_1 + \lambda_2\kappa_2]\!] = \lambda_1[\![\zeta, \kappa_1]\!] + \lambda_2[\![\zeta, \kappa_2]\!] \quad \textit{and}$$
$$[\![\lambda_1\kappa_1 + \lambda_2\kappa_2, \zeta]\!] = \lambda_1[\![\kappa_1, \zeta]\!] + \lambda_2[\![\kappa_2, \zeta]\!].$$

---

[18] This is our standing assumption in this part of the paper; see Assumption 9.5.

[19] If $\zeta = x \wedge y$ with $\omega(x, y) = 0$, then any $n$ works. However, if $\zeta = x \wedge y$ with $\omega(x, y) \in \{-1, 1\}$ then we must take $n \in \{-1, 0, 1\}$.

These are trivial if $\zeta = 0$, so we can assume that $\zeta \neq 0$. Also, these are special cases of the linearity relations from Theorem 14.2 if $\zeta$ is either a symplectic pair or an isotropic pair. The remaining case is where $\zeta = x \wedge y$ with $\omega(x, y) = -1$. In that case, using the linearity relations from Theorem 14.2 we have that $[\![x \wedge y, \lambda_1\kappa_1 + \lambda_2\kappa_2]\!]$ equals

$$-[\![y \wedge x, \lambda_1\kappa_1 + \lambda_2\kappa_2]\!] = -\lambda_1[\![y \wedge x, \kappa_1]\!] - \lambda_2[\![y \wedge x, \kappa_2]\!] = \lambda_1[\![x \wedge y, \kappa_1]\!] + \lambda_2[\![x \wedge y, \kappa_2]\!],$$

and similarly for $[\![\lambda_1\kappa_1 + \lambda_2\kappa_2, x \wedge y]\!]$.

**Step 3.** *Let $\zeta \in (\wedge^2 H)/\mathbb{Q}$ be a special pair, let $\kappa \in (\wedge^2 H)/\mathbb{Q}$ be sym-orthogonal to $\zeta$, and let $m \in \mathbb{Z}$ be such that such that $m\zeta$ is a special pair. Then:*

$$[\![m\zeta, \kappa]\!] = m[\![\zeta, \kappa]\!] \quad \textit{and}$$
$$[\![\kappa, m\zeta]\!] = m[\![\kappa, \zeta]\!].$$

This is trivial if $\zeta = 0$ or if $m = 0$, so assume that both are nonzero. This is immediate from the definitions if $\zeta = x \wedge y$ with $\omega(x, y) \in \{\pm 1\}$, in which case we necessarily also have $m \in \{\pm 1\}$. The remaining case is when $\zeta = x \wedge y$ is an isotropic pair. Write $\kappa = \sum_{i=1}^n \lambda_i \zeta_i$ with $\zeta_i$ a symplectic pair in $(x \wedge y)^\perp$. We also have $\zeta_i \in (mx \wedge y)^\perp$ for $1 \leq i \leq n$. By definition, we therefore have

$$[\![x \wedge y, \kappa]\!] = \sum_{i=1}^n \lambda_i [\![x \wedge y, \zeta_i]\!] \quad \text{and} \quad [\![mx \wedge y, \kappa]\!] = \sum_{i=1}^n \lambda_i [\![mx \wedge y, \zeta_i]\!].$$

The linearity relations from Theorem 14.2 imply that for $1 \leq i \leq n$ we have $[\![mx \wedge y, \zeta_i]\!] = m[\![x \wedge y, \zeta_i]\!]$. Plugging this into the above formulas, we therefore have

$$[\![mx \wedge y, \kappa]\!] = \sum_{i=1}^n \lambda_i m [\![x \wedge y, \zeta_i]\!] = m \sum_{i=1}^n \lambda_i [\![x \wedge y, \zeta_i]\!] = m[\![x \wedge y, \kappa]\!].$$

A similar argument shows that $[\![\kappa, m\zeta]\!] = m[\![\kappa, \zeta]\!]$. □

## Part 3. Verifying the presentation for the symmetric kernel, alternating version

Our goal in the rest of the paper is to prove Theorems F and G. This part of the paper proves Theorem F, while Part 4 proves Theorem G. See the introductory §16 for an outline of what we do in this part. Throughout, we make the following genus assumption:

**Assumption 15.2.** Throughout Part 3, unless otherwise specified we assume that $g \geq 4$. □

### 16. Symmetric kernel, alternating version: introduction

We start by recalling some results and definitions from earlier in the paper, and then outline what we prove in this part.

16.1. **Symmetric kernel and contraction.** Recall that $\omega$ is the symplectic form on $H$. The *symmetric contraction* is the alternating $\text{Sym}^2(H)$-valued alternating form $\mathfrak{c}$ on $(\wedge^2 H)/\mathbb{Q}$ defined via the formula

$$\mathfrak{c}(x \wedge y, z \wedge w) = \omega(x, z) y \cdot w - \omega(x, w) y \cdot z - \omega(y, z) x \cdot w + \omega(y, w) x \cdot z \text{ for } x, y, z, w \in H.$$

It induces a map $\wedge^2((\wedge^2 H)/\mathbb{Q}) \to \text{Sym}^2(H)$ whose kernel $\mathcal{K}_g^a$ is the *symmetric kernel*. Elements $\kappa_1, \kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ are *sym-orthogonal* if $\mathfrak{c}(\kappa_1, \kappa_2) = 0$, or equivalently if $\kappa_1 \wedge \kappa_2 \in \mathcal{K}_g^a$. The *sym-orthogonal complement* of $\kappa \in (\wedge^2 H)/\mathbb{Q}$ is the subspace $\kappa^\perp$ consisting of all elements that are sym-orthogonal to $\kappa$.

16.2. **Special pairs.** A *special pair* in $(\wedge^2 H_{\mathbb{Z}})/\mathbb{Z}$ is an element of the form $x \wedge y$ with $\omega(x,y) \in \{-1,0,1\}$. Examples include symplectic pairs and isotropic pairs. Lemmas 10.1 and 10.2 say that the sym-orthogonal complements in $(\wedge^2 H)/\mathbb{Q}$ of these are:

- for a symplectic pair $a \wedge b$, we have $(a \wedge b)^{\perp} = \overline{\wedge^2 \langle a,b\rangle_{\mathbb{Q}}^{\perp}}$; and
- for an isotropic pair $a \wedge a'$, we have $(a \wedge a')^{\perp} = \overline{\wedge^2 \langle a,a'\rangle_{\mathbb{Q}}^{\perp}}$.

16.3. **Non-symmetric presentation.** We will use the generators and relations for $\mathfrak{K}_g$ from Theorem 15.1, whose statement we recall:

**Theorem 15.1.** *For $g \geq 4$, the vector space $\mathfrak{K}_g$ has the following presentation:*

- **Generators**. *A generator $[\![\kappa_1,\kappa_2]\!]$ for all sym-orthogonal $\kappa_1,\kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ such that either $\kappa_1$ or $\kappa_2$ (or both) is a special pair.*
- **Relations**. *The following two families of relations:*
  - *For special pairs $\zeta \in (\wedge^2 H)/\mathbb{Q}$ and all $\kappa_1,\kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ that are sym-orthogonal to $\zeta$ and all $\lambda_1,\lambda_2 \in \mathbb{Q}$, the linearity relations*

$$[\![\zeta,\lambda_1\kappa_1+\lambda_2\kappa_2]\!] = \lambda_1[\![\zeta,\kappa_1]\!]+\lambda_2[\![\zeta,\kappa_2]\!] \quad \textit{and}$$
$$[\![\lambda_1\kappa_1+\lambda_2\kappa_2,\zeta]\!] = \lambda_1[\![\kappa_1,\zeta]\!]+\lambda_2[\![\kappa_2,\zeta]\!].$$

  - *For all special pairs $\zeta \in (\wedge^2 H)/\mathbb{Q}$ and all $\kappa \in (\wedge^2 H)/\mathbb{Q}$ that are sym-orthogonal to $\zeta$ and all $n \in \mathbb{Z}$ such that $n\zeta$ is a special pair, the relations*

$$[\![n\zeta,\kappa]\!] = n[\![\zeta,\kappa]\!] \quad \textit{and}$$
$$[\![\kappa,n\zeta]\!] = n[\![\kappa,\zeta]\!].$$

*Remark* 16.1. Our standing assumption is that $g \geq 4$ (Assumption 15.2). However, in a few places we will need to work with $g = 3$ for inductive proofs. In those cases, Theorem 15.1 does not apply. To fix this, in this part of the paper we will redefine $\mathfrak{K}_3$ to be the vector space given by the presentation from Theorem 15.1. Note that we will *not* extend Theorem F to $g = 3$, and we do not know if this $\mathfrak{K}_3$ is isomorphic to $\mathcal{K}_3$. □

16.4. **Anti-symmetrizing.** Recall from Lemma 10.6 that $\mathfrak{K}_g^a$ is the $-1$-eigenspace of the involution of $\mathfrak{K}_g$ that takes a generator $[\![\kappa_1,\kappa_2]\!]$ to $[\![\kappa_2,\kappa_1]\!]$. We anti-symmetrize a generator $[\![\kappa_1,\kappa_2]\!]$ of $\mathfrak{K}_g$ to

$$[\![\kappa_1,\kappa_2]\!]_a = \frac{1}{2}\left([\![\kappa_1,\kappa_2]\!]-[\![\kappa_2,\kappa_1]\!]\right) \in \mathfrak{K}_g^a.$$

The anti-symmetrized generators generate $\mathfrak{K}_g^a$. They satisfy the same relations as the generators of $\mathfrak{K}_g$, and also the anti-symmetry relation $[\![\kappa_2,\kappa_1]\!]_a = -[\![\kappa_1,\kappa_2]\!]_a$.

16.5. **Goal and outline.** We have a linearization map $\Phi\colon \mathfrak{K}_g^a \to \wedge^2((\wedge^2 H)/\mathbb{Q})$. On generators, it satisfies

$$\Phi([\![\kappa_1,\kappa_2]\!]_a) = \kappa_1 \wedge \kappa_2 \in \wedge^2((\wedge^2 H)/\mathbb{Q}).$$

Its image is contained in the symmetric kernel $\mathcal{K}_g^a$. Our goal in this part of the paper is to prove Theorem F, which says that $\Phi$ is an isomorphism from $\mathfrak{K}_g^a$ to $\mathcal{K}_g^a$. The proof uses the proof technique described in §3, and is modeled on the proofs of Theorems A–E. However, since the calculations are lengthy we spread them out over nine sections:

- In the preliminary §17, we identify some important subspaces of $\mathfrak{K}_g^a$.
- In §18 – §20, we construct a subset $S$ of $\mathfrak{K}_g^a$ and prove that $\langle S\rangle = \mathfrak{K}_g^a$. The proof of this uses the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathfrak{K}_g^a$: we first prove that the $\mathrm{Sp}_{2g}(\mathbb{Z})$-orbit of $S$ spans $\mathfrak{K}_g^a$, and then we prove that $\mathrm{Sp}_{2g}(\mathbb{Z})$ takes $\langle S\rangle$ to itself. This corresponds to

Steps 2 and 3 of the proof outline from §3. We do these steps first because they are easier than Step 1.[20]

- The set $S$ is the union of sets $S_{12}$ and $S_3$. In §21 – §24, we prove that $\Phi$ is an isomorphism by first proving that its restriction to $S_{12}$ is an isomorphism onto its image (§21 – §22) and then extending this to $S_3$ and hence all of $\langle S \rangle = \mathfrak{K}_g^a$ (§23 – §25). This roughly speaking corresponds to Step 1.

Throughout the following nine sections, $\Phi$ will always mean the linearization map $\Phi \colon \mathfrak{K}_g^a \to \wedge^2((\wedge^2 H)/\mathbb{Q})$. Also, $\mathfrak{c}$ will always mean the symmetric contraction.

## 17. Symmetric kernel, alternating version I: fixing the 1st coordinates of generators

In this preliminary section, we identify some important subspaces of $\mathfrak{K}_g^a$ and their images in $\wedge^2((\wedge^2 H)/\mathbb{Q})$ under the linearization map. Throughout this section, we relax our standing assumption that $g \geq 4$ (Assumption 15.2), so our results include explicit genus ranges when they are necessary. See Remark 16.1.

**Warning 17.1.** The vector space $\wedge^2((\wedge^2 H)/\mathbb{Q})$ is *not* a quotient of $\wedge^4 H$. Because of this, for $x, y, z, w \in H$ care must be taken when working with elements like $(x \wedge y) \wedge (z \wedge w) \in \wedge^2((\wedge^2 H)/\mathbb{Q})$. The wedges $\wedge$ cannot be rearranged like in $\wedge^4 H$; for instance, $(x \wedge y) \wedge (z \wedge w)$ is not equal to $-(x \wedge z) \wedge (y \wedge w)$. □

17.1. **Setup.** Let $a, a' \in H_{\mathbb{Z}}$ satisfy $\omega(a, a') = 0$. Define $\mathfrak{F}[a, a']$ to be the subspace of $\mathfrak{K}_g^a$ spanned by elements $[\![a \wedge x, a' \wedge y]\!]_a$ where $x, y \in H_{\mathbb{Z}}$ satisfy the following two conditions:

- (i) We have $\omega(a, y) = \omega(a', x) = \omega(x, y) = 0$. This ensures that $\mathfrak{c}(a \wedge x, a' \wedge y) = 0$.
- (ii) Both $a \wedge x$ and $a' \wedge y$ are special pairs, so in particular $[\![a \wedge x, a' \wedge y]\!]_a$ is defined. Equivalently, $\omega(a, x) \in \{-1, 0, 1\}$ and $\omega(a', y) \in \{-1, 0, 1\}$.

We will call these $[\![a \wedge x, a' \wedge y]\!]_a$ the *generators* of $\mathfrak{F}[a, a']$. Here are some easy properties of these subspaces:

**Lemma 17.2.** *Let $g \geq 3$ and let $a, a' \in H_{\mathbb{Z}}$ satisfy $\omega(a, a') = 0$. Then:*

- *(a)* $\mathfrak{F}[a', a] = \mathfrak{F}[a, a']$.
- *(b)* $\mathfrak{F}[-a, a'] = \mathfrak{F}[a, -a'] = \mathfrak{F}[-a, -a'] = \mathfrak{F}[a, a']$.

*Proof.* Conclusion (a) follows from the fact that for all generators $[\![a \wedge x, a' \wedge y]\!]_a$ of $\mathfrak{F}[a, a']$, the element $[\![a' \wedge y, a \wedge x]\!]_a$ is a generator of $\mathfrak{F}[a', a]$ satisfying

$$[\![a' \wedge y, a \wedge x]\!]_a = -[\![a \wedge x, a' \wedge y]\!]_a.$$

In light of (a), to prove (b) it is enough to prove that $\mathfrak{F}[-a, a'] = \mathfrak{F}[a, a']$. For this, note that if $[\![a \wedge x, a' \wedge y]\!]_a$ is a generator of $\mathfrak{F}[a, a']$ then $[\![-a \wedge x, a' \wedge y]\!]_a$ is a generator of $\mathfrak{F}[-a, a']$ and we have

$$[\![-a \wedge x, a' \wedge y]\!]_a = -[\![a \wedge x, a' \wedge y]\!]_a.$$ □

17.2. **Image.** Define $\mathcal{F}[a, a']$ to be the subspace of $\wedge^2((\wedge^2 H)/\mathbb{Q})$ spanned by elements of the form $(a \wedge x) \wedge (a' \wedge y)$ such that $[\![a \wedge x, a' \wedge y]\!]_a$ is a generator of $\mathfrak{F}[a, a']$. We thus have $\mathcal{F}[a, a'] \subset \mathcal{K}_g^a$ and

$$\Phi(\mathfrak{F}[a, a']) \subset \mathcal{F}[a, a'].$$

Our goal in the rest of this section is to prove that in two important cases the map $\Phi$ takes $\mathfrak{F}[a, a']$ isomorphically to $\mathcal{F}[a, a']$.

[20] Though $S$ will be infinite, it will follow from our results in these sections that $\langle S \rangle$ is finite-dimensional. At the end of §20, we will therefore already know that $\mathfrak{K}_g^a$ is a finite-dimensional representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$.

Before we do this, we introduce one further piece of notation. Define $\widehat{\mathcal{F}}[a,a']$ to be the subspace of $\wedge^2((\wedge^2 H)/\mathbb{Q})$ spanned by elements of the form $(a \wedge x) \wedge (a' \wedge y)$ such that:

- We have $\omega(a,y) = \omega(a',x) = 0$. Note that we are not requiring that $\omega(x,y) = 0$.

For such an element, we have

$$\mathfrak{c}(a \wedge x, a' \wedge y) = \omega(a,a')x{\cdot}y - \omega(a,y)x{\cdot}a' - \omega(x,a')a{\cdot}y + \omega(x,y)a{\cdot}a' = \omega(x,y)a{\cdot}a'.$$

It follows that $\mathfrak{c}$ takes $\widehat{\mathcal{F}}[a,a']$ to the 1-dimensional subspace of $\mathrm{Sym}^2(H)$ spanned by $a{\cdot}a'$. It is easy to see that we can find elements $(a \wedge x) \wedge (a' \wedge y)$ as above with $\omega(x,y) \neq 0$, so in fact the kernel $\mathcal{K}_g^a \cap \widehat{\mathcal{F}}[a,a']$ of $\mathfrak{c}$ restricted to $\widehat{\mathcal{F}}[a,a']$ has codimension 1. We have $\mathcal{F}[a,a'] \subset \mathcal{K}_g^a \cap \widehat{\mathcal{F}}[a,a']$, and later in this section we will prove that in two cases we have $\mathcal{F}[a,a'] = \mathcal{K}_g^a \cap \widehat{\mathcal{F}}[a,a']$.

17.3. **Identification I.** We now commence with identifying our subspaces. For the first, let $a \in H_{\mathbb{Z}}$ be primitive. We will identify $\mathfrak{F}[a,a]$. In a generator $[\![a \wedge x, a \wedge y]\!]_a$ of $\mathfrak{F}[a,a]$, both $x$ and $y$ are orthogonal to $a$ and also are only well-defined up to multiples of $a$. This suggests defining $\mathcal{U}(a) = \langle a \rangle_{\mathbb{Q}}^{\perp}/\langle a \rangle$. We can embed $\mathcal{U}(a)$ into $(\wedge^2 H)/\mathbb{Q}$ by taking $x \in \mathcal{U}(a)$ to $a \wedge x \in (\wedge^2 H)/\mathbb{Q}$. Using this, we have

$$\widehat{\mathcal{F}}[a,a] = \wedge^2 \mathcal{U}(a).$$

The symplectic form $\omega$ induces a symplectic form $\overline{\omega}$ on $\mathcal{U}(a)$. Let $\mathcal{K}(a)$ be the kernel of the map $\wedge^2 \mathcal{U}(a) \to \mathbb{Q}$ induced by $\overline{\omega}$. We then have:

**Lemma 17.3.** *Let $g \geq 3$ and let $a \in H_{\mathbb{Z}}$ be primitive. Then:*

*(a) $\widehat{\mathcal{F}}[a,a] = \wedge^2 \mathcal{U}(a)$; and*
*(b) $\mathcal{F}[a,a] = \mathcal{K}_g^a \cap \widehat{\mathcal{F}}[a,a]$; and*
*(c) the linearization map $\Phi\colon \mathfrak{K}_g^a \to \wedge^2((\wedge^2 H)/\mathbb{Q})$ takes $\mathfrak{F}[a,a]$ isomorphically onto $\mathcal{K}(a)$.*

*Proof.* We noted that (a) held right before the lemma. We will prove (c) and then (b).

**Step 1.** *Conclusion (c) holds: the linearization map $\Phi\colon \mathfrak{K}_g^a \to \wedge^2((\wedge^2 H)/\mathbb{Q})$ takes $\mathfrak{F}[a,a]$ isomorphically onto $\mathcal{K}(a)$.*

Endow $\mathbb{Z}^{2g-2}$ with the standard symplectic form. Let $\mu\colon \mathbb{Z}^{2g-2} \to \langle a \rangle^{\perp}/\langle a \rangle$ be an isomorphism of abelian groups equipped with symplectic forms. If $v_1, v_2 \in \mathbb{Z}^{2g-2}$ are orthogonal vectors, then $a \wedge \mu(v_1)$ and $a \wedge \mu(v_2)$ are both either isotropic pairs or 0, so we have a generator $[\![a \wedge \mu(v_1), a \wedge \mu(v_2)]\!]_a$ of $\mathfrak{F}[a,a]$.

Recall that we defined the vector space $\mathfrak{Z}_{g-1}^a$ in Definition 1.11. It is generated by elements $(\!|v_1, v_2|\!)_a$ with $v_1, v_2 \in \mathbb{Z}^{2g-2}$ orthogonal primitive vectors. Define a map $\psi\colon \mathfrak{Z}_{g-1}^a \to \mathfrak{F}[a,a]$ via the formula

$$\psi((\!|v_1,v_2|\!)_a) = [\![a \wedge \mu(v_1), a \wedge \mu(v_2)]\!]_a \quad \text{for orthogonal primitive vectors } v_1, v_2 \in \mathbb{Z}^{2g-2}.$$

This takes relations to relations, and thus gives a well-defined map.

We claim that $\psi$ is surjective. To see this, consider a generator $[\![a \wedge x, a \wedge y]\!]_a$ of $\mathfrak{F}[a,a]$. We must check that $[\![a \wedge x, a \wedge y]\!]_a$ is in the image of $\psi$. Let $w_1, w_2 \in \mathbb{Z}^{2g-2}$ be such that $\mu(w_1) = x$ and $\mu(w_2) = y$. Write $w_1 = \lambda_1 v_1$ and $w_2 = \lambda_2 v_2$ with $\lambda_1, \lambda_2 \in \mathbb{Z}$ and $v_1, v_2 \in \mathbb{Z}^{2g-2}$ primitive. We then have

$$\begin{aligned}\psi(\lambda_1\lambda_2(\!|v_1,v_2|\!)_a) &= \lambda_1\lambda_2[\![a \wedge \mu(v_1), a \wedge \mu(v_2)]\!]_a = [\![a \wedge (\lambda_1\mu(v_1)), a \wedge (\lambda_2\mu(v_2))]\!]_a \\ &= [\![a \wedge \mu(w_1), a \wedge \mu(w_2)]\!]_a = [\![a \wedge x, a \wedge y]\!]_a,\end{aligned}$$

as desired.

We can identify the composition

$$\Phi \circ \psi \colon \mathfrak{Z}_{g-1}^{a} \longrightarrow \mathcal{K}(a)$$

with the linearization map for $\mathfrak{Z}_{g-1}^{a}$. Our assumption $g \geq 3$ implies that $g-1 \geq 1$, so Theorem D says that $\Phi \circ \psi$ is an isomorphism. Since $\psi$ is a surjection, this implies that $\Phi$ takes $\mathfrak{F}[a,a]$ isomorphically to $\mathcal{K}(a)$, as desired.

**Step 2.** *Conclusion (b) holds:* $\mathcal{F}[a,a] = \mathcal{K}_g^a \cap \mathcal{F}[a,a] = \mathcal{K}(a)$.

That $\mathcal{F}[a,a] = \mathcal{K}(a)$ is immediate from (c). Conclusion (a) says that $\widehat{\mathcal{F}}[a,a] = \wedge^2 \mathcal{U}(a)$. Since $\mathcal{K}_g^a \cap \widehat{\mathcal{F}}[a,a]$ is a codimension-1 subspace of $\widehat{\mathcal{F}}[a,a] = \wedge^2\mathcal{U}(a)$ containing $\mathcal{F}[a,a]$ and $\mathcal{K}(a)$ is also a codimension-1 subspace of $\wedge^2\mathcal{U}(a)$, it follows that $\mathcal{F}[a,a] = \mathcal{K}(a)$ equals $\mathcal{K}_g^a \cap \widehat{\mathcal{F}}[a,a]$, as desired. □

17.4. **Identification II.** Let $(a,a')$ be a pair of elements of $H_{\mathbb{Z}}$ such that $\omega(a,a') = 0$ and such that $\{a,a'\}$ is a basis for a rank-2 direct summand of $H_{\mathbb{Z}}$. This latter condition implies that $a \wedge a'$ is a strong isotropic pair. We will call such a pair $(a,a')$ an *isotropic basis*. Our next goal is to identify $\mathfrak{F}[a,a']$.

In a generator $[\![a \wedge x, a' \wedge y]\!]_a$ of $\mathfrak{F}[a,a']$, we have that $x$ is orthogonal to $a'$ and $y$ is orthogonal to $a$. Moreover, $x$ is only well-defined up to multiples of $a$ and $y$ is only well-defined up to multiples of $a'$. This suggests defining

$$\mathcal{V}(a,a') = \langle a' \rangle_{\mathbb{Q}}^{\perp} / \langle a \rangle \quad \text{and} \quad \mathcal{W}(a,a') = \langle a \rangle_{\mathbb{Q}}^{\perp} / \langle a' \rangle.$$

We can embed $\mathcal{V}(a,a')$ and $\mathcal{W}(a,a')$ into $(\wedge^2 H)/\mathbb{Q}$ by taking $x \in \mathcal{V}(a,a')$ to $a \wedge x$ and $y \in \mathcal{W}(a,a')$ to $a' \wedge y$. Identifying $\mathcal{V}(a,a')$ and $\mathcal{W}(a,a')$ with the corresponding subspaces of $(\wedge^2 H)/\mathbb{Q}$, the intersection $\mathcal{V}(a,a') \cap \mathcal{W}(a,a')$ is spanned by $a \wedge a'$. Here $a \wedge a'$ corresponds to $a' \in \mathcal{V}(a,a')$ and $-a \in \mathcal{W}(a,a')$. It follows that as subspaces of $\wedge^2((\wedge^2 H)/\mathbb{Q})$ we have

$$\widehat{\mathcal{F}}[a,a'] = \mathcal{V}(a,a') \wedge \mathcal{W}(a,a') \cong \frac{\mathcal{V}(a,a') \otimes \mathcal{W}(a,a')}{\langle a' \otimes a \rangle}.$$

The symplectic form $\omega$ induces a bilinear pairing

$$\overline{\omega} \colon \mathcal{V}(a,a') \times \mathcal{W}(a,a') \longrightarrow \mathbb{Q}.$$

Let $\mathcal{K}(a,a')$ be the kernel of the map

$$\mathcal{V}(a,a') \otimes \mathcal{W}(a,a') \longrightarrow \mathbb{Q}$$

induced by $\overline{\omega}$. We have $a' \otimes a \in \mathcal{K}(a,a')$, and by the previous paragraph $\mathcal{K}(a,a')/\langle a' \otimes a \rangle$ is a subspace of $\wedge^2((\wedge^2 H)/\mathbb{Q})$. We then have:

**Lemma 17.4.** *Let $g \geq 3$ and let $(a,a')$ be an isotropic basis. Then:*

- *(a)* $\widehat{\mathcal{F}}[a,a'] = \mathcal{V}(a,a') \wedge \mathcal{W}(a,a') \cong (\mathcal{V}(a,a') \otimes \mathcal{W}(a,a'))/\langle a' \otimes a \rangle$*; and*
- *(b)* $\mathcal{F}[a,a'] = \mathcal{K}_g^a \cap \widehat{\mathcal{F}}[a,a'] = \mathcal{K}(a,a')/\langle a' \otimes a \rangle$*; and*
- *(c) the linearization map* $\Phi \colon \mathfrak{K}_g^a \to \wedge^2((\wedge^2 H)/\mathbb{Q})$ *takes* $\mathfrak{F}[a,a']$ *isomorphically onto* $\mathcal{K}(a,a')/\langle a' \otimes a \rangle$.

*Proof.* We noted that (a) held right before the lemma, and (b) follows from (c) just like in the proof of Lemma 17.3. We must prove (c). Since $(a,a')$ is an isotropic basis, we can find a symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $H_{\mathbb{Z}}$ such that $a_1 = a$ and $a_2 = a'$. Define $\mathcal{V}_{\mathbb{Z}} = \langle a_2 \rangle^{\perp} / \langle a_1 \rangle$ and $\mathcal{W}_{\mathbb{Z}} = \langle a_1 \rangle^{\perp} / \langle a_2 \rangle$. We can identify:

- $\mathcal{V}_{\mathbb{Z}}$ with $\langle b_1, a_2, a_3, b_3, \ldots, a_g, b_g \rangle \cong \mathbb{Z}^{2g-2}$; and
- $\mathcal{W}_{\mathbb{Z}}$ with $\langle a_1, b_2, a_3, b_3, \ldots, a_g, b_g \rangle \cong \mathbb{Z}^{2g-2}$.

Under these identifications, the bilinear pairing $\overline{\omega}$ between $\mathcal{V}_{\mathbb{Z}}$ and $\mathcal{W}_{\mathbb{Z}}$ induced by $\omega$ is identified with the bilinear pairing between $\langle b_1, a_2, a_3, b_3, \ldots, a_g, b_g\rangle$ and $\langle a_1, b_2, a_3, b_3, \ldots, a_g, b_g\rangle$ given by $\omega$. Let $\{e_1, \ldots, e_{2g-2}\}$ be the standard basis for $\mathbb{Z}^{2g-2}$ and $\{e_1^*, \ldots, e_{2g-2}^*\}$ be the corresponding dual basis for $(\mathbb{Z}^{2g-1})^*$. Let $\mu_1 \colon (\mathbb{Z}^{2g-2})^* \to \mathcal{V}_{\mathbb{Z}}$ and $\mu_2 \colon \mathbb{Z}^{2g-1} \to \mathcal{W}_{\mathbb{Z}}$ be the isomorphisms defined by

$$\begin{aligned} \mu_1(e_1^*) &= b_1, & \mu_1(e_2^*) &= a_2 \\ \mu_2(e_1) &= -a_1, & \mu_2(e_2) &= b_2 \end{aligned} \tag{17.1}$$

and

$$\begin{aligned} &\mu_1(e_{2i-3}^*) = a_i, \quad \mu_1(e_{2i-2}^*) = b_i && \text{for } 3 \leq i \leq g, \\ &\mu_2(e_{2i-3}) = b_i, \quad \mu_2(e_{2i-2}) = -a_i, && \text{for } 3 \leq i \leq g. \end{aligned}$$

We chose these isomorphisms in part because they ensure that

$$f(x) = \overline{\omega}(\mu_1(f), \mu_2(x)) \quad \text{for all } f \in (\mathbb{Z}^{2g-2})^* \text{ and } x \in \mathbb{Z}^{2g-2}.$$

The precise form of (17.1) will be important in the next paragraph.

Recall that we defined the vector space $\mathfrak{A}'_{2g-2}$ in Definition 6.1. It is generated by elements $[f, v]'_0$ with $f \in (\mathbb{Z}^{2g-2})^*$ and $v \in \mathbb{Z}^{2g-2}$ primitive vectors such that $f(v) = 0$ and such that:

- $f$ is $e_1^*$-standard, which means that the $e_1^*$-coordinate of $f$ lies in $\{-1, 0, 1\}$; and
- $v$ is $e_2$-standard, which means that the $e_2$-coordinate of $v$ lies in $\{-1, 0, 1\}$.

In light of (17.1), this implies that both $a_1 \wedge \mu_1(v)$ and $a_2 \wedge \mu_2(v)$ are special pairs. We can therefore define a map $\psi \colon \mathfrak{A}'_{2g-2} \to \mathfrak{F}[a_1, a_2]$ via the formula

$$\psi([f, v]'_0) = [\![a_1 \wedge \mu_1(f), a_2 \wedge \mu(v)]\!]_a \quad \text{for a generator } [f, v]'_0 \text{ of } \mathfrak{A}'_{2g-2}.$$

This takes relations to relations, and thus gives a well-defined map. We also have:

**Claim.** *The map $\psi$ is surjective.*

*Proof of claim.* Consider a generator $[\![a_1 \wedge x, a_2 \wedge y]\!]_a$ of $\mathfrak{F}[a_1, a_2]$. We must show that $[\![a_1 \wedge x, a_2 \wedge y]\!]_a$ is in the image of $\psi$. We can assume that our generator is nonzero, so $x \neq 0$ and $y \neq 0$. Write $x = \mu_1(f)$ and $y = \mu_2(v)$ with $f \in (\mathbb{Z}^{2g-2})^*$ and $v \in \mathbb{Z}^{2g-2}$. By definition, $f$ is $e_1^*$-standard and $v$ is $e_2$-standard. Write $f = \lambda f'$ and $v = \eta v'$ with $\lambda, \eta \in \mathbb{Z}$ and $f'$ and $v'$ primitive. The elements $f'$ and $v'$ are $e_1^*$- and $e_2$-standard, respectively. We therefore have a generator $[f', v']'_0$ of $\mathfrak{A}'_{2g-2}$, and

$$\begin{aligned} \psi(\lambda\eta[f', v']'_0) &= \lambda\eta[\![a_1 \wedge \mu_1(f'), a_2 \wedge \mu_2(v')]\!]_a = \lambda[\![a_1 \wedge \mu_1(f'), a_2 \wedge \big(\eta\mu_2(v')\big)]\!]_a \\ &= \lambda[\![a_1 \wedge \mu_1(f'), a_2 \wedge y]\!]_a = [\![a_1 \wedge \big(\lambda\mu_1(f')\big), a_2 \wedge y]\!]_a = [\![a_1 \wedge x, a_2 \wedge y]\!]_a, \end{aligned}$$

as desired. □

Since $g \geq 3$, we have $2g - 2 \geq 4$. Theorem B′ thus gives a linearization map $\widetilde{\Phi} \colon \mathfrak{A}'_{2g-2} \to \mathfrak{sl}_{2g-2}(\mathbb{Q})$ that is an isomorphism. Here $\mathfrak{sl}_{2g}(\mathbb{Q})$ is the kernel of the trace map $(\mathbb{Q}^{2g-2})^* \otimes \mathbb{Q}^{2g-2} \to \mathbb{Q}$. Recall that $\mathcal{K}(a_1, a_2)$ is the kernel of the symplectic pairing $\mathcal{V}(a_1, a_2) \otimes \mathcal{W}(a_1, a_2) \to \mathbb{Q}$. Identifying $\mu_1$ and $\mu_2$ with maps $(\mathbb{Q}^{2g-2})^* \to \mathcal{V}(a_1, a_2)$ and $\mathbb{Q}^{2g-2} \to \mathcal{W}(a_1, a_2)$, this all fits into a commutative diagram

$$\begin{array}{ccccc} \mathfrak{A}'_{2g-2} & \xrightarrow[\cong]{\widetilde{\Phi}} & \mathfrak{sl}_{2g-2}(\mathbb{Q}) & \xrightarrow[\cong]{\mu_1 \otimes \mu_2} & \mathcal{K}(a_1, a_2) \\ \Big\downarrow \psi & & & & \Big\downarrow \\ \mathfrak{F}[a_1, a_2] & & \xrightarrow{\quad\Phi\quad} & & \mathcal{K}(a_1, a_2)/\langle a_2 \otimes a_1 \rangle. \end{array}$$

By (17.1), we have $\psi([e_2^*, -e_1]_0') = [\![a_1 \wedge a_2, a_2 \wedge a_1]\!]_a = 0$. Since the isomorphism on the top row of this diagram takes $[e_2^*, -e_1]_0'$ to $a_2 \otimes a_1$, we conclude that the map $\Phi$ on the bottom row is an isomorphism, as desired. □

## 18. Symmetric kernel, alternating version II: the set $S$ and $\mathrm{SymSp}_g$

In the next three sections, we construct a set $S \subset \mathfrak{K}_g^a$ with $\langle S \rangle = \mathfrak{K}_g^a$. This section defines $S$ and studies its symmetries. In §19 we will construct a large number of elements in $\langle S \rangle$, and then finally in §20 we will prove that $\langle S \rangle = \mathfrak{K}_g^a$. Throughout this section, like in the last section we relax our standing assumption that $g \geq 4$ (Assumption 15.2), so our results include explicit genus ranges when they are necessary. See Remark 16.1.

18.1. **The set $S$.** Fix a symplectic basis $\mathcal{B} = \{a_1, b_1, \ldots, a_g, b_g\}$ for $H_{\mathbb{Z}}$. Define $S = S_{12} \cup S_3$, where[21]

$$S_{12} = \bigcup_{\substack{a,a' \in \mathcal{B} \\ \omega(a,a')=0}} \mathfrak{F}[a, a'] \quad \text{and} \quad S_3 = \bigcup_{\substack{1 \leq i,j \leq g \\ i \neq j}} \mathfrak{F}[a_i - b_j, b_i - a_j].$$

Just like we did here, we will write elements of $\langle S_{12} \rangle$ in purple and elements of $\langle S_3 \rangle$ in orange; for instance,

$$[\![a_1 \wedge a_3, a_2 \wedge (a_3 - b_4)]\!]_a \in S_{12},$$
$$[\![(a_1 - b_2) \wedge (a_3 + a_4), (b_1 - a_2) \wedge (b_3 - b_4)]\!]_a \in S_3.$$

The sets $S_{12}$ and $S_3$ are not disjoint, so some elements could be written in either color; for instance,

$$[\![a_1 \wedge (a_2 - b_3), a_1 \wedge (b_2 - a_3)]\!]_a \in S_{12},$$
$$[\![a_1 \wedge (a_2 - b_3), a_1 \wedge (b_2 - a_3)]\!]_a = [\![(a_2 - b_3) \wedge a_1, (b_2 - a_3) \wedge a_1]\!]_a \in S_3.$$

18.2. **Signed symmetric group.** Recall that $\mathrm{SymSp}_g$ consists of all $f \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that for all $x \in \mathcal{B}$, we have either $f(x) \in \mathcal{B}$ or $-f(x) \in \mathcal{B}$. This is a finite group. Associated to each $f \in \mathrm{Sp}_{2g}(\mathbb{Z})$ is a permutation $p$ of $\{1, \ldots, g\}$ such that for all $1 \leq i \leq g$ the pair $(f(a_i), f(b_i))$ is one of the following:

$$(a_{p(i)}, b_{p(i)}), \quad \text{or} \quad (-a_{p(i)}, -b_{p(i)}), \quad \text{or} \quad (b_{p(i)}, -a_{p(i)}), \quad \text{or} \quad (-b_{p(i)}, a_{p(i)}).$$

Our main goal in this section is to prove:

**Lemma 18.1.** *For all $g \geq 3$, the action of $\mathrm{SymSp}_g$ on $\mathfrak{K}_g^a$ takes $\langle S \rangle$ to $\langle S \rangle$.*

The proof of Lemma 18.1 is at the end of this section after some preliminaries.

18.3. **Symmetric group.** Embed the symmetric group $\mathfrak{S}_g$ on $g$ generators into $\mathrm{Sp}_{2g}(\mathbb{Z})$ by letting $p \in \mathfrak{S}_g$ act as $p(a_i) = a_{p(i)}$ and $p(b_i) = b_{p(i)}$ for $1 \leq i \leq g$. The group $\mathfrak{S}_g$ is a subgroup of $\mathrm{SymSp}_g$. We start with:

**Lemma 18.2.** *For all $g \geq 3$, the action of $\mathfrak{S}_g$ on $\mathfrak{K}_g^a$ takes $\langle S \rangle$ to $\langle S \rangle$.*

*Proof.* For $p \in \mathfrak{S}_g$, we have

$$p(\mathfrak{F}[a, a']) = \mathfrak{F}[p(a), p(a')] \subset S_{12} \qquad \text{for } a, a \in \mathcal{B} \text{ with } \omega(a, a') = 0,$$
$$p(\mathfrak{F}[a_i - b_j, b_i - a_j]) = \mathfrak{F}[a_{p(i)} - b_{p(j)}, b_{p(i)} - a_{p(j)}] \subset S_3 \qquad \text{for } 1 \leq i, j \leq g \text{ distinct}.$$

The lemma follows. □

[21]The reason for calling this set $S_{12}$ is that later it will be expressed as the union of sets $S_1$ and $S_2$, which also explains why the other set is $S_3$.

18.4. **Some elements, I.** To extend Lemma 18.2 to $\text{SymSp}_g$, we need to construct some elements in $\langle S \rangle$. We start with:

**Lemma 18.3.** *Let $g \geq 3$. For $1 \leq i \leq g$, we have $\mathfrak{F}[a_i + b_i, a_i + b_i] \subset \langle S \rangle$.*

*Proof.* By Lemma 18.2, we can apply any element of $\mathfrak{S}_g$ to $\mathfrak{F}[a_i+b_i, a_i+b_i]$ without changing whether or not it lies in $\langle S \rangle$. Applying an appropriate such element, we reduce to showing that $\mathfrak{F}[a_1 + b_1, a_1 + b_1] \subset \langle S \rangle$. Following the notation in §17.3, define

$$\mathcal{U} = \langle a_1 + b_1 \rangle_{\mathbb{Q}}^{\perp} / \langle a_1 + b_1 \rangle \cong \langle A \rangle_{\mathbb{Q}} \quad \text{with} \quad A = \{a_2, b_2, \ldots, a_g, b_g\}.$$

We proved in Lemma 17.3 that $\mathfrak{F}[a_1 + b_1, a_1 + b_1]$ is isomorphic to the kernel of the map $\wedge^2 \mathcal{U} \to \mathbb{Q}$ induced by the symplectic form $\omega$. Under this isomorphism, a generator $[\![(a_1 + b_1) \wedge x, (a_1 + b_1) \wedge y]\!]_a$ of $\mathfrak{F}[a_1 + b_1, a_1 + b_1]$ maps to $x \wedge y \in \wedge^2 \mathcal{U}$.

The kernel of $\wedge^2 \mathcal{U} \to \mathbb{Q}$ is spanned by $X = \{x \wedge y \mid x, y \in A, \, \omega(x,y) = 0\}$ and $Y = \{a_i \wedge b_i - a_j \wedge b_j \mid 2 \leq i < j \leq g\}$. Since for $2 \leq i < j \leq g$ we have

$$(a_i - b_j) \wedge (b_i - a_j) = a_i \wedge b_i - a_j \wedge b_j + \text{an element of } \langle X \rangle,$$

we can replace $Y$ by $\{(a_i - b_j) \wedge (b_i - a_j) \mid 2 \leq i < j \leq g\}$. It follows that $\mathfrak{F}[a_1 + b_1, a_1 + b_1]$ is generated by the following elements:

**Case 1.** $[\![(a_1 + b_1) \wedge x, (a_1 + b_1) \wedge y]\!]_a$ *for $x, y \in A$ with $\omega(x,y) = 0$.*

These equal $[\![x \wedge (a_1 + b_1), y \wedge (a_1 + b_1)]\!]_a \in S_{12}$.

**Case 2.** $[\![(a_1 + b_1) \wedge (a_i - b_j), (a_1 + b_1) \wedge (b_i - a_j)]\!]_a$ *for $2 \leq i < j \leq g$.*

These equal $[\![(a_i - b_j) \wedge (a_1 + b_1), (b_i - a_j) \wedge (a_1 + b_1)]\!]_a \in S_3$. ☐

18.5. **Some elements, II.** We next handle the following variants of the elements of $S_3$:

**Lemma 18.4.** *Let $g \geq 3$. For all distinct $1 \leq i, j \leq g$ and $\epsilon, \epsilon' \in \{\pm 1\}$, both $\mathfrak{F}[\epsilon a_i + \epsilon' b_j, \epsilon b_i + \epsilon' a_j]$ and $\mathfrak{F}[\epsilon a_i + \epsilon' a_j, \epsilon b_i - \epsilon' b_j]$ are subsets of $\langle S \rangle$.*

*Proof.* Since $\mathfrak{F}[-,-]$ is not changed when its entries are multiplied by $-1$ (Lemma 17.2), it is enough to prove this for $\epsilon = 1$. Also, using Lemma 18.2 we can multiply our elements by appropriate elements of the symmetric group $\mathfrak{S}_g$ and assume that $(i,j) = (1,2)$. Since we already know that $\mathfrak{F}[a_1 - b_2, b_1 - a_2] \subset S_3$, this reduces us to proving that $\mathfrak{F}[a_1 + b_2, b_1 + a_2]$ and $\mathfrak{F}[a_1 + a_2, b_1 - b_2]$ and $\mathfrak{F}[a_1 - a_2, b_1 + b_2]$ are contained in $\langle S \rangle$. We do this in Lemmas 18.5 and 18.6 and 18.5 below. ☐

**Lemma 18.5.** *Let $g \geq 3$. We have $\mathfrak{F}[a_1 + b_2, b_1 + a_2] \subset \langle S \rangle$.*

*Proof.* Following the notation in §17.4, define

$$\begin{aligned} \mathcal{V} &= \langle b_1 + a_2 \rangle_{\mathbb{Q}}^{\perp} / \langle a_1 + b_2 \rangle \cong \langle A_V \rangle_{\mathbb{Q}} \quad \text{with} \quad A_V = \{b_1, a_2, a_3, b_3, \ldots, a_g, b_g\}, \\ \mathcal{W} &= \langle a_1 + b_2 \rangle_{\mathbb{Q}}^{\perp} / \langle b_1 + a_2 \rangle \cong \langle A_W \rangle_{\mathbb{Q}} \quad \text{with} \quad A_W = \{a_1, b_2, a_3, b_3, \ldots, a_g, b_g\}. \end{aligned}$$

We proved in Lemma 17.4 that $\mathfrak{F}[a_1 + b_2, b_1 + a_2]$ is isomorphic to a quotient of the kernel of the map $\mathcal{V} \otimes \mathcal{W} \to \mathbb{Q}$ induced by the symplectic form $\omega$. Under this isomorphism, a generator $[\![(a_1 + b_2) \wedge x, (b_1 + a_2) \wedge y]\!]_a$ of $\mathfrak{F}[a_1 + b_2, b_1 + a_2]$ maps to $x \otimes y \in \mathcal{V} \otimes \mathcal{W}$.

The kernel of $\mathcal{V} \otimes \mathcal{W} \to \mathbb{Q}$ is spanned by $X \cup Y$ where[22]

$$\begin{aligned} X &= \{x \otimes y \mid x \in A_V, \, y \in A_W, \, \omega(x,y) = 0\}, \\ Y &= \{a_i \otimes b_i + b_j \otimes a_j \mid 3 \leq i, j \leq g\} \\ &\quad \cup \{a_3 \otimes b_3 + b_1 \otimes a_1, a_2 \otimes b_2 + b_3 \otimes a_3\}. \end{aligned}$$

[22]The first part of $Y$ along with the portion of $X$ lying in it spans the kernel of the map $\langle a_3, b_3, \ldots, a_g, b_g \rangle^{\otimes 2} \to \mathbb{Q}$. Indeed, quotienting $\langle a_3, b_3, \ldots, a_g, b_g \rangle^{\otimes 2}$ by the portion of $X$ lying in it results in

$$U = \langle a_3 \otimes b_3, b_3 \otimes a_3, \ldots, a_g \otimes b_g, b_g \otimes a_g \rangle.$$

Since for $1 \leq i, j \leq g$ distinct with either $i, j \geq 3$ or $(i,j) \in \{(3,1),(2,3)\}$ and for $3 \leq k \leq g$ we have

$$(a_i - b_j) \otimes (b_i - a_j) = a_i \otimes b_i + b_j \otimes a_j + \text{an element of } \langle X \rangle,$$
$$(a_k + b_k) \otimes (a_k + b_k) = a_k \otimes b_k + b_k \otimes a_k + \text{an element of } \langle X \rangle,$$

we can replace $Y$ by the set

$$\{(a_i - b_j) \otimes (b_i - a_j) \mid 1 \leq i, j \leq g \text{ distinct, either } i, j \geq 3 \text{ or } (i,j) \in \{(3,1),(2,3)\}\}$$
$$\cup \{(a_i + b_i) \otimes (a_i + b_i) \mid 3 \leq i \leq g\}.$$

From this, we see that $\mathfrak{F}[a_1 + b_2, b_1 + a_2]$ is generated by the following elements:[23]

**Case 1.** *$[\![(a_1 + b_2) \wedge x, (b_1 + a_2) \wedge y]\!]_a$ for $x \in A_V$ and $y \in A_W$ with $\omega(x,y) = 0$.*

These equal $[\![x \wedge (a_1 + b_2), y \wedge (b_1 + a_2)]\!]_a \in S_{12}$.

**Case 2.** *$[\![(a_1 + b_2) \wedge (a_i - b_j), (b_1 + a_2) \wedge (b_i - a_j)]\!]_a$ for $1 \leq i, j \leq g$ distinct with either $i, j \geq 3$ or $(i,j) \in \{(3,1),(2,3)\}$.*

These equal $[\![(a_i - b_j) \wedge (a_1 + b_2), (b_i - a_j) \wedge (b_1 + a_2)]\!]_a \in S_3$.

**Case 3.** *$[\![(a_1 + b_2) \wedge (a_i + b_i), (b_1 + a_2) \wedge (a_i + b_i)]\!]_a$ for $3 \leq i \leq g$.*

These equal $[\![(a_i + b_i) \wedge (a_1 + b_2), (a_i + b_i) \wedge (b_1 + a_2)]\!]_a$, which lie in $\langle S \rangle$ by Lemma 18.3. □

**Lemma 18.6.** *Let $g \geq 3$. We have $\mathfrak{F}[a_1 + a_2, b_1 - b_2] \subset \langle S \rangle$.*

*Proof.* This is similar to Lemma 18.5, so we just sketch the argument. Like in that lemma, define

$$\mathcal{V} = \langle b_1 - b_2 \rangle_{\mathbb{Q}}^{\perp} / \langle a_1 + a_2 \rangle \cong \langle A_V \rangle_{\mathbb{Q}} \quad \text{with} \quad A_V = \{b_1, b_2, a_3, b_3, \ldots, a_g, b_g\},$$
$$\mathcal{W} = \langle a_1 + a_2 \rangle_{\mathbb{Q}}^{\perp} / \langle b_1 - b_2 \rangle \cong \langle A_W \rangle_{\mathbb{Q}} \quad \text{with} \quad A_W = \{a_1, a_2, a_3, b_3, \ldots, a_g, b_g\}.$$

Elements of $\mathfrak{F}[a_1 + a_2, b_1 - b_2]$ correspond to elements of a quotient of the kernel of the map $\mathcal{V} \otimes \mathcal{W} \to \mathbb{Q}$ induced by the symplectic form $\omega$. Using this, just like in Lemma 18.5 we can reduce to the following three cases, each of which is handled just like in Lemma 18.5.

- $[\![(a_1 + a_2) \wedge x, (b_1 - b_2) \wedge y]\!]_a$ for $x \in A_V$ and $y \in A_W$ with $\omega(x,y) = 0$.
- $[\![(a_1 + a_2) \wedge (a_i - b_j), (b_1 - b_2) \wedge (b_i - a_j)]\!]_a$ for $1 \leq i, j \leq g$ distinct with either $i, j \geq 3$ or $(i,j) \in \{(3,1),(3,2)\}$.
- $[\![(a_1 + a_2) \wedge (a_i + b_i), (b_1 - b_2) \wedge (a_i + b_i)]\!]_a$ for $3 \leq i \leq g$. □

**Lemma 18.7.** *Let $g \geq 3$. We have $\mathfrak{F}[a_1 - a_2, b_1 + b_2] \subset \langle S \rangle$.*

*Proof.* This is also similar to Lemma 18.5, so we just sketch the argument. Like in that lemma, define

$$\mathcal{V} = \langle b_1 + b_2 \rangle_{\mathbb{Q}}^{\perp} / \langle a_1 - a_2 \rangle \cong \langle A_V \rangle_{\mathbb{Q}} \quad \text{with} \quad A_V = \{b_1, b_2, a_3, b_3, \ldots, a_g, b_g\},$$
$$\mathcal{W} = \langle a_1 - a_2 \rangle_{\mathbb{Q}}^{\perp} / \langle b_1 + b_2 \rangle \cong \langle A_W \rangle_{\mathbb{Q}} \quad \text{with} \quad A_W = \{a_1, a_2, a_3, b_3, \ldots, a_g, b_g\}.$$

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In $Y$, we have $a_i \otimes b_i + b_i \otimes a_i$ for all $3 \leq i \leq g$, and also for $3 \leq i, j \leq g$ distinct the elements

$$a_i \otimes b_i - a_j \otimes b_j = (a_i \otimes b_i + b_j \otimes a_j) - (a_j \otimes b_j + b_j \otimes a_j),$$
$$b_i \otimes a_i - b_j \otimes a_j = (a_j \otimes b_j + b_i \otimes a_i) - (a_j \otimes b_j + b_j \otimes a_j).$$

Quotienting $U$ by these gives $\mathbb{Q}$, as desired. We will silently use calculations like this in the next two sections.

[23] The elements $[\![(a_1 + b_2) \wedge z, (b_1 + a_2) \wedge w]\!]_a$ listed below each map to one of the generators for the kernel of $\mathcal{V} \otimes \mathcal{W} \to \mathbb{Q}$ we identified. It is important that these are indeed generators of $\mathfrak{F}[a_1 + b_2, b_1 + a_2]$, i.e., that both $(a_1 + b_2) \wedge z$ and $(b_1 + a_2) \wedge w$ are special pairs.

Elements of $\mathfrak{F}[a_1 - a_2, b_1 + b_2]$ correspond to elements of a quotient of the kernel of the map $\mathcal{V} \otimes \mathcal{W} \to \mathbb{Q}$ induced by the symplectic form $\omega$. Using this, just like in Lemma 18.5 we can reduce to the following three cases, each of which is handled just like in Lemma 18.5.

- $[\![(a_1 - a_2) \wedge x, (b_1 + b_2) \wedge y]\!]_a$ for $x \in A_V$ and $y \in A_W$ with $\omega(x, y) = 0$.
- $[\![(a_1 - a_2) \wedge (a_i - b_j), (b_1 + b_2) \wedge (b_i - a_j)]\!]_a$ for $1 \leq i, j \leq g$ distinct with either $i, j \geq 3$ or $(i, j) \in \{(3, 1), (3, 2)\}$.
- $[\![(a_1 - a_2) \wedge (a_i + b_i), (b_1 + b_2) \wedge (a_i + b_i)]\!]_a$ for $3 \leq i \leq g$. □

18.6. **Closure under signed symmetric group.** We now prove Lemma 18.1, whose statement we recall:

**Lemma 18.1.** *For all $g \geq 3$, the action of $\mathrm{SymSp}_g$ on $\mathfrak{K}_g^a$ takes $\langle S \rangle$ to $\langle S \rangle$.*

*Proof.* It is enough to prove that $\mathrm{SymSp}_g$ takes both $S_{12}$ and $S_3$ into $\langle S \rangle$:

**Claim 1.** *For $f \in \mathrm{SymSp}_g$ and $a, a' \in \mathcal{B}$ with $\omega(a, a') = 0$, we have $f(\mathfrak{F}[a, a']) \subset \langle S \rangle$.*

Write $f(a) = \epsilon a_0$ and $f(a') = \epsilon' a_0'$ with $a_0, a_0' \in \mathcal{B}$ and $\epsilon, \epsilon \in \{\pm 1\}$. We have $\omega(a_0, a_0') = 0$, and by Lemma 17.2 we have

$$f(\mathfrak{F}[a, a']) = \mathfrak{F}[\epsilon a_0, \epsilon' a_0'] = \mathfrak{F}[a_0, a_0'] \subset S_{12}.$$

**Claim 2.** *For $f \in \mathrm{SymSp}_g$ and $1 \leq i < j \leq g$, we have $f(\mathfrak{F}[a_i - b_j, b_i - a_j]) \subset \langle S \rangle$.*

For some distinct $1 \leq i_0, j_0 \leq g$ we have

$$f(a_i), f(b_i) \in \{\pm a_{i_0}, \pm b_{i_0}\} \quad \text{and} \quad f(a_j), f(b_j) \in \{\pm a_{j_0}, \pm b_{j_0}\}.$$

Since

$$\omega(f(a_i - b_i), f(b_i - a_j)) = \omega(a_i - b_i, b_i - a_j) = 0,$$

it follows that for some distinct $1 \leq i_0, j_0 \leq g$ and $\epsilon, \epsilon' \in \{\pm 1\}$ the set $f(\mathfrak{F}[a_i - b_j, b_i - a_j])$ equals either

$$\mathfrak{F}[\epsilon a_{i_0} + \epsilon' b_{j_0}, \epsilon b_{i_0} + \epsilon' a_{j_0}] \quad \text{or} \quad \mathfrak{F}[\epsilon a_{i_0} + \epsilon' a_{j_0}, \epsilon b_{i_0} - \epsilon' b_{j_0}].$$

Lemma 18.4 says that both of these are contained in $\langle S \rangle$. □

One useful consequence of Lemma 18.1 is the following generalization of Lemma 18.8:

**Lemma 18.8.** *Let $g \geq 3$. For $1 \leq i \leq g$ and $\epsilon \in \{\pm 1\}$, we have $\mathfrak{F}[a_i + \epsilon b_i, a_i + \epsilon b_i] \subset \langle S \rangle$.*

*Proof.* The group $\mathrm{SymSp}_g$ can take $a_i + b_i$ to $a_i - b_i$ by mapping $a_i$ to $-b_i$ and $b_i$ to $a_i$. From this and Lemma 18.1, the lemma reduces to Lemma 18.3. □

## 19. Symmetric kernel, alternating version III: eight elements

We now re-impose our standing assumption $g \geq 4$ (Assumption 15.2), which will remain in place until we say otherwise. We continue using all the notation from §18. Our goal in this section is to prove eight lemmas saying that certain elements lie in $\langle S \rangle$. They might appear random, but they are exactly[24] the ones needed in the next section (§20) to prove that $\langle S \rangle = \mathfrak{K}_g^a$, and as motivation a reader might first consult that proof.

*Remark* 19.1. We apologize for the repetitiveness of the proofs of our lemmas. They all follow the same pattern, but each has small twists and it is important to prove them in the right order so that earlier lemmas can be invoked during the proofs of later ones. □

[24] Except for two that are needed for the proofs of ones needed in the next section.

19.1. **Two vs one.** Our first three lemmas are about $\mathfrak{F}[a, a']$ where $a$ uses two generators and $a'$ is a single generator. We remark that during our proofs we will frequently silently be using the fact that $g \geq 4$ (Assumption 15.2).

**Lemma 19.2** (Two vs one, I)**.** *For all $x \in \mathcal{B} \setminus \{a_1, b_1\}$, both $\mathfrak{F}[a_1 + b_1, x]$ and $\mathfrak{F}[a_1 - b_1, x]$ are subsets of $\langle S \rangle$.*

*Proof.* Using Lemma 18.1, we can apply an appropriate element of $\mathrm{SymSp}_g$ and reduce ourselves to proving that $\mathfrak{F}[a_1 + b_1, a_2] \subset \langle S \rangle$. Following the notation in §17.4, define

$$\mathcal{V} = \langle a_2 \rangle_{\mathbb{Q}}^{\perp} / \langle a_1 + b_1 \rangle \cong \langle A_V, A_U \rangle_{\mathbb{Q}} \quad \text{with} \quad A_V = \{a_1, a_2, a_3, b_3, \ldots, a_g, b_g\},$$
$$\mathcal{W} = \langle a_1 + b_1 \rangle_{\mathbb{Q}}^{\perp} / \langle a_2 \rangle \cong \langle A_W, A_U \rangle_{\mathbb{Q}} \quad \text{with} \quad A_W = \{a_1 + b_1, b_2, a_3, b_3, \ldots, a_g, b_g\}.$$

We proved in Lemma 17.4 that $\mathfrak{F}[a_1 + b_1, a_2]$ is isomorphic to a quotient of the kernel of the map $\mathcal{V} \otimes \mathcal{W} \to \mathbb{Q}$ induced by the symplectic form $\omega$. Under this isomorphism, a generator $[\![(a_1 + b_1) \wedge x, a_2 \wedge y]\!]_a$ of $\mathfrak{F}[a_1 + b_1, a_2]$ maps to $x \otimes y \in \mathcal{V} \otimes \mathcal{W}$.

The kernel of $\mathcal{V} \otimes \mathcal{W} \to \mathbb{Q}$ is spanned by $X \cup Y$ where[25]

$$X = \{x \otimes y \mid x \in A_V,\, y \in A_W,\, \omega(x, y) = 0\},$$
$$Y = \{a_i \otimes b_i + b_j \otimes a_j \mid 3 \leq i, j \leq g\}$$
$$\cup \{a_2 \otimes b_2 - a_1 \otimes (a_1 + b_1), a_2 \otimes b_2 + b_3 \otimes a_3\}.$$

Since for $1 \leq i, j \leq g$ distinct with either $i, j \geq 3$ or $(i, j) = (2, 3)$ and for $3 \leq k \leq g$ we have

$$(a_i - b_j) \otimes (b_i - a_j) = a_i \otimes b_i + b_j \otimes a_j + \text{an element of } \langle X \rangle,$$
$$(a_k + b_k) \otimes (a_k + b_k) = a_k \otimes b_k + b_k \otimes a_k + \text{an element of } \langle X \rangle,$$
$$(a_1 - a_2) \otimes (a_1 + b_1 + b_2) = a_1 \otimes (a_1 + b_1) - a_2 \otimes b_2 + \text{an element of } \langle X \rangle,$$

we can replace $Y$ by the set

$$\{(a_i - b_j) \otimes (b_i - a_j) \mid 1 \leq i, j \leq g \text{ distinct, either } i, j \geq 3 \text{ or } (i, j) = (2, 3)\}$$
$$\cup \{(a_i + b_i) \otimes (a_i + b_i) \mid 3 \leq i \leq g\}$$
$$\cup \{(a_1 - a_2) \otimes (a_1 + b_1 + b_2)\}.$$

From this, we see that $\mathfrak{F}[a_1 + b_1, a_2]$ is generated by the following elements:

**Case 1.** $[\![(a_1 + b_1) \wedge x, a_2 \wedge y]\!]_a$ *for $x \in A_V$ and $y \in A_W$ with $\omega(x, y) = 0$.*

These equal $-[\![x \wedge (a_1 + b_1), a_2 \wedge y]\!]_a \in S_{12}$.

**Case 2.** $[\![(a_1 + b_1) \wedge (a_i - b_j), a_2 \wedge (b_i - a_j)]\!]_a$ *with $1 \leq i, j \leq g$ distinct and either $i, j \geq 3$ or $(i, j) = (2, 3)$.*

These equal $[\![(a_i - b_j) \wedge (a_1 + b_1), (b_i - a_j) \wedge a_2]\!]_a \in S_3$.

**Case 3.** $[\![(a_1 + b_1) \wedge (a_i + b_i), a_2 \wedge (a_i + b_i)]\!]_a$ *with $3 \leq i \leq g$.*

These equal $[\![(a_i + b_i) \wedge (a_1 + b_1), (a_i + b_i) \wedge a_2]\!]_a$, which lie in $\langle S \rangle$ by Lemma 18.3.

**Case 4.** $[\![(a_1 + b_1) \wedge (a_2 - a_1), a_2 \wedge (a_1 + b_1 + b_2)]\!]_a$.

[25] See the footnotes in the proof of Lemma 18.5 for more on this. The elements are carefully chosen such that all the elements $[\![(a_1 + b_1) \wedge z, a_2 \wedge w]\!]_a$ that appears in Cases 1 – 4 below are actually generators of $\mathfrak{F}[a_1 + b_1, a_2]$, i.e., both $(a_1 + b_1) \wedge z$ and $a_2 \wedge w$ are special pairs. The most delicate choice here is $a_2 \otimes b_2 - a_1 \otimes (a_1 + b_1)$, which is also chosen to make the calculation in Case 4 easier.

Since $\{a_1+b_1, a_2-a_1, a_2, a_1+b_1+b_2, a_3, b_3, \ldots, a_g, b_g\}$ is a symplectic basis for $H_{\mathbb{Z}}$, in $(\wedge^2 H)/\mathbb{Q}$ we have

$$(a_1+b_1)\wedge(a_2+a_1)+a_2\wedge(a_1+b_1+b_2)+a_3\wedge b_3+\cdots+a_g\wedge b_g=0.$$

Solving for $a_2\wedge(a_1+b_1+b_2)$ and plugging the result into the second entry of our element, we see that $[\![(a_1+b_1)\wedge(a_2-a_1), a_2\wedge(a_1+b_1+b_2)]\!]_a$ equals

$$\begin{aligned}&-[\![(a_1+b_1)\wedge(a_2-a_1),(a_1+b_1)\wedge(a_2-a_1)]\!]_a-\sum\nolimits_{i=3}^{g}[\![(a_1+b_1)\wedge(a_2-a_1),a_i\wedge b_i]\!]_a\\ =&-\sum\nolimits_{i=3}^{g}\left([\![a_1\wedge a_2,a_i\wedge b_i]\!]_a+[\![b_1\wedge a_2,a_i\wedge b_i]\!]_a-[\![b_1\wedge a_1,a_i\wedge b_i]\!]_a\right)\in\langle S_{12}\rangle.\end{aligned}$$ □

**Lemma 19.3** (Two vs one, II). *Both $\mathfrak{F}[a_1+b_2, b_2]$ and $\mathfrak{F}[a_2+b_1, b_1]$ are subsets of $\langle S\rangle$.*

*Proof.* The subsets differ by an element of $\mathrm{SymSp}_g$, so by Lemma 18.1 it is enough to prove that $\mathfrak{F}[a_1+b_2, b_2]\subset\langle S\rangle$. Following the notation in §17.4, define

$$\begin{aligned}\mathcal{V}=&\langle b_2\rangle_{\mathbb{Q}}^{\perp}/\langle a_1+b_2\rangle\cong\langle A_V\rangle_{\mathbb{Q}}\quad\text{with}\quad A_V=\{b_1,b_2,a_3,b_3,\ldots,a_g,b_g\},\\ \mathcal{W}=&\langle a_1+b_2\rangle_{\mathbb{Q}}^{\perp}/\langle b_2\rangle\cong\langle A_W\rangle_{\mathbb{Q}}\quad\text{with}\quad A_W=\{a_1,b_1+a_2,a_3,b_3,\ldots,a_g,b_g\}.\end{aligned}$$

We proved in Lemma 17.4 that $\mathfrak{F}[a_1+b_2, b_2]$ is isomorphic to a quotient of the kernel of the map $\mathcal{V}\otimes\mathcal{W}\to\mathbb{Q}$ induced by the symplectic form $\omega$. Under this isomorphism, a generator $[\![(a_1+b_2)\wedge x, b_2\wedge y]\!]_a$ of $\mathfrak{F}[a_1+b_2, b_2]$ maps to $x\otimes y\in\mathcal{V}\otimes\mathcal{W}$.

The kernel of $\mathcal{V}\otimes\mathcal{W}\to\mathbb{Q}$ is spanned by $X\cup Y$ where

$$\begin{aligned}X=&\{x\otimes y\mid x\in A_V,\, y\in A_W,\, \omega(x,y)=0\},\\ Y=&\{a_i\otimes b_i+b_j\otimes a_j\mid i,j\geq 3\}\\ &\cup\{a_3\otimes b_3+b_1\otimes a_1, b_1\otimes a_1-b_2\otimes(b_1+a_2)\}.\end{aligned}$$

Just like in the proof of Lemma 19.2, we can replace $Y$ by the set

$$\begin{aligned}&\{(a_i-b_j)\otimes(b_i-a_j)\mid 1\leq i,j\leq g\text{ distinct, either }i,j\geq 3\text{ or }(i,j)=(3,1)\}\\ &\cup\{(a_i+b_i)\otimes(a_i+b_i)\mid 3\leq i\leq g\}\\ &\cup\{(b_1-b_2)\otimes(a_1+b_1+a_2)\}.\end{aligned}$$

From this, we see that $\mathfrak{F}[a_1+b_2, b_2]$ is generated by the following elements:

**Case 1.** *$[\![(a_1+b_2)\wedge x, b_2\wedge y]\!]_a$ for $x\in A_V$ and $y\in A_W$ with $\omega(x,y)=0$.*

These equal $-[\![x\wedge(a_1+b_2), b_2\wedge y]\!]_a\in S_{12}$.

**Case 2.** *$[\![(a_1+b_2)\wedge(a_i-b_j), b_2\wedge(b_i-a_j)]\!]_a$ for $1\leq i,j\leq g$ distinct with either $i,j\geq 3$ or $(i,j)=(3,1)$.*

These equal $[\![(a_i-b_j)\wedge(a_1+b_2),(b_i-a_j)\wedge b_2]\!]_a\in S_3$.

**Case 3.** *$[\![(a_1+b_2)\wedge(a_i+b_i), b_2\wedge(a_i+b_i)]\!]_a$ for $3\leq i\leq g$.*

These equal $[\![(a_i+b_i)\wedge(a_1+b_2),(a_i+b_i)\wedge b_2]\!]_a$, which lie in $\langle S\rangle$ by Lemma 18.3.

**Case 4.** *$[\![(a_1+b_2)\wedge(b_1-b_2), b_2\wedge(a_1+b_1+a_2)]\!]_a$.*

This equals $[\![(a_1+b_1)\wedge(b_1-b_2), b_2\wedge(a_1+b_1+a_2)]\!]_a$, which lies in $\langle S\rangle$ by Lemma 19.2. □

**Lemma 19.4** (Two vs one, III). *The following hold:*

- *For $a'\in\mathcal{B}\setminus\{a_1,b_1,a_2,b_2\}$, we have $\mathfrak{F}[a_1+b_2, a']\subset\langle S\rangle$.*
- *For $a'\in\mathcal{B}\setminus\{a_1,a_2,a_2,b_2\}$, we have $\mathfrak{F}[a_2+b_1, a']\subset\langle S\rangle$.*

*Proof.* The two sets differ by an element of $\mathrm{SymSp}_g$, so by Lemma 18.1 it is enough to deal with $\mathfrak{F}[a_1+b_2, a']$. Applying a further element of $\mathrm{SymSp}_g$, we can reduce to the case $a' = a_3$, i.e., to $\mathfrak{F}[a_1+b_2, a_3]$. In fact, to simplify our notation we will apply yet another element of $\mathrm{SymSp}_g$ and transform our goal into proving that $\mathfrak{F}[a_1+a_3, a_2] \subset \langle S \rangle$. Following the notation in §17.4, define

$$\mathcal{V} = \langle a_2 \rangle_{\mathbb{Q}}^{\perp} / \langle a_1 + a_3 \rangle \cong \langle A_V \rangle_{\mathbb{Q}} \quad \text{with} \quad A_V = \{b_1, a_1, a_2, b_3, a_4, b_4, \ldots, a_g, b_g\},$$
$$\mathcal{W} = \langle a_1 + a_3 \rangle_{\mathbb{Q}}^{\perp} / \langle a_2 \rangle \cong \langle A_W \rangle_{\mathbb{Q}} \quad \text{with} \quad A_W = \{a_1, b_1 - b_3, b_2, a_3, a_4, b_4, \ldots, a_g, b_g\}.$$

We proved in Lemma 17.4 that $\mathfrak{F}[a_1+a_3, a_2]$ is isomorphic to a quotient of the kernel of the map $\mathcal{V} \otimes \mathcal{W} \to \mathbb{Q}$ induced by the symplectic form $\omega$. Under this isomorphism, a generator $[\![ (a_1+a_3) \wedge x, a_2 \wedge y ]\!]_a$ of $\mathfrak{F}[a_1+a_3, a_2]$ maps to $x \otimes y \in \mathcal{V} \otimes \mathcal{W}$.

The kernel of $\mathcal{V} \otimes \mathcal{W} \to \mathbb{Q}$ is spanned by $X \cup Y$ where

$$\begin{aligned} X &= \{x \otimes y \mid x \in A_V,\, y \in A_W,\, \omega(x,y) = 0\}, \\ Y &= \{a_i \otimes b_i + b_j \otimes a_j \mid 4 \le i,j \le g\} \\ &\quad \cup \{a_4 \otimes b_4 + b_1 \otimes a_1, b_1 \otimes a_1 + a_1 \otimes (b_1 - b_3), a_2 \otimes b_2 + b_4 \otimes a_4, a_4 \otimes b_4 + b_3 \otimes a_3\}. \end{aligned}$$

Just like in the proof of Lemma 19.2, we can replace $Y$ by

$$\begin{aligned} &\{(a_i - b_j) \otimes (b_i - a_j) \mid 1 \le i,j \le g \text{ distinct, either } i,j \ge 4 \text{ or } (i,j) \in \{(4,1),(2,4),(4,3)\}\} \\ &\cup \{(a_i + b_i) \otimes (a_i + b_i) \mid 4 \le i \le g\} \\ &\cup \{(a_1 + b_1) \otimes (a_1 + b_1 - b_3)\}. \end{aligned}$$

From this, we see that $\mathfrak{F}[a_1+a_3, a_2]$ is generated by the following elements:

**Case 1.** $[\![ (a_1+a_3) \wedge x, a_2 \wedge y ]\!]_a$ *for* $x \in A_V$ *and* $y \in A_W$ *with* $\omega(x,y) = 0$.

These equal $-[\![ x \wedge (a_1+a_3), a_2 \wedge y ]\!]_a \in S_{12}$.

**Case 2.** $[\![ (a_1+a_3) \wedge (a_i - b_j), a_2 \wedge (b_i - a_j) ]\!]_a$ *for* $1 \le i,j \le g$ *distinct with either* $i,j \ge 4$ *or* $(i,j) \in \{(4,1),(2,4),(4,3)\}$.

These equal $[\![ (a_i - b_j) \wedge (a_1+a_3), (b_i - a_j) \wedge a_2 ]\!]_a \in S_3$.

**Case 3.** $[\![ (a_1+a_3) \wedge (a_i + b_i), a_2 \wedge (a_i + b_i) ]\!]_a$ *for* $4 \le i \le g$.

These equal $[\![ (a_i + b_i) \wedge (a_1+a_3), (a_i + b_i) \wedge a_2 ]\!]_a$, which lie in $\langle S \rangle$ by Lemma 18.3.

**Case 4.** $[\![ (a_1+a_3) \wedge (a_1 + b_1), a_2 \wedge (a_1 + b_1 - b_3) ]\!]_a$.

This equals $-[\![ (a_1 + b_1) \wedge (a_1+a_3), a_2 \wedge (a_1 + b_1 - b_3) ]\!]_a$, which lies in $\langle S \rangle$ by Lemma 19.2. □

19.2. **Two vs two.** Our next three lemmas are about $\mathfrak{F}[a, a']$ where both $a$ and $a'$ involve two generators. Note that we already proved many such results in Lemma 18.4.

**Lemma 19.5** (Two vs two, I)**.** *Both* $\mathfrak{F}[a_1+b_2, a_1+b_2]$ *and* $\mathfrak{F}[a_2+b_1, a_2+b_1]$ *are subsets of* $\langle S \rangle$.

*Proof.* The pairs $(a_1+b_2, a_1+b_2)$ and $(a_2+b_1, a_2+b_1)$ differ by an element of $\mathrm{SymSp}_g$, so by Lemma 18.1 it is enough to prove that $\mathfrak{F}[a_1+b_2, a_1+b_2] \subset \langle S \rangle$. Following the notation in §17.3, define

$$\mathcal{U} = \langle a_1 + b_2 \rangle_{\mathbb{Q}}^{\perp} / \langle a_1 + b_2 \rangle \cong \langle A \rangle_{\mathbb{Q}} \quad \text{with} \quad A = \{b_1 + a_2, b_2, a_3, b_3, \ldots, a_g, b_g\}.$$

We proved in Lemma 17.3 that $\mathfrak{F}[a_1+b_2, a_1+b_2]$ is isomorphic to the kernel of the map $\wedge^2 \mathcal{U} \to \mathbb{Q}$ induced by the symplectic form $\omega$. Under this isomorphism, a generator $[\![ (a_1+b_2) \wedge x, (a_1+b_2) \wedge y ]\!]_a$ of $\mathfrak{F}[a_1+b_2, a_1+b_2]$ maps to $x \wedge y \in \wedge^2 \mathcal{U}$.

The kernel of $\wedge^2\mathcal{U} \to \mathbb{Q}$ is spanned by $X \cup Y$ with

$$X = \{x \wedge y \mid x, y \in A \text{ distinct}, \omega(x,y) = 0\},$$
$$Y = \{a_i \wedge b_i - a_j \wedge b_j \mid 3 \leq i < j \leq g\} \cup \{(b_1 + a_2) \wedge b_2 - a_3 \wedge b_3\}.$$

Just like in the proof of Lemma 19.2, we can replace $Y$ by

$$\{(a_i - b_j) \wedge (b_i - a_j) \mid 3 \leq i < j \leq g\} \cup \{(b_1 + a_2 + b_3) \wedge (b_2 + a_3)\}.$$

It follows that $\mathfrak{F}[a_1 + b_2, a_1 + b_2]$ is generated by the following elements:

**Case 1.** $[\![(a_1 + b_2) \wedge x, (a_1 + b_2) \wedge y]\!]_a$ *for* $x, y \in A$ *with* $\omega(x,y) = 0$.

If $x, y \in \{b_2, a_3, b_3, \ldots, a_g, b_g\}$, then

$$[\![(a_1 + b_2) \wedge x, (a_1 + b_2) \wedge y]\!]_a = [\![x \wedge (a_1 + b_2), y \wedge (a_1 + b_2)]\!]_a \in S_{12}.$$

If instead one of $x$ and $y$ equals $b_1 + a_2$, then swapping $x$ and $y$ if necessary we can assume that $y = b_1 + a_2$. In this case, using Lemma 18.4 we have

$$[\![(a_1 + b_2) \wedge x, (a_1 + b_2) \wedge (b_1 + a_2)]\!]_a = -[\![(a_1 + b_2) \wedge x, (b_1 + a_2) \wedge (a_1 + b_2)]\!]_a \in \langle S \rangle.$$

**Case 2.** $[\![(a_1 + b_2) \wedge (a_i - b_j), (a_1 + b_2) \wedge (b_i - a_j)]\!]_a$ *for* $3 \leq i < j \leq g$.

These equal $[\![(a_i - b_j) \wedge (a_1 + b_2), (b_i - a_j) \wedge (a_1 + b_2)]\!]_a \in S_3$.

**Case 3.** $[\![(a_1 + b_2) \wedge (b_1 + a_2 + b_3), (a_1 + b_2) \wedge (b_2 + a_3)]\!]_a$.

This element equals

$$[\![a_1 \wedge (b_1 + a_2 + b_3), (a_1 + b_2) \wedge (b_2 + a_3)]\!]_a + [\![b_2 \wedge (b_1 + a_2 + b_3), (a_1 + b_2) \wedge (b_2 + a_3)]\!]_a. \tag{19.1}$$

This is the sum of an element of $\mathfrak{F}[a_1, a_1 + b_2] = \mathfrak{F}[a_1 + b_2, a_1]$ and an element of $\mathfrak{F}[b_2, a_1 + b_2] = \mathfrak{F}[a_1 + b_2, b_2]$. Lemma 19.3 says that $\mathfrak{F}[a_1 + b_2, b_2] \subset \langle S \rangle$. The set $\mathfrak{F}[a_1 + b_2, a_1]$ differs from $\mathfrak{F}[a_1 + b_2, b_2]$ by an element of $\mathrm{SymSp}_g$, so by Lemma 18.1 it also lies in $\langle S \rangle$. We conclude that (19.1) also lies in $\langle S \rangle$, as desired. □

The next two lemmas are not needed in the next section, but will be invoked during proofs later in this section.

**Lemma 19.6** (Two vs two, II)**.** *We have* $\mathfrak{F}[a_1 + a_2, a_1 + a_3] \subset \langle S \rangle$.

*Proof.* Following the notation in §17.4, define

$$\mathcal{V} = \langle a_1 + a_3 \rangle_{\mathbb{Q}}^{\perp} / \langle a_1 + a_2 \rangle \cong \langle A_V \rangle_{\mathbb{Q}} \quad \text{with} \quad A_V = \{a_1, b_1 - b_3, b_2, a_3, a_4, b_4, \ldots, a_g, b_g\},$$
$$\mathcal{W} = \langle a_1 + a_2 \rangle_{\mathbb{Q}}^{\perp} / \langle a_1 + a_3 \rangle \cong \langle A_W \rangle_{\mathbb{Q}} \quad \text{with} \quad A_W = \{b_1 - b_2, a_1, a_2, b_3, a_4, b_4, \ldots, a_g, b_g\}.$$

We proved in Lemma 17.4 that $\mathfrak{F}[a_1 + a_2, a_1 + a_3]$ is isomorphic to a quotient of the kernel of the map $\mathcal{V} \otimes \mathcal{W} \to \mathbb{Q}$ induced by the symplectic form $\omega$. Under this isomorphism, a generator $[\![(a_1 + a_2) \wedge x, (a_1 + a_3) \wedge y]\!]_a$ of $\mathfrak{F}[a_1 + a_2, a_1 + a_3]$ maps to $x \otimes y \in \mathcal{V} \otimes \mathcal{W}$.

The kernel of $\mathcal{V} \otimes \mathcal{W} \to \mathbb{Q}$ is spanned by $X \cup Y$ where

$$X = \{x \otimes y \mid x \in A_V, \, y \in A_W, \, \omega(x,y) = 0\},$$
$$Y = \{a_i \otimes b_i + b_j \otimes a_j \mid 4 \leq i, j \leq g\}$$
$$\cup \{a_1 \otimes (b_1 - b_2) + b_2 \otimes a_2, a_3 \otimes b_3 + (b_1 - b_3) \otimes a_1,$$
$$a_4 \otimes b_4 + b_2 \otimes a_2, a_3 \otimes b_3 + b_4 \otimes a_4\}.$$

Just like in the proof of Lemma 19.2, we can replace $Y$ by

$$\{(a_i - b_j) \otimes (b_i - a_j) \mid 1 \leq i, j \leq g \text{ distinct, either } i, j \geq 4 \text{ or } (i,j) \in \{(4,2),(3,4)\}\}$$
$$\cup \{(a_i + b_i) \otimes (a_i + b_i) \mid 4 \leq i \leq g\}$$
$$\cup \{(a_1 + b_2) \otimes (b_1 + a_2 - b_2), (b_1 + a_3 - b_3) \otimes (a_1 + b_3)\}.$$

From this, we see that $\mathfrak{F}[a_1+a_2, a_1+a_3]$ is generated by the following elements:

**Case 1.** $[\![(a_1+a_2)\wedge x, (a_1+a_3)\wedge y]\!]_a$ *for* $x \in A_V$ *and* $y \in A_W$ *with* $\omega(x,y)=0$.

There are three cases:

- If $y = b_1 - b_2$, then this equals $-[\![(a_1+a_2)\wedge x, (b_1-b_2)\wedge(a_1+a_3)]\!]_a$, which lies in $\langle S\rangle$ by Lemma 18.4.
- If $x = b_1 - b_3$, then this equals $[\![(a_1+a_3)\wedge y, (b_1-b_3)\wedge(a_1+a_2)]\!]_a$, which lies in $\langle S\rangle$ by Lemma 18.4.
- If neither equality holds, then this equals $[\![x\wedge(a_1+a_2), y\wedge(a_1+a_3)]\!]_a \in S_{12}$.

**Case 2.** $[\![(a_1+a_2)\wedge(a_i-b_j), (a_1+a_3)\wedge(b_i-a_j)]\!]_a$ *for* $1 \leq i,j \leq g$ *distinct with either* $i,j \geq 4$ *or* $(i,j) \in \{(4,2),(3,4)\}$.

These equal $[\![(a_i-b_j)\wedge(a_1+a_2), (b_i-a_j)\wedge(a_1+a_3)]\!]_a \in S_3$.

**Case 3.** $[\![(a_1+a_2)\wedge(a_i+b_i), (a_1+a_3)\wedge(a_i+b_i)]\!]_a$ *for* $4 \leq i \leq g$.

These equal $[\![(a_i+b_i)\wedge(a_1+a_2), (a_i+b_i)\wedge(a_1+a_3)]\!]_a$, which lie in $\langle S\rangle$ by Lemma 18.3.

**Case 4.** *The following elements:*

*(a)* $[\![(a_1+a_2)\wedge(a_1+b_2), (a_1+a_3)\wedge(b_1+a_2-b_2)]\!]_a$
*(a)* $[\![(a_1+a_2)\wedge(b_1+a_3-b_3), (a_1+a_3)\wedge(a_1+b_3)]\!]_a$

The element in (a) equals

$$[\![(a_1+a_2)\wedge(a_1+b_2), a_1\wedge(b_1+a_2-b_2)]\!]_a + [\![(a_1+a_2)\wedge(a_1+b_2), a_3\wedge(b_1+a_2-b_2)]\!]_a \tag{19.2}$$

The first term lies in $\mathfrak{F}[a_1+a_2, a_1]$, which differs from $\mathfrak{F}[a_1+b_2, b_2]$ by an element of $\mathrm{SymSp}_g$. Lemma 19.3 says that $\mathfrak{F}[a_1+b_2, b_2] \subset \langle S\rangle$, so by Lemma 18.1 the set $\mathfrak{F}[a_1+a_2, a_1]$ lies in $\langle S\rangle$ too. The same argument (but using Lemma 19.4 instead of Lemma 19.3) shows that the second term in (19.2) also lies in $\langle S\rangle$, so (19.2) lies in $\langle S\rangle$.

The element in (b) equals

$$[\![a_1\wedge(b_1+a_3-b_3), (a_1+a_3)\wedge(a_1+b_3)]\!]_a + [\![a_2\wedge(b_1+a_3-b_3), (a_1+a_3)\wedge(a_1+b_3)]\!]_a.$$

Using the fact that $\mathfrak{F}[-,-]$ is symmetric in its inputs, this can be shown to lie in $\langle S\rangle$ just like in the previous paragraph. □

**Lemma 19.7** (Two vs two, III)**.** *We have* $\mathfrak{F}[a_1+a_2, a_3+a_4] \subset \langle S\rangle$.

*Proof.* Following the notation in §17.4, define

$$\mathcal{V} = \langle a_3+a_4\rangle_{\mathbb{Q}}^{\perp}/\langle a_1+a_2\rangle \cong \langle A_V\rangle_{\mathbb{Q}} \text{ with } A_V = \{a_1, b_1, b_2, a_3, a_4, b_3-b_4, a_5, b_5, \ldots, a_g, b_g\},$$
$$\mathcal{W} = \langle a_1+a_2\rangle_{\mathbb{Q}}^{\perp}/\langle a_3+a_4\rangle \cong \langle A_W\rangle_{\mathbb{Q}} \text{ with } A_W = \{b_1-b_2, a_1, a_2, b_3, b_4, a_3, a_5, b_5, \ldots, a_g, b_g\}.$$

We proved in Lemma 17.4 that $\mathfrak{F}[a_1+a_2, a_3+a_4]$ is isomorphic to a quotient of the kernel of the map $\mathcal{V}\otimes\mathcal{W} \to \mathbb{Q}$ induced by the symplectic form $\omega$. Under this isomorphism, a generator $[\![(a_1+a_2)\wedge x, (a_3+a_4)\wedge y]\!]_a$ of $\mathfrak{F}[a_1+a_2, a_3+a_4]$ maps to $x\otimes y \in \mathcal{V}\otimes\mathcal{W}$.

Recall that $g \geq 4$ (Assumption 15.2). The argument is slightly different when $g=4$ and when $g \geq 5$. Assume first that $g=4$. The kernel of $\mathcal{V}\otimes\mathcal{W}\to\mathbb{Q}$ is spanned by $X \cup Y_4$ where

$$\begin{aligned} X &= \{x\otimes y \mid x \in A_V,\ y \in A_W,\ \omega(x,y)=0\},\\ Y_4 &= \{a_1\otimes(b_1-b_2) + b_1\otimes a_1, b_1\otimes a_1 - b_2\otimes a_2, a_3\otimes b_3 + b_2\otimes a_2,\\ &\qquad a_3\otimes b_3 - a_4\otimes b_4, (b_3-b_4)\otimes a_3 + a_4\otimes b_4\}. \end{aligned}$$

When $g \geq 5$, you instead take $X \cup Y_4 \cup Y_5$ with

$$Y_5 = \{a_i\otimes b_i + b_j\otimes a_j \mid 5 \leq i,j \leq g\} \cup \{a_4\otimes b_4 + b_5\otimes a_5\}.$$

Just like in the proof of Lemma 19.2, you can replace $Y_4$ with

$$\{(a_1+b_1)\otimes(a_1+b_1-b_2),(b_1-b_2)\otimes(a_1+a_2),(b_2+a_3)\otimes(a_2+b_3),\\ (a_3+a_4)\otimes(b_3-b_4),(b_3+a_4-b_4)\otimes(a_3+b_4)\}$$

and $Y_5$ by

$$\{(a_i-b_j)\otimes(b_i-a_j)\mid 4\le i,j\le g \text{ distinct, either } i,j\ge 5 \text{ or } (i,j)=(4,5)\}\\ \cup\{(a_i+b_i)\otimes(a_i+b_i)\mid 5\le i\le g\}.$$

From this, we see that $\mathfrak{F}[a_1+a_2,a_3+a_4]$ is generated by the following elements:

**Case 1.** $[\![(a_1+a_2)\wedge x,(a_3+a_4)\wedge y]\!]_a$ *for* $x\in A_V$ *and* $Y\in A_W$ *with* $\omega(x,y)=0$.

There are three cases:

- If $y=b_1-b_2$, then this equals $-[\![(a_1+a_2)\wedge x,(b_1-b_2)\wedge(a_3+a_4)]\!]_a$, which lies in $\langle S\rangle$ by Lemma 18.4.
- If $x=b_3-b_4$, then this equals $[\![(a_3+a_4)\wedge y,(b_3-b_4)\wedge(a_1+a_2)]\!]_a$, which lies in $\langle S\rangle$ by Lemma 18.4.
- If neither equality holds, then this equals $[\![x\wedge(a_1+a_2),y\wedge(a_3+a_4)]\!]_a\in S_{12}$.

**Case 2.** *When* $g\ge 5$, *elements* $[\![(a_1+a_2)\wedge(a_i-b_j),(a_3+a_4)\wedge(b_i-a_j)]\!]_a$ *for either* $5\le i,j\le g$ *distinct or* $(i,j)=(4,5)$.

These equal $[\![(a_i-b_j)\wedge(a_1+a_2),(b_i-a_j)\wedge(a_3+a_4)]\!]_a\in S_3$.

**Case 3.** *When* $g\ge 5$, *elements* $[\![(a_1+a_2)\wedge(a_i+b_i),(a_3+a_4)\wedge(a_i+b_i)]\!]_a$ *for* $5\le i\le g$.

These equal $[\![(a_i+b_i)\wedge(a_1+a_2),(a_i+b_i)\wedge(a_3+a_4)]\!]_a$, which lie in $\langle S\rangle$ by Lemma 18.3.

**Case 4.** *The following elements:*

(a) $[\![(a_1+a_2)\wedge(a_1+b_1),(a_3+a_4)\wedge(a_1+b_1-b_2)]\!]_a$
(b) $[\![(a_1+a_2)\wedge(b_3+a_4-b_4),(a_3+a_4)\wedge(a_3+b_4)]\!]_a$
(c) $[\![(a_1+a_2)\wedge(b_1-b_2),(a_3+a_4)\wedge(a_1+a_2)]\!]_a$
(d) $[\![(a_1+a_2)\wedge(a_3+a_4),(a_3+a_4)\wedge(b_3-b_4)]\!]_a$
(e) $[\![(a_1+a_2)\wedge(b_2+a_3),(a_3+a_4)\wedge(a_2+b_3)]\!]_a$

The element in (a) equals

$$[\![(a_1+a_2)\wedge(a_1+b_1),a_3\wedge(a_1+b_1-b_2)]\!]_a+[\![(a_1+a_2)\wedge(a_1+b_1),a_4\wedge(a_1+b_1-b_2)]\!]_a. \tag{19.3}$$

The first term of (19.3) lies in $\mathfrak{F}[a_1+a_2,a_3]$, which differs from $\mathfrak{F}[a_1+b_2,a_3]$ by an element of $\mathrm{SymSp}_g$. Lemma 19.4 says that $\mathfrak{F}[a_1+b_2,a_3]\subset\langle S\rangle$, so by Lemma 18.1 the set $\mathfrak{F}[a_1+a_2,a_3]$ lies in $\langle S\rangle$ too. The same argument shows that the second term in (19.3) also lies in $\langle S\rangle$, so (19.3) lies in $\langle S\rangle$.

The element in (b) equals

$$[\![a_1\wedge(b_3+a_4-b_4),(a_3+a_4)\wedge(a_3+b_4)]\!]_a+[\![a_2\wedge(b_3+a_4-b_4),(a_3+a_4)\wedge(a_3+b_4)]\!]_a.$$

Since $[\![-,-]\!]_a$ is anti-symmetric, the same argument from the previous paragraph applies here too.

For the elements in (c) and (d), note that they equal

$$-[\![(a_1+a_2)\wedge(a_3+a_4),(b_1-b_2)\wedge(a_1+a_2)]\!]_a \quad\text{and}\\ [\![(a_3+a_4)\wedge(a_1+a_2),(b_3-b_4)\wedge(a_3+a_4)]\!]_a$$

Lemma 18.4 says that both of these lie in $\langle S\rangle$.

Finally, the element in (e) equals $-[\![(a_2+b_3)\wedge(a_3+a_4),(b_2+a_3)\wedge(a_1+a_2)]\!]_a$. Lemma 18.4 says that this lies in $\langle S\rangle$. □

19.3. **Three vs two.** Our final two results are about $\mathfrak{F}[a,a']$ where $a$ uses three generators and $a'$ uses two:

**Lemma 19.8** (Three vs two, I)**.** *For all $j \geq 2$, both $\mathfrak{F}[a_1+b_1-b_j, b_1-a_j]$ and $\mathfrak{F}[a_1-b_1-b_j, b_1-a_j]$ are subsets of $\langle S \rangle$.*

*Proof.* These two subsets do not differ by an element of $\mathrm{SymSp}_g$, but the proofs that they are contained in $\langle S \rangle$ are almost identical. We will therefore give the details for $\mathfrak{F}[a_1+b_1-b_j, b_1-a_j]$ and leave the other case to the reader. By Lemma 18.1, we can apply an appropriate element of $\mathrm{SymSp}_g$ and reduce ourselves to proving that $\mathfrak{F}[a_1+b_1-b_2, b_1-a_2] \subset \langle S \rangle$. Following the notation in §17.4, define

$$\mathcal{V} = \langle b_1 - a_2 \rangle_{\mathbb{Q}}^{\perp} / \langle a_1 + b_1 - b_2 \rangle \cong \langle A_V \rangle_{\mathbb{Q}} \quad \text{with} \quad A_V = \{b_1, a_2, a_3, b_3, \ldots, a_g, b_g\},$$
$$\mathcal{W} = \langle a_1 + b_1 - b_2 \rangle_{\mathbb{Q}}^{\perp} / \langle b_1 - a_2 \rangle \cong \langle A_W \rangle_{\mathbb{Q}} \quad \text{with} \quad A_W = \{a_1 + b_1, b_2, a_3, b_3, \ldots, a_g, b_g\}.$$

We proved in Lemma 17.4 that $\mathfrak{F}[a_1+b_1-b_2, b_1-a_2]$ is isomorphic to a quotient of the kernel of the map $\mathcal{V} \otimes \mathcal{W} \to \mathbb{Q}$ induced by the symplectic form $\omega$. Under this isomorphism, a generator $[\![(a_1+b_1-b_2) \wedge x, (b_1-a_2) \wedge y]\!]_a$ of $\mathfrak{F}[a_1+b_1-b_2, b_1-a_2]$ maps to $x \otimes y \in \mathcal{V} \otimes \mathcal{W}$.

The kernel of $\mathcal{V} \otimes \mathcal{W} \to \mathbb{Q}$ is spanned by $X \cup Y$ where

$$\begin{aligned} X &= \{x \otimes y \mid x \in A_V,\, y \in A_W,\, \omega(x,y) = 0\}, \\ Y &= \{a_i \otimes b_i + b_j \otimes a_j \mid 3 \leq i,j \leq g\} \\ &\quad \cup \{a_3 \otimes b_3 + b_1 \otimes (a_1 + b_1), a_2 \otimes b_2 + b_3 \otimes a_3\}. \end{aligned}$$

Just like in the proof of Lemma 19.2, we can replace $Y$ by

$$\begin{aligned} &\{(a_i - b_j) \otimes (b_i - a_j) \mid 1 \leq i,j \leq g \text{ distinct, either } i,j \geq 3 \text{ or } (i,j) = (2,3)\} \\ &\cup \{(a_i + b_i) \otimes (a_i + b_i) \mid 3 \leq i \leq g\} \\ &\cup \{(b_1 + a_3) \otimes (a_1 + b_1 + b_3)\}. \end{aligned}$$

From this, we see that $\mathfrak{F}[a_1+b_1-b_2, b_1-a_2]$ is generated by the following elements:

**Case 1.** $[\![(a_1+b_1-b_2) \wedge x, (b_1-a_2) \wedge y]\!]_a$ *for $x \in A_V$ and $y \in A_W$ with $\omega(x,y) = 0$.*

These equal $[\![(b_1-a_2) \wedge y, x \wedge (a_1+b_1-b_2)]\!]_a \in \mathfrak{F}[b_1-a_2, x]$. There are three cases:

- $x = b_1$. In this case, $\mathfrak{F}[b_1-a_2, b_1]$ differs from $\mathfrak{F}[a_1+b_2, b_2]$ by an element of $\mathrm{SymSp}_g$. Lemma 19.3 says that $\mathfrak{F}[a_1+b_2, b_2] \subset \langle S \rangle$, so by Lemma 18.1 so does $\mathfrak{F}[b_1-a_2, b_1]$.
- $x = a_2$. In this case, $\mathfrak{F}[b_1-a_2, a_2] = \mathfrak{F}[b_1-a_2, -a_2]$ by Lemma 17.2. The set $\mathfrak{F}[b_1-a_2, -a_2]$ differs from $\mathfrak{F}[a_1+b_2, b_2]$ by an element of $\mathrm{SymSp}_g$, so just like in the case $x = b_1$ it follows that $\mathfrak{F}[b_1-a_2, -a_2]$ lies in $\langle S \rangle$.
- $x \in \{a_3, b_3, \ldots, a_g, b_g\}$. In this case, $\mathfrak{F}[b_1-a_2, x]$ differs from $\mathfrak{F}[a_1+b_2, a_3]$ by an element of $\mathrm{SymSp}_g$. Lemma 19.4 says that $\mathfrak{F}[a_1+b_2, a_3] \subset \langle S \rangle$, so by Lemma 18.1 so does $\mathfrak{F}[b_1-a_2, x]$.

**Case 2.** $[\![(a_1+b_1-b_2) \wedge (a_i-b_j), (b_1-a_2) \wedge (b_i-a_j)]\!]_a$ *for $1 \leq i,j \leq g$ distinct with either $i,j \geq 3$ or $(i,j) = (2,3)$.*

These equal $[\![(a_i-b_j) \wedge (a_1+b_1-b_2), (b_i-a_j) \wedge (b_1-a_2)]\!]_a \in S_3$.

**Case 3.** $[\![(a_1+b_1-b_2) \wedge (a_i+b_i), (b_1-a_2) \wedge (a_i+b_i)]\!]_a$ *for $3 \leq i \leq g$.*

These equal $[\![(a_i+b_i) \wedge (a_1+b_1-b_2), (a_i+b_i) \wedge (b_1-a_2)]\!]_a$, which lie in $\langle S \rangle$ by Lemma 18.3.

**Case 4.** $[\![(a_1+b_1-b_2) \wedge (b_1+a_3), (b_1-a_2) \wedge (a_1+b_1+b_3)]\!]_a$.

This equals $[\![(b_1+a_3)\wedge(a_1+b_1-b_2),(b_1-a_2)\wedge(a_1+b_1+b_3)]\!]_a \in \mathfrak{F}[b_1+a_3,b_1-a_2]$ The set $\mathfrak{F}[b_1+a_3,b_1-a_2]$ differs from $\mathfrak{F}[a_1+a_2,a_1+a_3]$ by an element of $\mathrm{SymSp}_g$. Lemma 19.6 says that $\mathfrak{F}[a_1+a_2,a_1+a_3]$ is contained in $\langle S\rangle$, so by Lemma 18.1 the set $\mathfrak{F}[b_1+a_3,b_1-a_2]$ is as well. □

**Lemma 19.9** (Three vs two, II)**.** *For $3\leq j\leq g$, both $\mathfrak{F}[b_1+a_2-b_j,b_2-a_j]$ and $\mathfrak{F}[a_1+b_2-b_j,b_1-a_j]$ are subsets of $\langle S\rangle$.*

*Proof.* The two sets differ by an element of $\mathrm{SymSp}_g$, so by Lemma 18.1 it is enough to deal with $\mathfrak{F}[b_1+a_2-b_j,b_2-a_j]$. Applying a further element of $\mathrm{SymSp}_g$, we can reduce to the case $j=3$, i.e., to $\mathfrak{F}[b_1+a_2-b_3,b_2-a_3]$. In fact, to simplify our notation we will apply yet another element of $\mathrm{SymSp}_g$ and transform our goal into proving that $\mathfrak{F}[a_1+a_2+b_3,b_1+a_3]\subset\langle S\rangle$. Following the notation in §17.4, define

$$\mathcal{V}=\langle b_1+a_3\rangle_{\mathbb{Q}}^{\perp}/\langle a_1+a_2+b_3\rangle\cong\langle A_V\rangle_{\mathbb{Q}}\quad\text{with}\quad A_V=\{b_1,a_2,b_2,a_3,a_4,b_4,\ldots,a_g,b_g\},$$
$$\mathcal{W}=\langle a_1+a_2+b_3\rangle_{\mathbb{Q}}^{\perp}/\langle b_1+a_3\rangle\cong\langle A_W\rangle_{\mathbb{Q}}\quad\text{with}\quad A_W=\{a_1,b_2-b_1,a_2,b_3,a_4,b_4,\ldots,a_g,b_g\}.$$

We proved in Lemma 17.4 that $\mathfrak{F}[a_1+a_2+b_3,b_1+a_3]$ is isomorphic to a quotient of the kernel of the map $\mathcal{V}\otimes\mathcal{W}\to\mathbb{Q}$ induced by the symplectic form $\omega$. Under this isomorphism, a generator $[\![(a_1+a_2+b_3)\wedge x,(b_1+a_3)\wedge y]\!]_a$ of $\mathfrak{F}[a_1+a_2+b_3,b_1+a_3]$ maps to $x\otimes y\in\mathcal{V}\otimes\mathcal{W}$.

The kernel of $\mathcal{V}\otimes\mathcal{W}\to\mathbb{Q}$ is spanned by $X\cup Y$ where

$$X=\{x\otimes y \mid x\in A_V,\ y\in A_W,\ \omega(x,y)=0\},$$
$$Y=\{a_i\otimes b_i+b_j\otimes a_j \mid 4\leq i,j\leq g\}$$
$$\cup\{a_4\otimes b_4+b_1\otimes a_1,\ a_2\otimes(b_2-b_1)+b_4\otimes a_4,\ a_4\otimes b_4+b_2\otimes a_2,\ a_3\otimes b_3+b_4\otimes a_4\}.$$

Just like in the proof of Lemma 19.2, we can replace $Y$ by

$$\{(a_i-b_j)\otimes(b_i-a_j) \mid 1\leq i,j\leq g \text{ distinct, either } i,j\geq 4 \text{ or } (i,j)\in\{(4,1),(4,2),(3,4)\}\}$$
$$\cup\{(a_i+b_i)\otimes(a_i+b_i) \mid 4\leq i\leq g\}$$
$$\cup\{(a_2+b_4)\otimes(b_2-b_1+a_4)\}.$$

From this, we see that $\mathfrak{F}[a_1+a_2+b_3,b_1+a_3]$ is generated by the following elements:

**Case 1.** $[\![(a_1+a_2+b_3)\wedge x,(b_1+a_3)\wedge y]\!]_a$ *for $x\in A_V$ and $y\in A_W$ with $\omega(x,y)=0$.*

These equal $[\![(b_1+a_3)\wedge y,x\wedge(a_1+a_2+b_3)]\!]_a\in\mathfrak{F}[b_1+a_3,x]$. There are two cases:

- $x=b_1$ or $x=a_3$. In this case, $\mathfrak{F}[b_1+a_3,x]$ differs from $\mathfrak{F}[a_1+b_2,b_2]$ by an element of $\mathrm{SymSp}_g$. Lemma 19.3 says that $\mathfrak{F}[a_1+b_2,b_2]\subset\langle S\rangle$, so by Lemma 18.1 so does $\mathfrak{F}[b_1+a_3,x]$.
- $x\in\{a_2,b_2,a_4,b_4,\ldots,a_g,b_g\}$. In this case, $\mathfrak{F}[b_1+a_3,x]$ differs from $\mathfrak{F}[a_1+b_2,a_3]$ by an element of $\mathrm{SymSp}_g$. Lemma 19.4 says that $\mathfrak{F}[a_1+b_2,a_3]\subset\langle S\rangle$, so by Lemma 18.1 so does $\mathfrak{F}[b_1+a_3,x]$.

**Case 2.** $[\![(a_1+a_2+b_3)\wedge(a_i-b_j),(b_1+a_3)\wedge(b_i-a_j)]\!]_a$ *for $1\leq i,j\leq g$ distinct with either $i,j\geq 4$ or $(i,j)\in\{(4,1),(4,2),(3,4)\}$.*

These equal $[\![(a_i-b_j)\wedge(a_1+a_2+b_3),(b_i-a_j)\wedge(b_1+a_3)]\!]_a\in S_3$.

**Case 3.** $[\![(a_1+a_2+b_3)\wedge(a_i+b_i),(b_1+a_3)\wedge(a_i+b_i)]\!]_a$ *for $4\leq i\leq g$.*

These equal $[\![(a_i+b_i)\wedge(a_1+a_2+b_3),(a_i+b_i)\wedge(b_1+a_3)]\!]_a$, which lie in $\langle S\rangle$ by Lemma 18.3.

**Case 4.** $[\![(a_1+a_2+b_3)\wedge(a_2+b_4),(b_1+a_3)\wedge(b_2-b_1+a_4)]\!]_a$.

This equals $-[\![(a_2+b_4)\wedge(a_1+a_2+b_3),(b_1+a_3)\wedge(b_2-b_1+a_4)]\!]_a\in\mathfrak{F}[a_2+b_4,b_1+a_3]$ The set $\mathfrak{F}[a_2+b_4,b_1+a_3]$ differs from $\mathfrak{F}[a_1+a_2,a_3+a_4]$ by an element of $\mathrm{SymSp}_g$. Lemma 19.7 says that $\mathfrak{F}[a_1+a_2,a_3+a_4]$ is contained in $\langle S\rangle$, so by Lemma 18.1 the set $\mathfrak{F}[a_2+b_4,b_1+a_3]$ is as well. □

## 20. Symmetric kernel, alternating version IV: the set $S$ spans $\mathfrak{K}_g^a$

We continue using all the notation from §18 – §19. Recall that $S = S_{12} \cup S_3$, where

$$S_{12} = \bigcup_{\substack{a,a' \in \mathcal{B} \\ \omega(a,a')=0}} \mathfrak{F}[a, a'] \quad \text{and} \quad S_3 = \bigcup_{\substack{1 \leq i,j \leq g \\ i \neq j}} \mathfrak{F}[a_i - b_j, b_i - a_j].$$

Our goal in this section is to prove the following lemma:

**Lemma 20.1.** *The set $S$ spans $\mathfrak{K}_g^a$.*

*Proof.* We first claim that the $\mathrm{Sp}_{2g}(\mathbb{Z})$-orbit of $S$ spans $\mathfrak{K}_g^a$. Indeed, using our original generating set for $\mathfrak{K}_g^a$ from Definition 1.15 together with Lemma 11.2, we see that $\mathfrak{K}_g^a$ is generated by elements of the form $[\![a \wedge b, a' \wedge b']\!]_a$, where $a \wedge b$ and $a' \wedge b'$ are symplectic pairs such that $\langle a, b \rangle$ and $\langle a', b' \rangle$ are orthogonal. The group $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts transitively on such elements. The set $S$ contains many elements of this form; for instance, it contains $[\![a_1 \wedge b_1, a_2 \wedge b_2]\!]_a \in \mathfrak{F}[a_1, a_2]$. It follows that $\mathrm{Sp}_{2g}(\mathbb{Z})$-orbit of $S$ spans $\mathfrak{K}_g^a$, as claimed.

To prove the lemma, therefore, we must prove that the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathfrak{K}_g^a$ takes $\langle S \rangle$ to itself. By Corollary 7.3, the group $\mathrm{Sp}_{2g}(\mathbb{Z})$ is generated as a monoid by the set $\Lambda = \mathrm{SymSp}_g \cup \{X_1, X_1^{-1}, Y_{12}\}$. We will recall the definitions of these elements as we use them. We must prove that for $f \in \Lambda$ and $s \in S = S_{12} \cup S_3$ we have $f(s) \in \langle S \rangle$. We already did this for $\mathrm{SymSp}_g$ in Lemma 18.1, so we must handle the other generators. We divide this into four claims.

**Claim 1.** *Recall that $X_1 \in \mathrm{Sp}_{2g}(\mathbb{Z})$ takes $a_1$ to $a_1 + b_1$ and fixes all other generators in $\mathcal{B}$. For $\epsilon \in \{\pm 1\}$ and*

$$s \in S_{12} = \bigcup_{\substack{a,a' \in \mathcal{B} \\ \omega(a,a')=0}} \mathfrak{F}[a, a'],$$

*we have $X_1^\epsilon(s) \in \langle S \rangle$.*

We have $s \in \mathfrak{F}[a, a']$ for some $a, a' \in \mathcal{B}$ with $\omega(a, a') = 0$, so $X_1^\epsilon(s) \in \mathfrak{F}[X_1^\epsilon(a), X_1^\epsilon(a')]$. We must show that $\mathfrak{F}[X_1^\epsilon(a), X_1^\epsilon(a')] \subset \langle S \rangle$. There are three cases:

- If $a, a' \neq a_1$, then
$$\mathfrak{F}[X_1^\epsilon(a), X_1^\epsilon(a')] = \mathfrak{F}[a, a'] \subset \langle S \rangle.$$
- If one of $a$ and $a'$ equals $a_1$ and the other is not $a_1$, then since $\mathfrak{F}[-,-]$ is symmetric in its entries (Lemma 17.2) we can assume without loss of generality that $a = a_1$ and $a' \neq a_1$. Since $\omega(a_1, a') = 0$, we also have $a' \neq b_1$. By Lemma 19.2,
$$\mathfrak{F}[X_1^\epsilon(a_1), X_1^\epsilon(a')] = \mathfrak{F}[a_1 + \epsilon b_1, a'] \subset \langle S \rangle.$$
- If both $a$ and $a'$ equal $a_1$ then we can apply Lemma 18.8 to see that
$$\mathfrak{F}[X_1^\epsilon(a_1), X_1^\epsilon(a_1)] = \mathfrak{F}[a_1 + \epsilon b_1, a_1 + \epsilon b_1] \subset \langle S \rangle.$$

The claim follows.

**Claim 2.** *For $\epsilon \in \{\pm 1\}$ and*

$$s \in S_3 = \bigcup_{1 \leq i < j \leq g} \mathfrak{F}[a_i - b_j, b_i - a_j],$$

*we have $X_1^\epsilon(s) \in \langle S \rangle$.*

We have $s \in \mathfrak{F}[a_i - b_j, b_i - a_j]$ for some $1 \leq i < j \leq g$, so $X_1^\epsilon(s) \in \mathfrak{F}[X_1^\epsilon(a_i - b_j), X_1^\epsilon(b_i - a_j)]$. We must show that $\mathfrak{F}[X_1^\epsilon(a_i - b_j), X_1^\epsilon(b_i - a_j)] \subset \langle S \rangle$. There are two cases:

- If $i \geq 2$, then $j \geq 2$ as well, so $X_1^\epsilon$ fixes both $a_i - b_j$ and $b_i - a_j$. It follows that
$$\mathfrak{F}[X_1^\epsilon(a_i - b_j), X_1^\epsilon(b_i - a_j)] = \mathfrak{F}[a_i - b_j, b_i - a_j] \subset \langle S \rangle.$$
- If $i = 1$, then $j \geq 2$, so $X_1^\epsilon$ fixes both $a_j$ and $b_j$. By Lemma 19.8,
$$\mathfrak{F}[X_1^\epsilon(a_1 - b_j), X_1^\epsilon(b_1 - a_j)] = \mathfrak{F}[a_1 + \epsilon b_1 - b_j, b_1 - a_j] \subset \langle S \rangle.$$

The claim follows.

**Claim 3.** *Recall that $Y_{12} \in \mathrm{Sp}_{2g}(\mathbb{Z})$ takes $a_1$ to $a_1 + b_2$ and $a_2$ to $a_2 + b_1$ and fixes all other generators in $\mathcal{B}$. For*
$$s \in S_{12} = \bigcup_{\substack{a, a' \in \mathcal{B} \\ \omega(a,a')=0}} \mathfrak{F}[a, a'],$$
*we have $Y_{12}(s) \in \langle S \rangle$.*

We have $s \in \mathfrak{F}[a, a']$ for some $a, a' \in \mathcal{B}$ with $\omega(a, a') = 0$, so $Y_{12}(s) \in \mathfrak{F}[Y_{12}(a), Y_{12}(a')]$. We must show that $\mathfrak{F}[Y_{12}(a), Y_{12}(a')] \subset \langle S \rangle$. There are six cases:

- If $a, a' \in \mathcal{B} \setminus \{a_1, a_2\}$, then both $a$ and $a'$ are fixed by $Y_{12}$, so
$$\mathfrak{F}[Y_{12}(a), Y_{12}(a')] = \mathfrak{F}[a, a'] \subset \langle S \rangle.$$
- If one of $a$ and $a'$ is $a_1$ and the other lies in $\mathcal{B} \setminus \{a_1, a_2\}$, then since $\mathfrak{F}[-,-]$ is symmetric in its entries (Lemma 17.2) we can assume without loss of generality that $a = a_1$ and $a' \in \mathcal{B} \setminus \{a_1, a_2\}$. Since $\omega(a_1, a') = 0$, we also have $a' \neq b_1$. By Lemmas 19.3 and 19.4,
$$\mathfrak{F}[Y_{12}(a_1), Y_{12}(a')] = \mathfrak{F}[a_1 + b_2, a'] \subset \langle S \rangle.$$
- If one of $a$ and $a'$ is $a_2$ and the other lies in $\mathcal{B} \setminus \{a_1, a_2\}$, then since $\mathfrak{F}[-,-]$ is symmetric in its entries (Lemma 17.2) we can assume without loss of generality that $a = a_2$ and $a' \in \mathcal{B} \setminus \{a_1, a_2\}$. Since $\omega(a_1, a') = 0$, we also have $a' \neq b_2$. By Lemmas 19.3 and 19.4,
$$\mathfrak{F}[Y_{12}(a_2), Y_{12}(a')] = \mathfrak{F}[a_2 + b_1, a'] \subset \langle S \rangle.$$
- If $a = a' = a_1$, then by Lemma 19.5
$$\mathfrak{F}[Y_{12}(a_1), Y_{12}(a_1)] = \mathfrak{F}[a_1 + b_2, a_1 + b_2] \subset \langle S \rangle.$$
- If $a = a' = a_2$, then by Lemma 19.5
$$\mathfrak{F}[Y_{12}(a_2), Y_{12}(a_2)] = \mathfrak{F}[a_2 + b_1, a_2 + b_1] \subset \langle S \rangle.$$
- If one of $a$ and $a'$ is $a_1$ and the other is $a_2$, then since $\mathfrak{F}[-,-]$ is symmetric in its entries (Lemma 17.2) we can assume without loss of generality that $a = a_1$ and $a' = a_2$. By Lemma 18.4,
$$\mathfrak{F}[Y_{12}(a_1), Y_{12}(a_2)] = \mathfrak{F}[a_1 + b_2, b_1 + a_2] \subset \langle S \rangle.$$

The claim follows.

**Claim 4.** *For*
$$s \in S_3 = \bigcup_{1 \leq i < j \leq g} \mathfrak{F}[a_i - b_j, b_i - a_j],$$
*we have $Y_{12}(s) \in \langle S \rangle$.*

We have $s \in \mathfrak{F}[a_i - b_j, b_i - a_j]$ for some $1 \leq i < j \leq g$, so $Y_{12}(s) \in \mathfrak{F}[Y_{12}(a_i - b_j), Y_{12}(b_i - a_j)]$. We must show that $\mathfrak{F}[Y_{12}(a_i - b_j), Y_{12}(b_i - a_j)] \subset \langle S \rangle$. There are four cases:

- If $i \geq 3$, then $j \geq 3$ as well and thus $Y_{12}$ fixes $a_i - b_j$ and $b_i - a_j$. Therefore,
$$\mathfrak{F}[Y_{12}(a_i - b_j), Y_{12}(b_i - a_j)] = \mathfrak{F}[a_i - b_j, b_i - a_j] \subset \langle S \rangle.$$

- If $i = 2$, then $j \geq 3$ and $Y_{12}$ fixes $a_j$ and $b_j$. By Lemma 19.9, we have
$$\mathfrak{F}[Y_{12}(a_2 - b_j), Y_{12}(b_2 - a_j)] = \mathfrak{F}[b_1 + a_2 - b_j, b_2 - a_j] \subset \langle S \rangle.$$
- If $i = 1$ and $j \geq 3$, then $Y_{12}$ fixes $a_j$ and $b_j$. By Lemma 19.9, we have
$$\mathfrak{F}[Y_{12}(a_1 - b_j), Y_{12}(b_1 - a_j)] = \mathfrak{F}[a_1 + b_2 - b_j, b_1 - a_j] \subset \langle S \rangle.$$
- If $i = 1$ and $j = 2$, then[26]
$$\mathfrak{F}[Y_{12}(a_1 - b_2), Y_{12}(b_1 - a_2)] = \mathfrak{F}[a_1, a_2] \subset \langle S_{12} \rangle.$$

The claim follows. □

## 21. Symmetric kernel, alternating version V: $S_1$ and structure of target

We will use the notation from §17 - §20, and in particular the set $S = S_{12} \cup S_3$. We now turn to Theorem F, which says that the linearization map $\Phi \colon \mathfrak{K}_g^a \to \mathcal{K}_g^a$ is an isomorphism. We will prove this by first handling $S_{12}$ in the next two sections (§21 – §22), and then in the final two sections (§23 – §24) extending this to $S_3$ and hence to all of $\langle S_{12}, S_3 \rangle = \mathfrak{K}_g^a$.

The vector space $\mathcal{K}_g^a$ is the kernel of the symmetric contraction $\mathfrak{c}\colon \wedge^2((\wedge^2 H)/\mathbb{Q}) \to \mathrm{Sym}^2(H)$, and this section studies $\wedge^2((\wedge^2 H)/\mathbb{Q})$ and constructs a subset $S_1$ of $S_{12}$ such that the restriction of $\Phi$ to $\langle S_1 \rangle$ is an isomorphism onto its image.

### 21.1. Generators and relations for target.

Let $\prec$ be the following total order on $\mathcal{B}$:
$$a_1 \prec b_1 \prec a_2 \prec b_2 \prec \cdots \prec a_g \prec b_g.$$
Using this ordering, order $\{x \wedge y \mid x, y \in \mathcal{B},\ x \prec y\}$ lexicographically, so $(x_1 \wedge y_1) \prec (x_2 \wedge y_2)$ if either $x_1 \prec x_2$ or if $x_1 = x_2$ and $y_1 \prec y_2$. Define $T = T_1 \cup T_2 \cup T_3$, where:
$$\begin{aligned} T_1 = \{&(x \wedge y) \wedge (z \wedge w) \mid x, y, z, w \in \mathcal{B},\ x \prec y,\ z \prec w,\ (x \wedge y) \prec (z \wedge w), \\ &\text{and } \omega(x, z) = \omega(x, w) = \omega(y, z) = \omega(y, w) = 0\}, \\ T_2 = \{&(a \wedge a_i) \wedge (a' \wedge b_i) \mid 1 \leq i \leq g,\ a \in \mathcal{B} \setminus \{a_i\},\ a' \in \mathcal{B} \setminus \{b_i\}, \\ &\text{and } \omega(a, a') = \omega(a_i, a') = \omega(b_i, a) = 0\}, \\ T_3 = \{&(a_i \wedge a_j) \wedge (b_i \wedge b_j),\ (a_i \wedge b_j) \wedge (b_i \wedge a_j) \mid 1 \leq i < j \leq g\}. \end{aligned}$$
The set $T$ generates $\wedge^2((\wedge^2 H)/\mathbb{Q})$, so every element of $\wedge^2((\wedge^2 H)/\mathbb{Q})$ can be written as a linear combination of elements of $T$. If we were working with $\wedge^2(\wedge^2 H)$, then $T$ would be a basis; however, since we are working with $\wedge^2((\wedge^2 H)/\mathbb{Q})$ there are relations between elements of $T$. These relations are linear combinations of elements of $T$. It is often awkward to write these linear combinations while maintaining the orderings on the terms of $T$, so we introduce the following convention:

*Convention* 21.1. Consider an expression
$$\sum_{i=1}^{n} \lambda_i (x_i \wedge y_i) \wedge (z_i \wedge w_i) \quad \text{with each } \lambda_i \in \mathbb{Q} \text{ and } x_i, y_i, z_i, w_i \in \mathcal{B}.$$
We regard this as a linear combination of elements of $T$ in the following way:

- First, delete all terms where either $x_i = y_i$ or where $z_i = w_i$. These terms vanish in $\wedge^2((\wedge^2 H)/\mathbb{Q})$.
- Second, delete all terms where $\{x_i, y_i\} = \{z_i, w_i\}$ as unordered 2-element sets. These terms also vanish in $\wedge^2((\wedge^2 H)/\mathbb{Q})$.
- Finally, replace each term $(x_i \wedge y_i) \wedge (z_i \wedge w_i)$ with $\epsilon t$ for some $\epsilon \in \{\pm 1\}$ and $t \in T$. This involves possibly flipping $x_i$ and $y_i$, flipping $z_i$ and $w_i$, and flipping $x_i \wedge y_i$ and $z_i \wedge w_i$. Each flip introduces a sign. □

[26] This is the one easy case!

Using this convention, the relations between elements of $T$ are generated by the set

$$R = \left\{ \sum_{i=1}^{g} (a_i \wedge b_i) \wedge (x \wedge y) \mid x, y \in \mathcal{B},\, x \prec y \right\}$$

of linear combinations of elements of $T$.

21.2. **Lifting elements of $T_1$.** For $(x \wedge y) \wedge (z \wedge w) \in T_1$, we have

$$\mathfrak{c}(x \wedge y, z \wedge w) = \omega(x, z) y \cdot w - \omega(x, w) y \cdot z - \omega(y, z) x \cdot w + \omega(y, w) x \cdot z = 0.$$

It follows that $(x \wedge y) \wedge (z \wedge w) \in \mathcal{K}_g^a$, and thus can be lifted to $\mathfrak{K}_g^a$. Indeed, define

$$S_1 = \{ [\![x \wedge y, z \wedge w]\!]_a \mid x, y, z, w \in \mathcal{B},\, x \prec y,\, z \prec w,\, (x \wedge y) \prec (z \wedge w), \\ \text{and } \omega(x, z) = \omega(x, w) = \omega(y, z) = \omega(y, w) = 0\}.$$

Like we did here, we will write elements of $\langle S_1 \rangle$ in blue. A generator $[\![x \wedge y, z \wedge w]\!]_a$ of $S_1$ lies in $\mathfrak{F}[x, z] \subset S_{12}$, so $S_1 \subset S_{12}$. For $[\![x \wedge y, z \wedge w]\!]_a \in S_1$, we have

$$\Phi([\![x \wedge y, z \wedge w]\!]_a) = (x \wedge y) \wedge (z \wedge w) \in T_1.$$

The map $\Phi$ restricts to a bijection between $S_1$ and $T_1$.

21.3. **Restricting linearization to $S_1$.** Recall that our goal is to prove Theorem F, which says that the linearization map $\Phi \colon \mathfrak{K}_g^a \to \mathcal{K}_g^a$ is an isomorphism. We now prove the following partial result in this direction:

**Lemma 21.2.** *The linearization map $\Phi$ takes $\langle S_1 \rangle$ isomorphically to $\langle T_1 \rangle$.*

*Proof.* Let $R_1$ be the subset of the relations $R$ consisting of relations between elements of $T_1$. The set $R_1$ consists of relations of the form

$$\sum_{i=1}^{g} (a_i \wedge b_i) \wedge (a_k \wedge b_k) \text{ with } 1 \leq k \leq g.$$

Set $R_2 = R \setminus R_1$. Each element of $R_2$ involves an element of $T \setminus T_1$ that appears in no other relations in $R$. For instance, for $1 \leq k < \ell \leq g$ the set $R_2$ contains the relation

$$\sum_{i=1}^{g} (a_i \wedge b_i) \wedge (a_k \wedge a_\ell),$$

and no other relation in $R$ uses the generator[27] $(a_k \wedge b_k) \wedge (a_k \wedge a_\ell)$. This implies that the subspace of $\wedge^2((\wedge^2 H)/\mathbb{Q})$ spanned by $T_1$ is generated by $T_1$ subject to only the relations in $R_1$. The map $\Phi$ takes $S_1$ bijectively to $T_1$. The relations in $R_1$ lift to relations between the elements of $S_1$ due to the bilinearity relations in $\mathfrak{K}_g^a$:

$$\sum_{i=1}^{g} [\![a_i \wedge b_i, a_k \wedge b_k]\!]_a = [\![\sum_{i=1}^{g} a_i \wedge b_i, a_k \wedge b_k]\!]_a = [\![0, a_k \wedge b_k]\!]_a = 0.$$

Combining all of this, we conclude that $\Phi$ takes $\langle S_1 \rangle$ isomorphically to $\langle T_1 \rangle$, as desired. □

## 22. Symmetric kernel, alternating version VI: $S_2$ and $S_{12}$

We continue using all the notation from §17 – §21. Recall that our goal is to prove Theorem F, which says that the linearization map $\Phi \colon \mathfrak{K}_g^a \to \mathcal{K}_g^a$ is an isomorphism. In the last section, we constructed a set $S_1 \subset S_{12}$ and proved Lemma 21.2, which says that $\Phi$ restricts to an isomorphism between $\langle S_1 \rangle$ and $\langle T_1 \rangle$. In this section, we prove that $\Phi$ restricts to an isomorphism between $\langle S_{12} \rangle$ and $\mathcal{K}_g^a \cap \langle T_1, T_2 \rangle$.

[27]Though $a_k$ appears twice in $(a_k \wedge b_k) \wedge (a_k \wedge a_\ell)$, this element is not 0. See Warning 17.1.

22.1. **The set $T_2$.** Consider an element $(a \wedge a_i) \wedge (a' \wedge b_i)$ in

$$T_2 = \{(a \wedge a_i) \wedge (a' \wedge b_i) \mid 1 \leq i \leq g,\, a \in \mathcal{B} \setminus \{a_i\},\, a' \in \mathcal{B} \setminus \{b_i\}, \\ \text{and } \omega(a,a') = \omega(a_i,a') = \omega(b_i,a) = 0\},$$

The image of $(a \wedge a_i) \wedge (a' \wedge b_i)$ in $\mathrm{Sym}^2(H)$ under the symmetric contraction is

$$\mathfrak{c}(a \wedge a_i, a' \wedge b_i) = \omega(a,a')a_i{\cdot}b_i - \omega(a,b_i)a_i{\cdot}a' - \omega(a_i,a')a{\cdot}b_i + \omega(a_i,b_i)a{\cdot}a' = a{\cdot}a'.$$

The set $\{a{\cdot}a' \mid a,a' \in \mathcal{B}\}$ is a basis for $\mathrm{Sym}^2(H)$, and this calculation suggests dividing $T_2$ into subsets mapping to different basis elements. Define $S_0^2(\mathcal{B}) = \{a{\cdot}a' \mid a,a' \in \mathcal{B},\, \omega(a,a') = 0\}$. For $a{\cdot}a' \in S_0^2(\mathcal{B})$, define

$$T_2(a{\cdot}a') = \{(a \wedge a_i) \wedge (a' \wedge b_i) \mid 1 \leq i \leq g,\, a_i \neq a,\, b_i \neq a',\, \omega(a_i,a') = \omega(b_i,a) = 0\} \\ \cup \{(a' \wedge a_i) \wedge (a \wedge b_i) \mid 1 \leq i \leq g,\, a_i \neq a',\, b_i \neq a,\, \omega(a_i,a) = \omega(b_i,a') = 0\}.$$

Note that this is symmetric in $a$ and $a'$. We have

$$T_2 = \bigcup_{a{\cdot}a' \in S_0^2(\mathcal{B})} T_2(a{\cdot}a').$$

By our calculation above, $\mathfrak{c}$ takes each element of $T_2(a{\cdot}a')$ to $a{\cdot}a'$. Since we will need it later, we record the following consequence the above calculations:

**Lemma 22.1.** *The symmetric contraction $\mathfrak{c}\colon \wedge^2((\wedge^2 H)/\mathbb{Q}) \to \mathrm{Sym}^2(H)$ takes $\langle T_1, T_2\rangle$ surjectively onto $\langle S_0^2(\mathcal{B})\rangle$.*

*Proof.* Immediate from the above calculation of $\mathfrak{c}$ on generators for $T_2(a{\cdot}a')$ along with the fact that $\mathfrak{c}$ vanishes on $T_1$. □

22.2. **Lifting $T_2(a{\cdot}a')$.** Consider $a{\cdot}a' \in S_0^2(\mathcal{B})$. Recall from §17.2 that $\widehat{\mathcal{F}}[a,a']$ is the subspace of $\wedge^2((\wedge^2 H)/\mathbb{Q})$ spanned by elements of the form $(a \wedge x) \wedge (a' \wedge y)$ with $x, y \in H_{\mathbb{Z}}$ satisfying $\omega(x,y) = 0$. Each element of $T_2(a{\cdot}a')$ is a generator of $\widehat{\mathcal{F}}[a,a']$. It follows that

$$\langle T_2(a{\cdot}a')\rangle \subset \widehat{\mathcal{F}}[a,a'].$$

Since $\mathfrak{c}$ takes every element of $T_2(a{\cdot}a')$ to $a{\cdot}a' \in \mathrm{Sym}^2(H)$, the intersection of the symmetric kernel $\mathcal{K}_g^a = \ker(\mathfrak{c})$ with $\langle T_2(a{\cdot}a')\rangle$ has codimension 1 in $\langle T_2(a{\cdot}a')\rangle$. Indeed, it is spanned by elements of the form $t_1 - t_2$ with $t_1, t_2 \in T_2(a{\cdot}a')$. Letting[28] $\mathfrak{F}[a,a']$ be as in §17.1, define

$$S_2(a{\cdot}a') = \left\{\eta \in \mathfrak{F}[a,a'] \mid \Phi(\eta) \in \langle T_2(a{\cdot}a')\rangle\right\}.$$

By construction, this is a subspace of $\mathfrak{F}[a,a']$. We will prove below that $\Phi$ takes $S_2(a{\cdot}a')$ isomorphically onto $\mathcal{K}_g^a \cap \langle T_2(a{\cdot}a')\rangle$. In fact, we will do more than this. Define

$$S_1(a{\cdot}a') = \{[\![a \wedge x, a' \wedge y]\!]_a \mid x \in \mathcal{B} \setminus \{a\},\, y \in \mathcal{B} \setminus \{a'\},\, \{a,x\} \neq \{a',y\}, \\ \text{and } \omega(a,y) = \omega(x,a') = \omega(x,y) = 0\}.$$

For each $\eta \in S_1(a{\cdot}a')$, either $\eta$ or $-\eta$ lies in $S_1$. Just like for $S_2(a{\cdot}a')$, we have $S_1(a{\cdot}a') \subset \mathfrak{F}[a,a']$. We will prove:

**Lemma 22.2.** *Consider $a{\cdot}a' \in S_0^2(\mathcal{B})$. Then:*

*(a) The linearization map $\Phi$ restricts to an isomorphism between $S_2(a{\cdot}a')$ and $\mathcal{K}_g^a \cap \langle T_2(a{\cdot}a')\rangle$.*
*(b) We have $\langle S_1(a{\cdot}a'), S_2(a{\cdot}a')\rangle = \mathfrak{F}[a,a']$.*

*Proof.* Conclusion (a) is immediate from Lemmas 17.3 and 17.4, which together imply that $\Phi$ restricts to an isomorphism between $\mathfrak{F}[a,a']$ and $\mathcal{K}_g^a \cap \widehat{\mathcal{F}}[a,a']$. We must prove (b). There are two cases:

[28]The set $\mathfrak{F}[a,a']$ is purple since it lies in $S_{12}$.

**Case 1.** $a = a'$.

To simplify our notation, we will assume that $a \in \mathcal{B} = \{a_1, b_1, \ldots, a_g, b_g\}$ equals $a_1$. The other cases are identical up to changes in indices. Following the notation in §17.3, define

$$\mathcal{U} = \langle a_1 \rangle_{\mathbb{Q}}^{\perp} / \langle a_1 \rangle \cong \langle A \rangle_{\mathbb{Q}} \quad \text{with} \quad A = \{a_2, b_2, \ldots, a_g, b_g\}.$$

We proved in Lemma 17.3 that $\mathfrak{F}[a_1, a_1]$ is isomorphic to the kernel of the map $\wedge^2 \mathcal{U} \to \mathbb{Q}$ induced by the symplectic form $\omega$. Under this isomorphism, a generator $[\![a_1 \wedge x, a_1 \wedge y]\!]_a$ of $\mathfrak{F}[a_1, a_1]$ maps to $x \wedge y \in \wedge^2 \mathcal{U}$.

The kernel of $\wedge^2 \mathcal{U} \to \mathbb{Q}$ is spanned by $X \cup Y$ where

$$X = \{x \wedge y \mid x, y \in A, \, \omega(x, y) = 0\},$$
$$Y = \{a_i \wedge b_i - a_j \wedge b_j \mid 2 \leq i < j \leq g\}.$$

Since for $2 \leq i < j \leq g$ we have

$$(a_i - b_j) \wedge (b_i - a_j) = a_i \wedge b_i - a_j \wedge b_j + \text{an element of } \langle X \rangle,$$

we can replace $Y$ by

$$\{(a_i - b_j) \wedge (b_i - a_j) \mid 2 \leq i < j \leq g\}.$$

It follows that $\mathfrak{F}[a_1, a_1]$ is generated by the following elements:

- $[\![a_1 \wedge x, a_1 \wedge y]\!]_a$ for $x, y \in A$ with $\omega(x, y) = 0$. These are elements of $S_1(a_1 \cdot a_1)$.
- $[\![a_1 \wedge (a_i - b_j), a_1 \wedge (b_i - a_j)]\!]_a$ for $2 \leq i < j \leq g$.

It is thus enough to prove that for $2 \leq i < j \leq g$ the element $[\![a_1 \wedge (a_i - b_j), a_1 \wedge (b_i - a_j)]\!]_a$ lies in the span of $S_1(a_1 \cdot a_1)$ and $S_2(a_1 \cdot a_1)$. For this, note that

$$\begin{aligned}
&\Phi([\![a_1 \wedge (a_i - b_j), a_1 \wedge (b_i - a_j)]\!]_a + [\![a_1 \wedge a_i, a_1 \wedge a_j]\!]_a + [\![a_1 \wedge b_j, a_1 \wedge b_i]\!]_a) \\
=&(a_1 \wedge (a_i - b_j)) \wedge (a_1 \wedge (b_i - a_j)) + (a_1 \wedge a_i) \wedge (a_1 \wedge a_j) + (a_1 \wedge b_j) \wedge (a_1 \wedge b_i) \\
=&(a_1 \wedge a_i) \wedge (a_1 \wedge b_i) - (a_1 \wedge a_j) \wedge (a_1 \wedge b_j) \in \langle T_2(a_1 \cdot a_1) \rangle.
\end{aligned}$$

It follows that

$$[\![a_1 \wedge (a_i - b_j), a_1 \wedge (b_i - a_j)]\!]_a + [\![a_1 \wedge a_i, a_1 \wedge a_j]\!]_a + [\![a_1 \wedge b_j, a_1 \wedge b_i]\!]_a \in S_2(a_1 \cdot a_1).$$

Since

$$[\![a_1 \wedge a_i, a_1 \wedge a_j]\!]_a + [\![a_1 \wedge b_j, a_1 \wedge b_i]\!]_a \in \langle S_1(a_1 \cdot a_1) \rangle,$$

the case follows.

**Case 2.** $a \neq a'$.

To simplify our notation, we will assume that $a, a' \in \mathcal{B} = \{a_1, b_1, \ldots, a_g, b_g\}$ are $a = a_1$ and $a' = a_2$. The other cases are identical up to changes in indices. Following the notation in §17.4, define

$$\begin{aligned}
\mathcal{V} =& \langle a_2 \rangle_{\mathbb{Q}}^{\perp} / \langle a_1 \rangle \cong \langle A_V \rangle_{\mathbb{Q}} \quad \text{with} \quad A_V = \{b_1, a_2, a_3, b_3, \ldots, a_g, b_g\}, \\
\mathcal{W} =& \langle a_1 \rangle_{\mathbb{Q}}^{\perp} / \langle a_2 \rangle \cong \langle A_W \rangle_{\mathbb{Q}} \quad \text{with} \quad A_W = \{a_1, b_2, a_3, b_3, \ldots, a_g, b_g\}.
\end{aligned}$$

We proved in Lemma 17.4 that $\mathfrak{F}[a_1, a_2]$ is isomorphic to a quotient of the kernel of the map $\mathcal{V} \otimes \mathcal{W} \to \mathbb{Q}$ induced by the symplectic form $\omega$. Under this isomorphism, a generator $[\![a_1 \wedge x, a_2 \wedge y]\!]_a$ of $\mathfrak{F}[a_1, a_2]$ maps to $x \otimes y \in \mathcal{V} \otimes \mathcal{W}$.

The kernel of $\mathcal{V} \otimes \mathcal{W} \to \mathbb{Q}$ is spanned by $X \cup Y$ where[29]

$$X = \{x \otimes y \mid x \in A_V,\, y \in A_W,\, \omega(x,y) = 0\},$$
$$Y = \{a_i \otimes b_i + b_j \otimes a_j \mid 1 \leq i,j \leq g \text{ distinct},\, i \neq 1,\, j \neq 2\}.$$

Since for $1 \leq i,j \leq g$ distinct with $i \neq 1$ and $j \neq 2$ we have

$$(a_i - b_j) \otimes (b_i - a_j) = a_i \otimes b_i + b_j \otimes a_j + \text{an element of } \langle X \rangle,$$

we can replace $Y$ by the set

$$\{(a_i - b_j) \otimes (b_i - a_j) \mid 1 \leq i,j \leq g \text{ distinct},\, i \neq 1,\, j \neq 2\}.$$

From this, we see that $\mathfrak{F}[a_1, a_2]$ is generated by the following elements:

- $[\![a_1 \wedge x, a_2 \wedge y]\!]_a$ for $x \in A_V$ and $y \in A_W$ with $\omega(x,y) = 0$. These are elements of $S_1(a_1 \cdot a_2)$.
- $[\![a_1 \wedge (a_i - b_j), a_2 \wedge (b_i - a_j)]\!]_a$ for $1 \leq i < j \leq g$ distinct with $i \neq 1$ and $j \neq 2$.

It is thus enough to prove that for $1 \leq i < j \leq g$ distinct with $i \neq 1$ and $j \neq 2$, the element $[\![a_1 \wedge (a_i - b_j), a_2 \wedge (b_i - a_j)]\!]_a$ lies in the span of $S_1(a_1 \cdot a_2)$ and $S_2(a_1 \cdot a_2)$. The proof of this is similar the argument we gave in Case 1, so we omit it. □

22.3. **The set $S_2$.** Define

$$S_2 = \bigcup_{a,a' \in \mathcal{B}} S_2(a \cdot a').$$

Recall that we proved in Lemma 21.2 that the linearization map $\Phi$ takes $\langle S_1 \rangle$ isomorphically to $\langle T_1 \rangle$. We have $\langle T_1 \rangle \subset \mathcal{K}_g^a$, but $\langle T_1, T_2 \rangle$ is not contained in $\mathcal{K}_g^a$. We now prove:

**Lemma 22.3.** *The linearization map $\Phi$ takes $\langle S_1, S_2 \rangle$ isomorphically to $\mathcal{K}_g^a \cap \langle T_1, T_2 \rangle$.*

*Proof.* Recall from §21.1 that $T = T_1 \cup T_2 \cup T_3$ where

$$\begin{aligned} T_1 = \{(x \wedge y) \wedge (z \wedge w) \mid\ & x,y,z,w \in \mathcal{B},\, x \prec y,\, z \prec w,\, x \wedge y \prec z \wedge w, \\ & \text{and } \omega(x,z) = \omega(x,w) = \omega(y,z) = \omega(y,w) = 0\}, \\ T_2 = \{(a \wedge a_i) \wedge (a' \wedge b_i) \mid\ & 1 \leq i \leq g,\, a \in \mathcal{B} \setminus \{a_i\},\, a' \in \mathcal{B} \setminus \{b_i\}, \\ & \text{and } \omega(a,a') = \omega(a_i,a') = \omega(b_i,a) = 0\}, \\ T_3 = \{(a_i \wedge a_j) \wedge (b_i \wedge b_j),\ & (a_i \wedge b_j) \wedge (b_i \wedge a_j) \mid 1 \leq i < j \leq g\}. \end{aligned}$$

Moreover, $\wedge^2((\wedge^2 H)/\mathbb{Q})$ is the $\mathbb{Q}$-vector space with generators $T$ subject to the relations

$$R = \left\{ \sum\nolimits_{i=1}^{g} (a_i \wedge b_i) \wedge (x \wedge y) \mid x,y \in \mathcal{B},\, x \prec y \right\}.$$

Here each element of $R$ should be interpreted as a linear combination of elements of $T$ using Convention 21.1. Another way of stating this is that $\wedge^2((\wedge^2 H)/\mathbb{Q})$ is the quotient of[30] $\langle T_1 \rangle \oplus \langle T_2 \rangle \oplus \langle T_3 \rangle$ by the span of elements corresponding to $R$. We have:

**Claim 1.** *Each $r \in R$ corresponds to an element of $\langle T_1 \rangle \oplus \langle T_2 \rangle$.*

[29]After reading all the proofs in §18 – §19, the reader might expect the word "distinct" to not appear in $Y$. To make some of the proofs in §18 – §19 work (e.g., the proof of Lemma 18.3), the set $Y$ needs to contain elements of the form $a_i \otimes b_i + b_i \otimes a_i$. Here, however, we can require $i$ and $j$ be distinct. Indeed, consider $3 \leq i \leq g$. We want to prove that $a_i \otimes b_i + b_i \otimes a_i$ is in the span of $Y$. For this, note that

$$a_i \otimes b_i + b_i \otimes a_i = (a_i \otimes b_i + b_1 \otimes a_1) - (a_2 \otimes b_2 + b_1 \otimes a_1) + (a_2 \otimes b_2 + a_i \otimes b_i).$$

[30]Here the $\langle T_i \rangle$ are subspaces of $\wedge^2((\wedge^2 H)/\mathbb{Q})$, so possibly some relations in $R$ already hold in $\langle T_1 \rangle \oplus \langle T_2 \rangle \oplus \langle T_3 \rangle$; indeed, as we will see this is in fact the case.

*Proof of claim.* Consider $x, y \in \mathcal{B}$ with $x \prec y$, so we have an element

$$r = \sum_{i=1}^{g} (a_i \wedge b_i) \wedge (x \wedge y) \in R.$$

There are two cases. The first is that $\omega(x, y) \neq 0$, so $(x, y) = (a_k, b_k)$ for some $1 \leq k \leq g$. In this case, $r$ is actually an element of $\langle T_1 \rangle$, or rather a linear combination of elements of $T_1$ that vanishes in $\langle T_1 \rangle$. The point here is that our convention is that the term $(a_k \wedge a_k) \wedge (a_k \wedge b_k)$ is deleted, and the rest of the terms clearly lie in $T_1$.

The second case is that $\omega(x, y) = 0$. There are a number of cases, so we will explain how to deal with $x = a_k$ and $y = a_\ell$ for some $1 \leq k < \ell \leq g$. The other cases are similar (but with slightly different notation). We have

$$\begin{aligned} r &= \sum_{i=1}^{g} (a_i \wedge b_i) \wedge (a_k \wedge a_\ell) \\ &= (a_k \wedge b_k) \wedge (a_k \wedge a_\ell) + (a_\ell \wedge b_\ell) \wedge (a_k \wedge a_\ell) + \sum_{\substack{1 \leq i \leq g \\ i \neq k, \ell}} (a_i \wedge b_i) \wedge (a_k \wedge a_\ell). \end{aligned}$$

The blue terms lie in $\langle T_1 \rangle$, while the remaining terms lie in $\langle T_2 \rangle$ since

$$\begin{aligned} &(a_k \wedge b_k) \wedge (a_k \wedge a_\ell) + (a_\ell \wedge b_\ell) \wedge (a_k \wedge a_\ell) \\ =&(a_\ell \wedge a_k) \wedge (a_k \wedge b_k) - (a_k \wedge a_\ell) \wedge (a_\ell \wedge b_\ell). \end{aligned}$$

The claim follows. □

The set $T_2$ is the disjoint union of the $T_2(a{\cdot}a')$ as $a{\cdot}a'$ ranges over elements of $S_0^2(\mathcal{B}) = \{a{\cdot}a' \in \mathrm{Sym}^2(H) \mid a, a' \in \mathcal{B}, \omega(a, a') = 0\}$. Using the above claim, we deduce that $\langle T_1, T_2 \rangle$ is the quotient of the direct sum

$$\langle T_1 \rangle \oplus \bigoplus_{s \in S_0^2(\mathcal{B})} \langle T_2(s) \rangle$$

by the subspace generated by the relations in $R$. The subspace $\mathcal{K}_g^a$ is the kernel of the symmetric contraction $\mathfrak{c}\colon \wedge^2((\wedge^2 H)/\mathbb{Q}) \to \mathrm{Sym}^2(H)$. The symmetric contraction $\mathfrak{c}$ vanishes on $T_1$, and for $s \in S_0^2(\mathcal{B})$ it takes elements of $T_2(s)$ to $s$ (see §22.1). Since $S_0^2(\mathcal{B})$ is a linearly independent subset of $\mathrm{Sym}^2(H)$, we deduce that $\mathcal{K}_g^a \cap \langle T_1, T_2 \rangle$ is the quotient of

$$\langle T_1 \rangle \oplus \bigoplus_{s \in S_0^2(\mathcal{B})} \mathcal{K}_g^a \cap \langle T_2(s) \rangle \tag{22.1}$$

by the relations in $R$. We remark that the relations in $R$ must lie in the above direct sum since otherwise they would map to nontrivial elements of $\mathrm{Sym}^2(H)$ under $\mathfrak{c}$.

We proved in Lemma 21.2 that $\Phi$ restricts to an isomorphism between $\langle S_1 \rangle$ and $\langle T_1 \rangle$. For $s \in S_0^2(\mathcal{B})$, recall that $S_2(s)$ is a vector space. We proved in Lemma 22.2 that $\Phi$ restricts to an isomorphism between $S_2(s)$ and $\mathcal{K}_g^a \cap \langle T_2(s) \rangle$. From this and in light of the previous paragraph,[31] to prove that $\Phi$ restricts to an isomorphism between $\langle S_1, S_2 \rangle$ and $\langle T_1, T_2 \rangle$, it is enough to prove that each relation in $R$ lifts to a relation in $\langle S_1, S_2 \rangle$.

The relations of the form

$$r = \sum_{i=1}^{g} (a_i \wedge b_i) \wedge (a_k \wedge b_k)$$

for some $1 \leq k \leq g$ are relations between elements of $T_1$, and since $\Phi$ restricts to an isomorphism between $\langle S_1 \rangle$ and $\langle T_1 \rangle$ these relations lift[32] to relations in $\langle S_1 \rangle$. We must therefore only deal with the relations in the following claim:

[31] In particular, the decomposition (22.1).

[32] See the proof of Lemma 21.2 for explicit lifts.

**Claim 2.** *Consider $x, y \in \mathcal{B}$ with $x \prec y$. Then the relation*

$$r = \sum_{i=1}^{g} (a_i \wedge b_i) \wedge (x \wedge y)$$

*lifts to a relation in $\mathfrak{K}_g^a$.*

*Proof of claim.* We will give the details for $(x, y) = (a_1, a_2)$. The other cases are similar but require worse notation. Write our relation as

$$\begin{aligned} r &= \sum_{i=1}^{g} (a_i \wedge b_i) \wedge (a_1 \wedge a_2) \\ &= -(a_1 \wedge b_1) \wedge (a_2 \wedge a_1) - (a_1 \wedge a_2) \wedge (a_2 \wedge b_2) + \sum_{i=3}^{g} (a_i \wedge b_i) \wedge (a_1 \wedge a_2). \end{aligned}$$

The blue sum lifts to the following element of $\langle S_1 \rangle$:

$$\sum_{i=3}^{g} [\![a_i \wedge b_i, a_1 \wedge a_2]\!]_a.$$

We claim that the remaining part lifts to the following element of $\langle S_1, S_2 \rangle$:

$$-[\![a_1 \wedge (b_1 + a_2), a_2 \wedge (a_1 + b_2)]\!]_a + [\![a_1 \wedge b_1, a_2 \wedge b_2]\!]_a. \tag{22.2}$$

To see this, note that $\Phi$ maps (22.2) to[33]

$$\begin{aligned} &- (a_1 \wedge (b_1 + a_2)) \wedge (a_2 \wedge (a_1 + b_2)) + (a_1 \wedge b_1) \wedge (a_2 \wedge b_2) \\ =&- (a_1 \wedge b_1) \wedge (a_2 \wedge a_1) - (a_1 \wedge a_2) \wedge (a_2 \wedge a_1) - (a_1 \wedge b_1) \wedge (a_2 \wedge b_2) \\ &- (a_1 \wedge a_2) \wedge (a_2 \wedge b_2) + (a_1 \wedge b_1) \wedge (a_2 \wedge b_2) \\ =&- (a_1 \wedge b_1) \wedge (a_2 \wedge a_1) - (a_1 \wedge a_2) \wedge (a_2 \wedge b_2), \end{aligned}$$

as desired.

Combining the above lifts, we see that the relation in $\mathfrak{K}_g^a$ we must verify is

$$0 = -[\![a_1 \wedge (b_1 + a_2), a_2 \wedge (a_1 + b_2)]\!]_a + [\![a_1 \wedge b_1, a_2 \wedge b_2]\!]_a + \sum_{i=3}^{g} [\![a_i \wedge b_i, a_1 \wedge a_2]\!]_a. \tag{22.3}$$

For this, note that $\{a_1, b_1 + a_2, a_2, a_1 + b_2, a_3, b_3, \ldots, a_g, b_g\}$ is a symplectic basis. In $(\wedge^2 H)/\mathbb{Q}$, we therefore have

$$a_1 \wedge (b_1 + a_2) + a_2 \wedge (a_1 + b_2) + a_3 \wedge b_3 + \cdots + a_g \wedge b_g = 0.$$

By plugging this into its first term, we calculate that $[\![a_1 \wedge (b_1 + a_2), a_2 \wedge (a_1 + b_2)]\!]_a$ equals

$$\begin{aligned} &- [\![a_2 \wedge (a_1 + b_2), a_2 \wedge (a_1 + b_2)]\!]_a - \sum_{i=3}^{g} [\![a_i \wedge b_i, a_2 \wedge (a_1 + b_2)]\!]_a \\ =&- \sum_{i=3}^{g} \left([\![a_i \wedge b_i, a_2 \wedge a_1]\!]_a + [\![a_i \wedge b_i, a_2 \wedge b_2]\!]_a\right) \\ =&- \left(\sum_{i=3}^{g} [\![a_i \wedge b_i, a_2 \wedge b_2]\!]_a\right) + \left(\sum_{i=3}^{g} [\![a_i \wedge b_i, a_1 \wedge a_2]\!]_a\right). \end{aligned}$$

In $(\wedge^2 H)/\mathbb{Q}$, we have $\sum_{i=1}^{g} a_i \wedge b_i = 0$. Plugging this into the first term of our formula, the formula becomes

$$\begin{aligned} &[\![a_1 \wedge b_1, a_2 \wedge b_2]\!]_a + [\![a_2 \wedge b_2, a_2 \wedge b_2]\!]_a + \sum_{i=3}^{g} [\![a_i \wedge b_i, a_1 \wedge a_2]\!]_a \\ =&[\![a_1 \wedge b_1, a_2 \wedge b_2]\!]_a + \sum_{i=3}^{g} [\![a_i \wedge b_i, a_1 \wedge a_2]\!]_a. \end{aligned}$$

Since this equals $[\![a_1 \wedge (b_1 + a_2), a_2 \wedge (a_1 + b_2)]\!]_a$, the relation (22.3) follows. ☐

This completes the proof of the lemma. ☐

[33] Part of this calculation is that $(a_1 \wedge a_2) \wedge (a_2 \wedge a_1) = 0$.

22.4. **Relations in $T_3$.** We extract a useful consequence of the above proof:

**Lemma 22.4.** *The quotient of $\wedge^2((\wedge^2 H)/\mathbb{Q})$ by $\langle T_1, T_2\rangle$ has dimension $g(g-1)$.*

*Proof.* Claim 1 of the proof of Lemma 22.3 says that all relations between $T = T_1 \cup T_2 \cup T_3$ are actually relations between $T_1 \cup T_2$. Since $T$ generates $\wedge^2((\wedge^2 H)/\mathbb{Q})$, it follows that the indicated quotient has dimension $|T_3|$. Since

$$T_3 = \{(a_i \wedge a_j) \wedge (b_i \wedge b_j),\ (a_i \wedge b_j) \wedge (b_i \wedge a_j) \mid 1 \leq i < j \leq g\},$$

the set $T_3$ has cardinality $2\binom{g}{2} = g(g-1)$. The lemma follows. □

22.5. **The set $S_{12}$.** Recall that

$$S_{12} = \bigcup_{a \cdot a' \in S_0^2(\mathcal{B})} \mathfrak{F}[a, a'].$$

Our goal in the rest of Part 3 is to prove Theorem F, which says that $\Phi$ is an isomorphism from $\mathfrak{K}_g^a = \langle S\rangle = \langle S_{12}, S_3\rangle$ to

$$\mathcal{K}_g^a \subset \wedge^2((\wedge^2 H)/\mathbb{Q}) = \langle T_1, T_2, T_3\rangle.$$

We close this section by proving the following partial result in this direction:

**Lemma 22.5.** *The linearization map $\Phi$ takes $\langle S_{12}\rangle$ isomorphically to $\mathcal{K}_g^a \cap \langle T_1, T_2\rangle$.*

*Proof.* Lemma 22.3 says that $\Phi$ takes $\langle S_1, S_2\rangle$ isomorphically to $\mathcal{K}_g^a \cap \langle T_1, T_2\rangle$, so it is enough to prove that $\langle S_{12}\rangle = \langle S_1, S_2\rangle$. For $a \cdot a' \in S_0^2(\mathcal{B})$, we proved in Lemma 22.2 that $S_1(a \cdot a') \cup S_2(a \cdot a')$ spans $\mathfrak{F}[a, a']$. For $i = 1, 2$, we have

$$S_i = \bigcup_{a \cdot a' \in S_0^2(\mathcal{B})} S_i(a \cdot a').$$

This is a disjoint union for $i = 2$, but the $S_1(a \cdot a')$ for different $a \cdot a' \in S_0^2(\mathcal{B})$ overlap. Combining these two facts, we see that

$$\langle S_1, S_2\rangle = \langle \bigcup_{a \cdot a' \in S_0^2(\mathcal{B})} S_1(a \cdot a') \cup S_2(a \cdot a')\rangle = \langle \bigcup_{a \cdot a' \in S_0^2(\mathcal{B})} \mathfrak{F}[a, a']\rangle = \langle S_{12}\rangle. \qquad \square$$

## 23. Symmetric kernel, alternating version VII: structure of $S_3$

We will continue using all the notation from §17 – §22. Having proved in Lemma 22.5 that $\Phi$ takes $\langle S_{12}\rangle$ isomorphically to $\mathcal{K}_g^a \cap \langle T_1, T_2\rangle$, our remaining task in Part 3 is to extend this to $\langle S\rangle = \langle S_{12}, S_3\rangle$ and prove Theorem F. We will do this in §25. This section and the next one contain some preliminary results about $S_3$.

23.1. **Quotients.** Define

$$\mathfrak{T}_g = \mathfrak{K}_g^a / \langle S_{12}\rangle,$$
$$\mathcal{T}_g = \mathcal{K}_g^a / (\mathcal{K}_g^a \cap \langle T_1, T_2\rangle).$$

Lemma 22.5 says that the linearization map $\Phi \colon \mathfrak{K}_g^a \to \mathcal{K}_g^a$ takes $\langle S_{12}\rangle$ isomorphically to $\mathcal{K}_g^a \cap \langle T_1, T_2\rangle$. It follows that $\Phi$ descends to a map $\overline{\Phi} \colon \mathfrak{T}_g \to \mathcal{T}_g$. Our goal is to prove that $\Phi$ is an isomorphism (Theorem F). Since $\Phi$ restricts to an isomorphism from $\langle S_{12}\rangle$ to $\mathcal{K}_g^a \cap \langle T_1, T_2\rangle$, this is equivalent to proving that $\overline{\Phi}$ is an isomorphism. That $\overline{\Phi}$ is surjective is easy:

**Lemma 23.1.** *The map $\overline{\Phi} \colon \mathfrak{T}_g \to \mathcal{T}_g$ is surjective.*

*Proof.* This follows from the fact that $\Phi\colon \mathfrak{K}_g^a \to \mathcal{K}_g^a$ is surjective. This could be proved directly, but another approach is to note that

$$\mathcal{K}_g^a = \ker(\wedge^2((\wedge^2 H)/\mathbb{Q}) \xrightarrow{\mathfrak{c}} \mathrm{Sym}^2(H))$$

is an irreducible algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$. Such representations are indexed by partitions with at most $g$ parts (see [2, §17]), and $\mathcal{K}_g^a$ is the one corresponding to the partition $2+1+1$. Since $\Phi$ is not the zero map, its image is a nonzero subrepresentation of the irreducible representation $\mathcal{K}_g^a$, and hence its image must be $\mathcal{K}_g^a$. □

23.2. **Dimension of target.** We now prove:

**Lemma 23.2.** *The vector space $\mathcal{T}_g$ is $g(g-2)$-dimensional.*

*Proof.* The vector space $\mathcal{K}_g^a$ is the kernel of the symmetric contraction $\mathfrak{c}\colon \wedge^2((\wedge^2 H)/\mathbb{Q}) \to \mathrm{Sym}^2(H)$. Let:

- $V$ be be the quotient of $\wedge^2((\wedge^2 H)/\mathbb{Q})$ by $\langle T_1, T_2\rangle$; and
- $W$ be the quotient of $\mathrm{Sym}^2(H)$ by $\mathfrak{c}(\langle T_1, T_2\rangle)$.

The symmetric contraction induces a map $\bar{\mathfrak{c}}\colon V \to W$, and $\mathcal{T}_g$ is isomorphic to $\ker(\bar{\mathfrak{c}})$.

Lemma 22.4 says that $\dim(V) = g(g-1)$. To calculate $\dim(W)$, note that Lemma 22.1 implies that $\mathfrak{c}(\langle T_1, T_2\rangle)$ is the subspace of $\mathrm{Sym}^2(H)$ spanned by $\{a\cdot a' \mid a, a' \in \mathcal{B},\ \omega(a,a') = 0\}$. This implies that the set $\{a_1\cdot b_1, \ldots, a_g \cdot b_g\}$ is a basis for a complement to $\mathfrak{c}(\langle T_1, T_2\rangle)$, so $\dim(W) = g$.

The map $\mathfrak{c}$ is surjective: this could be proved directly, but just like in the proof of Lemma 23.1 it also follows from the fact that $\mathrm{Sym}^2(H)$ is an irreducible algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$. The corresponding partition is simply 2. This implies that $\bar{\mathfrak{c}}$ is also surjective. Consequently,

$$\dim(\mathcal{T}_g) = \dim(V) - \dim(W) = g(g-1) - g = g(g-2).$$ □

23.3. **Proof strategy.** Recall that we want to prove that $\overline{\Phi}\colon \mathfrak{T}_g \to \mathcal{T}_g$ is an isomorphism. Lemmas 23.1 and 23.2 say that $\overline{\Phi}$ is a surjective map to a $g(g-2)$-dimensional vector space. To prove that $\overline{\Phi}$ is an isomorphism, it is enough to prove that $\mathfrak{T}_g$ is at most $g(g-2)$-dimensional. We will do this via a calculation involving generators and relations. The rest of this section is devoted to constructing a generating set for $\mathfrak{T}_g$. We will then give some relations in $\mathfrak{T}_g$ in §24, and complete the proof in §25.

23.4. **Basic elements.** Recall that

$$S_3 = \bigcup_{\substack{1 \leq i,j \leq g \\ i \neq j}} \mathfrak{F}[a_i - b_j, b_i - a_j].$$

Elements of $S_3$ are written in orange. For $\eta \in \mathfrak{K}_g^a$, let $\overline{\eta}$ be its image in $\mathfrak{T}_g = \mathfrak{K}_g^a / \langle S_{12}\rangle$. For $1 \leq i,j,k \leq g$ distinct, define

$$\Delta^i_{jk} = \overline{[\![(a_i - b_j) \wedge (a_k - b_i), (b_i - a_j) \wedge (b_k - a_i)]\!]_a} \in \mathfrak{T}_g.$$

We call $\Delta^i_{jk}$ a *basic element* of $\mathfrak{T}_g$. These satisfy:

**Lemma 23.3.** *For $1 \leq i,j,k \leq g$ distinct, we have $\Delta^i_{kj} = -\Delta^i_{jk}$.*

*Proof.* Immediate from the fact that

$$[\![(a_i - b_j) \wedge (a_k - b_i), (b_i - a_j) \wedge (b_k - a_i)]\!]_a = -[\![(a_i - b_k) \wedge (b_i - a_j), (b_i - a_k) \wedge (a_i - b_j)]\!]_a.$$ □

23.5. **Generation by basic elements.** We now prove:

**Lemma 23.4.** *The vector space* $\mathfrak{T}_g$ *is spanned by* $\left\{\Delta^i_{jk} \mid 1 \leq i, j, k \leq g \text{ distinct}\right\}$.

*Proof.* Lemma 20.1 says that $\mathfrak{K}^a_g$ is spanned by $S = S_{12} \cup S_3$. It follows that $\mathfrak{T}_g = \mathfrak{K}^a_g / \langle S_{12} \rangle$ is spanned by the image of $S_3$. Fixing some $1 \leq i, j \leq g$ distinct, it is therefore enough to prove that the image of $\mathfrak{F}[a_i - b_j, b_i - a_j]$ in $\mathfrak{T}_g$ is contained in the span of the indicated generating set.

In Lemma 18.2, we proved that the action of the symmetric group $\mathfrak{S}_g$ on $\mathfrak{K}^a_g$ takes $\langle S \rangle$ to itself. It follows from the proof of that lemma that this action also takes $\langle S_{12} \rangle$ to itself, so we get an induced action of $\mathfrak{S}_g$ on $\mathfrak{T}_g$. Applying an appropriate of $\mathfrak{S}_g$, we reduce ourselves to proving that the image of $\mathfrak{F}[a_1 - b_2, b_1 - a_2]$ is contained in the span of the indicated generating set.

We construct generators for $\mathfrak{F}[a_1 - b_2, b_1 - a_2]$ in the now-familiar way and then show that their images in $\mathfrak{T}_g$ are in the span of the indicated generating set. Following the notation in §17.4, define

$$\mathcal{V} = \langle b_1 - a_2 \rangle^{\perp}_{\mathbb{Q}} / \langle a_1 - b_2 \rangle \cong \langle A_V \rangle_{\mathbb{Q}} \quad \text{with} \quad A_V = \{b_1, a_2, a_3, b_3, \ldots, a_g, b_g\},$$
$$\mathcal{W} = \langle a_1 - b_2 \rangle^{\perp}_{\mathbb{Q}} / \langle b_1 - a_2 \rangle \cong \langle A_W \rangle_{\mathbb{Q}} \quad \text{with} \quad A_W = \{a_1, b_2, a_3, b_3, \ldots, a_g, b_g\}.$$

We proved in Lemma 17.4 that $\mathfrak{F}[a_1 - b_2, b_1 - a_2]$ is isomorphic to a quotient of the kernel of the map $\mathcal{V} \otimes \mathcal{W} \to \mathbb{Q}$ induced by the symplectic form $\omega$. Under this isomorphism, a generator $[\![(a_1 - b_2) \wedge x, (b_1 - a_2) \wedge y]\!]_a$ of $\mathfrak{F}[a_1 - b_2, b_1 - a_2]$ maps to $x \otimes y \in \mathcal{V} \otimes \mathcal{W}$.

The kernel of $\mathcal{V} \otimes \mathcal{W} \to \mathbb{Q}$ is spanned by $X \cup Y$ where

$$\begin{aligned} X &= \{x \otimes y \mid x \in A_V,\, y \in A_W,\, \omega(x, y) = 0\}, \\ Y &= \{a_k \otimes b_k + b_1 \otimes a_1,\, a_2 \otimes b_2 + b_k \otimes a_k \mid 3 \leq k \leq g\} \\ &\quad \cup \{a_2 \otimes b_2 + b_1 \otimes a_1\}. \end{aligned}$$

Since for $3 \leq k \leq g$ we have

$$\begin{aligned} (a_k - b_1) \otimes (b_k - a_1) &= a_k \otimes b_k + b_1 \otimes a_1 + \text{an element of } \langle X \rangle, \\ (a_2 - b_k) \otimes (b_2 - a_k) &= a_2 \otimes b_2 + b_k \otimes a_k + \text{an element of } \langle X \rangle, \\ (a_1 - b_2) \otimes (b_1 - a_2) &= a_1 \otimes b_1 + b_2 \otimes a_2 + \text{an element of } \langle X \rangle, \end{aligned}$$

we can replace $Y$ by the set

$$\begin{aligned} &\{(a_k - b_1) \otimes (b_k - a_1),\, (a_2 - b_k) \otimes (b_2 - a_k) \mid 3 \leq k \leq g\}, \\ &\quad \cup \{(a_2 - b_1) \otimes (b_2 - a_1)\}. \end{aligned}$$

From this, we see that $\mathfrak{F}[a_1 - b_2, b_1 - a_2]$ is generated by the elements listed in the following cases. To prove the lemma, we must prove that each of these generators maps to something in the span of the indicated generators of $\mathfrak{T}_g$.

**Case 1.** $[\![(a_1 - b_2) \wedge x, (b_1 - a_2) \wedge y]\!]_a$ *for* $x \in A_V$ *and* $y \in A_W$ *with* $\omega(x, y) = 0$.

These equal $[\![x \wedge (a_1 - b_2), y \wedge (b_1 - a_2)]\!]_a \in S_{12}$, and thus go to zero in $\mathfrak{T}_g$.

**Case 2.** $[\![(a_1 - b_2) \wedge (a_k - b_1), (b_1 - a_2) \wedge (b_k - a_1)]\!]_a$ *with* $3 \leq k \leq g$.

This maps to $\Delta^1_{2k} \in \mathfrak{T}_g$.

**Case 3.** $[\![(a_1 - b_2) \wedge (a_2 - b_k), (b_1 - a_2) \wedge (b_2 - a_k)]\!]_a$ *with* $3 \leq k \leq g$.

This equals $-[\![(a_2 - b_1) \wedge (a_k - b_2), (b_2 - a_1) \wedge (b_k - a_2)]\!]_a$, which maps to $-\Delta^2_{1k} \in \mathfrak{T}_g$.

**Case 4.** $[\![(a_1 - b_2) \wedge (a_2 - b_1), (b_1 - a_2) \wedge (b_2 - a_1)]\!]_a$.

Since $[\![-,-]\!]$ is alternating, this equals $-[\![(a_1-b_2)\wedge(a_2-b_1),(a_1-b_2)\wedge(a_2-b_1)]\!]_a=0$. We remark that this is why we always insist that $\mathfrak{F}[a_1-b_2,b_1-a_2]$ is a *quotient* of the kernel of the map $\mathcal{V}\otimes\mathcal{W}\to\mathbb{Q}$; see Lemma 17.4. □

## 24. Symmetric kernel, alternating version VIII: relations between basic elements

We will continue using all the notation from §17 – §23. This section contains three relations involving our basic elements $\Delta^i_{jk}$. We will use these relations to prove Theorem F in §25.

24.1. **Relation I.** The first is:

**Lemma 24.1.** *Let $1\leq i,j,k,\ell\leq g$ be distinct. Then $\Delta^k_{ij}+\Delta^i_{\ell k}=\Delta^\ell_{ij}+\Delta^j_{\ell k}$.*

*Proof.* As in the proof of Lemma 23.4, we can apply an appropriate element of the symmetric group and reduce ourselves to proving that $\Delta^3_{12}+\Delta^1_{43}=\Delta^4_{12}+\Delta^2_{43}$. What we will prove is that both sides of this identity equal $\overline{[\![(a_3-b_1)\wedge(a_2-b_4),(b_3-a_1)\wedge(b_2-a_4)]\!]}$:

**Claim 1.** $\Delta^3_{12}+\Delta^1_{43}=\overline{[\![(a_3-b_1)\wedge(a_2-b_4),(b_3-a_1)\wedge(b_2-a_4)]\!]}$.

We have

$$\begin{aligned}\Delta^3_{12}&=\overline{[\![(a_3-b_1)\wedge(a_2-b_3),(b_3-a_1)\wedge(b_2-a_3)]\!]},\\ \Delta^1_{43}&=\overline{[\![(a_1-b_4)\wedge(a_3-b_1),(b_1-a_4)\wedge(b_3-a_1)]\!]}\\ &=\overline{[\![(a_3-b_1)\wedge(a_1-b_4),(b_3-a_1)\wedge(b_1-a_4)]\!]}.\end{aligned}$$

We must therefore prove that

$$[\![(a_3-b_1)\wedge(a_2-b_3),(b_3-a_1)\wedge(b_2-a_3)]\!]+[\![(a_3-b_1)\wedge(a_1-b_4),(b_3-a_1)\wedge(b_1-a_4)]\!]\tag{24.1}$$

equals

$$[\![(a_3-b_1)\wedge(a_2-b_4),(b_3-a_1)\wedge(b_2-a_4)]\!]\tag{24.2}$$

modulo elements of $\langle S_{12}\rangle$. These both live in $\mathfrak{F}[a_3-b_1,b_3-a_1]$, so we work there.

Following the notation in §17.4, define

$$\begin{aligned}\mathcal{V}=&\langle b_3-a_1\rangle^{\perp}_{\mathbb{Q}}/\langle a_3-b_1\rangle\cong\langle A_V\rangle_{\mathbb{Q}}\quad\text{with}\quad A_V=\{b_3,a_1,a_2,b_2,a_3,b_3,\ldots,a_g,b_g\},\\ \mathcal{W}=&\langle a_3-b_1\rangle^{\perp}_{\mathbb{Q}}/\langle b_3-a_1\rangle\cong\langle A_W\rangle_{\mathbb{Q}}\quad\text{with}\quad A_W=\{a_3,b_1,a_2,b_2,a_3,b_3,\ldots,a_g,b_g\}.\end{aligned}$$

We proved in Lemma 17.4 that $\mathfrak{F}[a_3-b_1,b_3-a_1]$ is isomorphic to a quotient of the kernel of the map $\mathcal{V}\otimes\mathcal{W}\to\mathbb{Q}$ induced by the symplectic form $\omega$. Under this isomorphism, a generator $[\![(a_3-b_1)\wedge x,(b_3-a_1)\wedge y]\!]_a$ of $\mathfrak{F}[a_3-b_1,b_3-a_1]$ maps to $x\otimes y\in\mathcal{V}\otimes\mathcal{W}$.

The elements of $\mathcal{V}\otimes\mathcal{W}$ corresponding to elements of $S_{12}$ are

$$X=\{x\otimes y\mid x\in A_V,\,y\in A_W,\,\omega(x,y)=0\}.$$

We can therefore work modulo $\langle X\rangle$. Let $\equiv$ denote equality modulo $\langle X\rangle$. The element of $\mathcal{V}\otimes\mathcal{W}$ corresponding to (24.1) is

$$\begin{aligned}(a_2-b_3)\otimes(b_2-a_3)+(a_1-b_4)\otimes(b_1-a_4)&\equiv(a_2\otimes b_2+b_3\otimes a_3)+(a_1\otimes b_1+b_4\otimes a_4)\\ &=(a_1\otimes b_1+b_3\otimes a_3)+(a_2\otimes b_2+b_4\otimes a_4)\\ &\equiv(a_1-b_3)\otimes(b_1-a_3)+(a_2-b_4)\otimes(b_2-a_4).\end{aligned}$$

The element $(a_2-b_4)\otimes(b_2-a_4)$ corresponds to (24.2), so what we must prove is that $(a_1-b_3)\otimes(b_1-a_3)$ corresponds to 0. In fact, this corresponds to

$$[\![(a_3-b_1)\wedge(a_1-b_3),(b_3-a_1)\wedge(b_1-a_3)]\!]_a.$$

Since $[\![-,-]\!]_a$ is alternating, this vanishes. The claim follows.

**Claim 2.** $\Delta^4_{12} + \Delta^2_{43} = \overline{[\![(a_3 - b_1) \wedge (a_2 - b_4), (b_3 - a_1) \wedge (b_2 - a_4)]\!]}$.

We have

$$\begin{aligned}\Delta^4_{12} &= \overline{[\![(a_4 - b_1) \wedge (a_2 - b_4), (b_4 - a_1) \wedge (b_2 - a_4)]\!]} \\ &= \overline{[\![(a_2 - b_4) \wedge (a_4 - b_1), (b_2 - a_4) \wedge (b_4 - a_1)]\!]}, \\ \Delta^2_{43} &= \overline{[\![(a_2 - b_4) \wedge (a_3 - b_2), (b_2 - a_4) \wedge (b_3 - a_2)]\!]}.\end{aligned}$$

and

$$\overline{[\![(a_3 - b_1) \wedge (a_2 - b_4), (b_3 - a_1) \wedge (b_2 - a_4)]\!]} = \overline{[\![(a_2 - b_4) \wedge (a_3 - b_1), (b_2 - a_4) \wedge (b_3 - a_1)]\!]}.$$

We must therefore prove that

$$[\![(a_2 - b_4) \wedge (a_4 - b_1), (b_2 - a_4) \wedge (b_4 - a_1)]\!] + [\![(a_2 - b_4) \wedge (a_3 - b_2), (b_2 - a_4) \wedge (b_3 - a_2)]\!]$$

equals

$$[\![(a_2 - b_4) \wedge (a_3 - b_1), (b_2 - a_4) \wedge (b_3 - a_1)]\!]$$

modulo elements of $\langle S_{12}\rangle$. These both live in $\mathfrak{F}[a_2 - b_4, b_2 - a_4]$. The calculation is similar to the one from Claim 1, so we omit it. □

24.2. **Relation II.** The second relation is:

**Lemma 24.2.** *For $1 \leq i, j, k \leq g$ distinct, we have*

$$\Delta^i_{jk} = \overline{[\![(a_i - b_j) \wedge (a_k + a_j), (b_i - a_j) \wedge (b_k - b_j)]\!]_a}.$$

*Proof.* As in the proof of Lemma 23.4, we can apply an appropriate element of the symmetric group and reduce ourselves to proving that

$$\Delta^1_{23} = \overline{[\![(a_1 - b_2) \wedge (a_3 + a_2), (b_1 - a_2) \wedge (b_3 - b_2)]\!]_a}.$$

By the definition of $\Delta^1_{23}$ and $\mathfrak{T}_g$, this lemma is asserting that

$$\begin{aligned} &[\![(a_1 - b_2) \wedge (a_3 - b_1), (b_1 - a_2) \wedge (b_3 - a_1)]\!]_a \\ &- [\![(a_1 - b_2) \wedge (a_3 + a_2), (b_1 - a_2) \wedge (b_3 - b_2)]\!]_a \end{aligned} \tag{24.3}$$

lies in the span of $S_{12}$. This difference lies in $\mathfrak{F}[a_1 - b_2, b_1 - a_2]$, so we work there.

Following the notation in §17.4, define

$$\begin{aligned}\mathcal{V} =& \langle b_1 - a_2\rangle^{\perp}_{\mathbb{Q}} / \langle a_1 - b_2\rangle \cong \langle A_V\rangle_{\mathbb{Q}} \quad \text{with} \quad A_V = \{b_1, a_2, a_3, b_3, \ldots, a_g, b_g\}, \\ \mathcal{W} =& \langle a_1 - b_2\rangle^{\perp}_{\mathbb{Q}} / \langle b_1 - a_2\rangle \cong \langle A_W\rangle_{\mathbb{Q}} \quad \text{with} \quad A_W = \{a_1, b_2, a_3, b_3, \ldots, a_g, b_g\}.\end{aligned}$$

We proved in Lemma 17.4 that $\mathfrak{F}[a_1 - b_2, b_1 - a_2]$ is isomorphic to a quotient of the kernel of the map $\mathcal{V} \otimes \mathcal{W} \to \mathbb{Q}$ induced by the symplectic form $\omega$. Under this isomorphism, a generator $[\![(a_1 - b_2) \wedge x, (b_1 - a_2) \wedge y]\!]_a$ of $\mathfrak{F}[a_1 - b_2, b_1 - a_2]$ maps to $x \otimes y \in \mathcal{V} \otimes \mathcal{W}$.

The elements of $\mathcal{V} \otimes \mathcal{W}$ corresponding to elements of $S_{12}$ are

$$X = \{x \otimes y \mid x \in A_V,\, y \in A_W,\, \omega(x, y) = 0\}\,.$$

We can therefore work modulo $\langle X\rangle$. Let $\equiv$ denote equality modulo $\langle X\rangle$. The difference (24.3) corresponds to the following element of $\mathcal{V} \otimes \mathcal{W}$:

$$\begin{aligned}&(a_3 - b_1) \otimes (b_3 - a_1) - (a_3 + a_2) \otimes (b_3 - b_2) \equiv (a_3 \otimes b_3 + b_1 \otimes a_1) - (a_3 \otimes b_3 - a_2 \otimes b_2) \\ =& a_2 \otimes b_2 + b_1 \otimes a_1 \equiv (a_2 - b_1) \otimes (b_2 - a_1).\end{aligned}$$

This corresponds to the element

$$[\![(a_1 - b_2) \wedge (a_2 - b_1), (b_1 - a_2) \wedge (b_2 - a_1)]\!]_a = -[\![(a_1 - b_2) \wedge (a_2 - b_1), (a_1 - b_2) \wedge (a_2 - b_1)]\!]_a.$$

Since $[\![-, -]\!]_a$ is alternating, this is 0. The lemma follows. □

24.3. **Relation III.** Our third and final relation is:

**Lemma 24.3.** *Let $1 \leq i, j, k, \ell \leq g$ be distinct. Then*

$$\Delta^i_{jk} + \Delta^i_{k\ell} = \Delta^i_{j\ell} + \overline{[\![(a_\ell + a_j) \wedge (b_k - b_j), (b_\ell - b_j) \wedge (a_k + a_j)]\!]_a}.$$

*Proof.* As in the proof of Lemma 23.4, we can apply an appropriate element of the symmetric group and reduce ourselves to proving that

$$\Delta^1_{23} + \Delta^1_{34} = \Delta^1_{24} + \overline{[\![(a_4 + a_2) \wedge (b_3 - b_2), (b_4 - b_2) \wedge (a_3 + a_2)]\!]_a}.$$

We will prove that both sides of this equal $\overline{[\![(a_1 - b_3) \wedge (a_4 + a_2), (b_1 - a_3) \wedge (b_4 - b_2)]\!]}$.

**Claim 1.** $\Delta^1_{23} + \Delta^1_{34} = \overline{[\![(a_1 - b_3) \wedge (a_4 + a_2), (b_1 - a_3) \wedge (b_4 - b_2)]\!]}$.

We have

$$\begin{aligned}
\Delta^1_{23} &= \overline{[\![(a_1 - b_2) \wedge (a_3 - b_1), (b_1 - a_2) \wedge (b_3 - a_1)]\!]} \\
&= -\overline{[\![(a_1 - b_3) \wedge (b_1 - a_2), (b_1 - a_3) \wedge (a_1 - b_2)]\!]}, \\
\Delta^1_{34} &= \overline{[\![(a_1 - b_3) \wedge (a_4 - b_1), (b_1 - a_3) \wedge (b_4 - a_1)]\!]}.
\end{aligned}$$

We must therefore prove that

$$-[\![(a_1 - b_3) \wedge (b_1 - a_2), (b_1 - a_3) \wedge (a_1 - b_2)]\!] + [\![(a_1 - b_3) \wedge (a_4 - b_1), (b_1 - a_3) \wedge (b_4 - a_1)]\!]$$

equals

$$[\![(a_1 - b_3) \wedge (a_4 + a_2), (b_1 - a_3) \wedge (b_4 - b_2)]\!]$$

modulo elements of $\langle S_{12} \rangle$. These both live in $\mathfrak{F}[a_1 - b_3, b_1 - a_3]$. The calculation is similar to the one from Claim 1 of the proof of Lemma 24.1, so we omit it.

**Claim 2.** *The elements*

$$\Delta^1_{24} + \overline{[\![(a_4 + a_2) \wedge (b_3 - b_2), (b_4 - b_2) \wedge (a_3 + a_2)]\!]_a}$$

*and*

$$\overline{[\![(a_1 - b_3) \wedge (a_4 + a_2), (b_1 - a_3) \wedge (b_4 - b_2)]\!]}$$

*are equal.*

Lemma 24.2 says that

$$\begin{aligned}
\Delta^1_{24} &= \overline{[\![(a_1 - b_2) \wedge (a_4 + a_2), (b_1 - a_2) \wedge (b_4 - b_2)]\!]_a} \\
&= \overline{[\![(a_4 + a_2) \wedge (a_1 - b_2), (b_4 - b_2) \wedge (b_1 - a_2)]\!]_a}.
\end{aligned}$$

Also,

$$\overline{[\![(a_1 - b_3) \wedge (a_4 + a_2), (b_1 - a_3) \wedge (b_4 - b_2)]\!]} = \overline{[\![(a_4 + a_2) \wedge (a_1 - b_3), (b_4 - b_2) \wedge (b_1 - a_3)]\!]}.$$

We must therefore prove that

$$[\![(a_4 + a_2) \wedge (a_1 - b_2), (b_4 - b_2) \wedge (b_1 - a_2)]\!]_a + [\![(a_4 + a_2) \wedge (b_3 - b_2), (b_4 - b_2) \wedge (a_3 + a_2)]\!]_a$$

equals

$$[\![(a_4 + a_2) \wedge (a_1 - b_3), (b_4 - b_2) \wedge (b_1 - a_3)]\!]$$

modulo elements of $\langle S_{12} \rangle$. These both live in $\mathfrak{F}[a_4 + a_2, b_4 - b_2]$. The calculation is similar to the one from Claim 1 of the proof of Lemma 24.1, so we omit it. □

## 25. Symmetric kernel, alternating version IX: the proof of Theorem F

We will continue using all the notation from §17 – §24. We finally prove Theorem F, whose statement we recall.

**Theorem F.** *For $g \geq 4$, the linearization map $\Phi\colon \mathfrak{K}_g^a \to \mathcal{K}_g^a$ is an isomorphism.*

*Proof.* Recall that Lemma 20.1 says that $\mathfrak{K}_g^a$ is generated by $S = S_{12} \cup S_3$. Lemma 22.5 says that $\Phi$ takes $\langle S_{12} \rangle$ isomorphically onto its image. Letting $\mathfrak{T}_g = \mathfrak{K}_g^a / \langle S_{12} \rangle$ and letting $\mathcal{T}_g$ be the quotient of $\mathcal{K}_g^a$ by $\langle \Phi(S_{12}) \rangle$, it is therefore enough to prove that the induced map $\overline{\Phi}\colon \mathfrak{T}_g \to \mathcal{T}_g$ is an isomorphism. Lemmas 23.1 and 23.2 say that $\overline{\Phi}$ is a surjective map to a $g(g-2)$-dimensional vector space. It is therefore enough to prove that $\mathfrak{T}_g$ is at most $g(g-2)$-dimensional.

Define

$$R_g = \left\{ \Delta^i_{jk} \in \mathfrak{T}_g \mid 1 \leq i, j, k \leq g \text{ distinct} \right\}.$$

By Lemma 23.4, the set $R_g$ spans $\mathfrak{T}_g$ for $g \geq 4$. The key to the proof is the following smaller generating set:

**Claim.** *For all $g \geq 4$, the vector space $\mathfrak{T}_g$ is spanned by $R'_g \cup R_g[\leq g-1]$, where:*

$$R'_g = \{\Delta^g_{1i} \mid 2 \leq i \leq g-1\} \cup \left\{\Delta^1_{gi} \mid 2 \leq i \leq g-1\right\} \cup \{\Delta^2_{g1}\},$$
$$R_g[\leq g-1] = \left\{\Delta^i_{jk} \mid 1 \leq i,j,k \leq g-1 \textit{ distinct}\right\}.$$

We will prove this claim in a moment, but let us first see why it implies that $\mathfrak{T}_g$ is at most $g(g-2)$-dimensional. The proof is by induction on $g$. The base case is $g = 4$, where we have to prove that $\mathfrak{T}_4$ is at most $4(4-2) = 8$ dimensional. For this, note that

$$R'_4 = \{\Delta^4_{12}, \Delta^4_{13}, \Delta^1_{42}, \Delta^1_{43}, \Delta^2_{41}\}$$

has 5 elements. The set

$$R_4[\leq 3] = \{\Delta^1_{23}, \Delta^1_{32}, \Delta^2_{13}, \Delta^2_{31}, \Delta^3_{12}, \Delta^3_{21}\}$$

has 6 elements, but since $\Delta^i_{kj} = -\Delta^i_{jk}$ (see Lemma 23.3) only 3 are needed to span $\mathfrak{T}_4$. We conclude that $\mathfrak{T}_4$ is at most $5 + 3 = 8$ dimensional, as desired.

For the inductive step, assume that $g \geq 5$ and that $\mathfrak{T}_{g-1}$ is at most $(g-1)((g-1)-2)$-dimensional. We must prove that $\mathfrak{T}_g$ is at most $g(g-2)$-dimensional. The natural map $\mathfrak{K}^a_{g-1} \to \mathfrak{K}^a_g$ taking a generator $[\![\kappa_1, \kappa_2]\!]_a$ of $\mathfrak{K}^a_{g-1}$ to the same generator of $\mathfrak{K}^a_g$ induces a map $\iota\colon \mathfrak{T}_{g-1} \to \mathfrak{T}_g$. The map $\iota$ takes a basic element $\Delta^i_{jk} \in \mathfrak{T}_{g-1}$ to the same basic element of $\mathfrak{T}_g$. It follows that the image of $\iota$ is the span of $R_g[\leq g-1]$. In particular, $\langle R_g[\leq g-1] \rangle$ is at most $(g-1)((g-1)-2)$-dimensional. The set $R'_g$ from the above claim has

$$(g-2) + (g-2) + 1 = 2g - 3$$

elements, so $\langle R'_g \rangle$ is at most $2g-3$ dimensional. Since $\mathfrak{T}_g$ is spanned by $R'_g$ and $R_g[\leq g-1]$, we conclude that $\mathfrak{T}_g$ is at most

$$(2g-3) + (g-1)((g-1)-2) = (2g-3) + (g^2 - 4g + 3) = g^2 - 2g = g(g-2)$$

dimensional, as desired.

It remains to prove the above claim:

*Proof of claim.* It is enough to prove that every element $\Delta$ of the generating set

$$R_g = \left\{ \Delta^i_{jk} \in \mathfrak{T}_g \mid 1 \leq i, j, k \leq g \text{ distinct} \right\}.$$

that does not lie in $R_g[\leq g-1]$ can be written as a linear combination of elements of $R'_g$ and $R_g[\leq g-1]$. There are two families of elements of $R_g$ that do not lie in $R_g[\leq g-1]$.

The first are elements of the form $\Delta^g_{ij}$ with $1 \leq i, j \leq g-1$ distinct. Since $\Delta^g_{ji} = -\Delta^g_{ij}$ (Lemma 23.3), we can assume that $i < j$. If $i = 1$, then $\Delta^g_{ij} \in R'_g$, so we can assume that $2 \leq i < j \leq g-1$. Lemma 24.3 gives a relation

$$\Delta^g_{i1} + \Delta^g_{1j} = \Delta^g_{ij} + \overline{[\![(a_j + a_i) \wedge (b_1 - b_i), (b_j - b_i) \wedge (a_1 + a_i)]\!]_a}. \tag{25.1}$$

Set

$$\kappa = [\![(a_j + a_i) \wedge (b_1 - b_i), (b_j - b_i) \wedge (a_1 + a_i)]\!]_a.$$

Since $1, i, j \leq g-1$, we can view $\kappa$ as an element of $\mathfrak{K}^a_{g-1}$. If $g \geq 5$, then Lemma 23.4 says that $\mathfrak{T}_{g-1}$ is generated by basic elements, so $\overline{\kappa}$ is in the span of $R_g[\leq g-1]$. This argument does not work if $g = 4$; however, in this case Lemma 18.4 (which does work in genus 3) says that $\kappa \in \mathfrak{K}^a_{g-1}$ can be written[34] as a linear combination of elements of our basis $S$. By the proof of Lemma 23.4 these map to linear combinations of basic elements in $\mathfrak{T}_{g-1}$, so again we deduce that $\overline{\kappa}$ is in the span of $R_g[\leq g-1]$.

In either case, since $\Delta^g_{i1} = -\Delta^g_{1i}$ (Lemma 23.3) we can rearrange (25.1) and see that

$$\Delta^g_{ij} = \Delta^g_{1j} - \Delta^g_{1i} - \overline{\kappa}$$

lies in the span of $R_g[\leq g-1]$ and $R'_g$, as desired.

The other family of elements of $R_g$ that do not lie in $R_g[\leq g-1]$ are those of the form $\Delta^i_{jg}$ and $\Delta^i_{gj}$ for $1 \leq i, j \leq g-1$ distinct. We must show that these lie in the span of $R_g[\leq g-1]$ and $R'_g$. Since $\Delta^i_{jg} = -\Delta^i_{gj}$ (Lemma 23.3), it is enough to deal with $\Delta^i_{gj}$. If $i = 1$ then $\Delta^i_{gj} \in R'_g$, so we can assume that $i \neq 1$. Since we also have $\Delta^2_{g1} \in R'_g$, we can assume that if $i = 2$ then $j \neq 1$. In other words, we can assume that $2 \leq i, j \leq g-1$. Lemma 24.1 gives a relation

$$\Delta^1_{gj} + \Delta^g_{i1} = \Delta^i_{gj} + \Delta^j_{i1}.$$

We have $\Delta^1_{gj} \in R'_g$ and $\Delta^j_{i1} \in R_g[\leq g-1]$, and we already proved that $\Delta^g_{i1}$ is in the span of $R'_g$ and $R_g[\leq g-1]$. We conclude that $\Delta^i_{gj}$ is also in the span of $R'_g$ and $R_g[\leq g-1]$, as desired. □

This completes the proof of the theorem. □

## Part 4. Verifying the presentation for the symmetric kernel, symmetric version

We now turn to Theorem G. See the introductory §26 for an outline of what we do in this part. Throughout, we make the following genus assumption:

**Assumption 25.1.** Throughout Part 4, we assume that $g \geq 4$. □

## 26. Symmetric kernel, symmetric version: introduction

We start by recalling some results and definitions from earlier in the paper, and then outline what we prove in this part.

26.1. **Symmetric contraction.** Recall that $\omega$ is the symplectic form on $H$. The *symmetric contraction* is the alternating $\text{Sym}^2(H)$-valued alternating form $\mathfrak{c}$ on $(\wedge^2 H)/\mathbb{Q}$ defined via the formula

$$\mathfrak{c}(x \wedge y, z \wedge w) = \omega(x,z)y{\cdot}w - \omega(x,w)y{\cdot}z - \omega(y,z)x{\cdot}w + \omega(y,w)x{\cdot}z \text{ for } x, y, z, w \in H.$$

Elements $\kappa_1, \kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ are *sym-orthogonal* if $\mathfrak{c}(\kappa_1, \kappa_2) = 0$. The *sym-orthogonal complement* of $\kappa \in (\wedge^2 H)/\mathbb{Q}$ is the subspace $\kappa^{\perp}$ consisting of all elements that are sym-orthogonal to $\kappa$.

[34] It is enlightening to go through the proof and work this out explicitly.

26.2. **Special pairs.** A *special pair* in $(\wedge^2 H_{\mathbb{Z}})/\mathbb{Z}$ is an element of the form $x \wedge y$ with $\omega(x,y) \in \{-1,0,1\}$. Examples include symplectic pairs and isotropic pairs. Lemmas 10.1 and 10.2 say that the sym-orthogonal complements in $(\wedge^2 H)/\mathbb{Q}$ of these are:

- for a symplectic pair $a \wedge b$, we have $(a \wedge b)^{\perp} = \overline{\wedge^2 \langle a, b\rangle_{\mathbb{Q}}^{\perp}}$; and
- for an isotropic pair $a \wedge a'$, we have $(a \wedge a')^{\perp} = \overline{\wedge^2 \langle a, a'\rangle_{\mathbb{Q}}^{\perp}}$.

26.3. **Non-symmetric presentation.** We will use the generators and relations for $\mathfrak{K}_g$ from Theorem 15.1, whose statement we recall:

**Theorem 15.1.** *For $g \geq 4$, the vector space $\mathfrak{K}_g$ has the following presentation:*

- **Generators**. *A generator $[\![\kappa_1, \kappa_2]\!]$ for all sym-orthogonal $\kappa_1, \kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ such that either $\kappa_1$ or $\kappa_2$ (or both) is a special pair.*
- **Relations**. *The following two families of relations:*
  - *For special pairs $\zeta \in (\wedge^2 H)/\mathbb{Q}$ and all $\kappa_1, \kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ that are sym-orthogonal to $\zeta$ and all $\lambda_1, \lambda_2 \in \mathbb{Q}$, the linearity relations*
  $$[\![\zeta, \lambda_1\kappa_1 + \lambda_2\kappa_2]\!] = \lambda_1[\![\zeta, \kappa_1]\!] + \lambda_2[\![\zeta, \kappa_2]\!] \quad \text{and}$$
  $$[\![\lambda_1\kappa_1 + \lambda_2\kappa_2, \zeta]\!] = \lambda_1[\![\kappa_1, \zeta]\!] + \lambda_2[\![\kappa_2, \zeta]\!].$$
  - *For all special pairs $\zeta \in (\wedge^2 H)/\mathbb{Q}$ and all $\kappa \in (\wedge^2 H)/\mathbb{Q}$ that are sym-orthogonal to $\zeta$ and all $n \in \mathbb{Z}$ such that $n\zeta$ is a special pair, the relations*
  $$[\![n\zeta, \kappa]\!] = n[\![\zeta, \kappa]\!] \quad \text{and}$$
  $$[\![\kappa, n\zeta]\!] = n[\![\kappa, \zeta]\!].$$

26.4. **Symmetrizing.** Recall from Lemma 10.6 that $\mathfrak{K}_g^s$ is the $+1$-eigenspace of the involution of $\mathfrak{K}_g$ that takes a generator $[\![\kappa_1, \kappa_2]\!]$ to $[\![\kappa_2, \kappa_1]\!]$. We symmetrize a generator $[\![\kappa_1, \kappa_2]\!]$ of $\mathfrak{K}_g$ to

$$[\![\kappa_1, \kappa_2]\!]_s = \frac{1}{2}\left([\![\kappa_1, \kappa_2]\!] + [\![\kappa_2, \kappa_1]\!]\right) \in \mathfrak{K}_g^s.$$

The symmetrized generators generate $\mathfrak{K}_g^s$. They satisfy the same relations as the generators of $\mathfrak{K}_g$, and also the symmetry relation $[\![\kappa_2, \kappa_1]\!]_s = [\![\kappa_1, \kappa_2]\!]_s$.

26.5. **Goal and outline.** We have a linearization map $\Phi\colon \mathfrak{K}_g^s \to \mathrm{Sym}^2((\wedge^2 H)/\mathbb{Q})$. On generators, it satisfies

$$\Phi([\![\kappa_1, \kappa_2]\!]_s) = \kappa_1 \cdot \kappa_2 \in \mathrm{Sym}^2((\wedge^2 H)/\mathbb{Q}).$$

Our goal in this part of the paper is to prove Theorem G, which says that $\Phi$ is an isomorphism from $\mathfrak{K}_g^s$ to $\mathrm{Sym}^2((\wedge^2 H)/\mathbb{Q})$. The proof uses the proof technique described in §3, and is modeled on the proofs of Theorems A–E. However, since the calculations are lengthy we spread them out over nine sections:

- In §27 – §30, we construct a subset $S$ of $\mathfrak{K}_g^s$ such that $\Phi$ restricted to $\langle S\rangle$ is an isomorphism (Step 1). This calculation is lengthy since it depends on the construction of three important families of elements of $\mathfrak{K}_g^s$ (the $\Theta$-, the $\Lambda$-, and the $\Omega$-elements), and it takes work to prove their basic properties.
- In §31, we prove that the $\mathrm{Sp}_{2g}(\mathbb{Z})$-orbit of $S$ spans $\mathfrak{K}_g^s$ (Step 2). We also outline the proof that $\mathrm{Sp}_{2g}(\mathbb{Z})$ takes takes $\langle S\rangle$ to itself (Step 3).
- Finally, in §32 – 35 we complete the outlined proof that $\mathrm{Sp}_{2g}(\mathbb{Z})$ takes $\langle S\rangle$ to itself. Together with Step 2 this implies that $\langle S\rangle = \mathfrak{K}_g^s$, so by Step 1 we conclude that $\Phi$ is an isomorphism.

Throughout the following nine sections, $\Phi$ will always mean the linearization map $\Phi\colon \mathfrak{K}_g^s \to \mathrm{Sym}^2((\wedge^2 H)/\mathbb{Q})$. Also, $\mathfrak{c}$ will always mean the symmetric contraction. Finally, we will fix a symplectic basis $\mathcal{B} = \{a_1, b_1, \ldots, a_g, b_g\}$ for $H_{\mathbb{Z}}$.

## 27. Symmetric kernel, symmetric version I: $S_1$ and structure of target

We discuss the structure of $\mathrm{Sym}^2((\wedge^2 H)/\mathbb{Q})$, and then begin our construction of $S$.

27.1. **Generators and relations for target.** Let $\prec$ be the following total order on $\mathcal{B}$:

$$a_1 \prec b_1 \prec a_2 \prec b_2 \prec \cdots \prec a_g \prec b_g.$$

Set

$$T = \{(x \wedge y)\cdot(z \wedge w) \mid x, y, z, w \in \mathcal{B},\ x \prec y,\ z \prec w\} \subset \mathrm{Sym}^2((\wedge^2 H)/\mathbb{Q}).$$

The set $T$ generates $\mathrm{Sym}^2((\wedge^2 H)/\mathbb{Q})$, and the relations between elements of $T$ are generated by the set

$$R = \left\{ \sum\nolimits_{i=1}^{g} (a_i \wedge b_i)\cdot(x \wedge y) \mid x, y \in \mathcal{B},\ x \prec y \right\}.$$

27.2. **Lifting easy generators.** Some elements of $T$ are easily lifted to $\mathfrak{K}_g^s$. Define

$$S_1 = \{ [\![x \wedge y, z \wedge w]\!]_s \mid x, y, z, w \in \mathcal{B},\ x \prec y,\ z \prec w,\ \mathfrak{c}(x \wedge y, z \wedge w) = 0 \}.$$

For $[\![x \wedge y, z \wedge w]\!]_s \in S_1$, we have

$$\Phi([\![x \wedge y, z \wedge w]\!]_s) = (x \wedge y)\cdot(z \wedge w) \in T.$$

Like we did above, we will write elements of $S_1 \subset \mathfrak{K}_g^s$ in blue. More generally, we will use blue to write elements of $\mathfrak{K}_g^s$ that lie in $\langle S_1 \rangle$. The set $S_1$ consists of two kinds of elements:

- those of the form $[\![x \wedge y, z \wedge w]\!]_s$ for $x, y, z, w \in \mathcal{B}$ with $x \prec y$ and $z \prec w$ and $\omega(x, z) = \omega(x, w) = \omega(y, z) = \omega(y, w) = 0$; and
- those of the form $[\![a_i \wedge b_i, a_i \wedge b_i]\!]_s$ for some $1 \leq i \leq g$. These lie in $S_1$ since $\mathfrak{c}$ is alternating, or more concretely due to the calculation

$$\mathfrak{c}(a_i \wedge b_i, a_i \wedge b_i) = -\omega(a_i, b_i)(b_i\cdot a_i) - \omega(b_i, a_i)(a_i\cdot b_i) = -(b_i\cdot a_i) + (a_i\cdot b_i) = 0.$$

Let $T_1 \subset T$ be the image of $S_1$.

27.3. **Lifting easy relations.** Let $R_1$ be the subset of $R$ consisting of relations between elements of $T_1$. Thus $R_1$ consists of relations of the form

$$\sum\nolimits_{i=1}^{g} (a_i \wedge b_i)\cdot(a_k \wedge b_k) \text{ with } 1 \leq k \leq g.$$

These lift to relations between the elements of $S_1$ due to the bilinearity relations in $\mathfrak{K}_g^s$:

$$\sum_{i=1}^{g} [\![a_i \wedge b_i, a_k \wedge b_k]\!]_s = [\![\sum_{i=1}^{g} a_i \wedge b_i, a_k \wedge b_k]\!]_s = [\![0, a_k \wedge b_k]\!]_s = 0.$$

27.4. **Other relations do not affect $T_1$.** Set $R_2 = R \setminus R_1$. Each element of $R_2$ involves an element of $T$ that appears in no other relations in $R$. For instance, for $1 \leq k < \ell \leq g$ the set $R_2$ contains the relation

$$\sum_{i=1}^{g} (a_i \wedge b_i)\cdot(a_k \wedge a_\ell),$$

and no other relation in $R$ uses the generator $(a_k \wedge b_k)\cdot(a_k \wedge a_\ell)$. This implies that the subspace of $\mathrm{Sym}^2((\wedge^2 H)/\mathbb{Q})$ spanned by $T_1$ is generated by $T_1$ subject to only the relations in $R_1$. This implies:

**Lemma 27.1.** *The linearization map $\Phi$ takes $\langle S_1 \rangle$ isomorphically to $\langle T_1 \rangle$.*

*Proof.* Immediate from the fact $\Phi$ takes $S_1$ bijectively to $T_1$ and each relation in $R_1$ lifts to a relation between the elements of $S_1$. $\square$

27.5. **Remaining generators.** Define

$$T_2 = \{(a_i \wedge b_i)\cdot(x \wedge b_i),\, (a_i \wedge b_i)\cdot(a_i \wedge y) \mid 1 \leq i \leq g,\, x, y \in \mathcal{B} \setminus \{a_i, b_i\}\}\,,$$
$$T_3 = \{(a_i \wedge y)\cdot(x \wedge b_i) \mid 1 \leq i \leq g,\, x, y \in \mathcal{B} \setminus \{a_i, b_i\},\, \omega(x,y) = 0\}\,,$$
$$T_4 = \{(a_i \wedge a_j)\cdot(b_i \wedge b_j),\, (a_i \wedge b_j)\cdot(b_i \wedge a_j) \mid 1 \leq i < j \leq g\}\,.$$

The set $T_2 \cup T_3 \cup T_4$ is almost equal to $T \setminus T_1$. The only difference is that for some $\eta \in T \setminus T_1$ we have $-\eta \in T_2 \cup T_3 \cup T_4$. For instance, we have $(a_1 \wedge b_1)\cdot(b_1 \wedge a_2) \in T \setminus T_1$ but

$$(a_1 \wedge b_1)\cdot(a_2 \wedge b_1) = -(a_1 \wedge b_1)\cdot(b_1 \wedge a_2) \in T_2.$$

In any case, we have that $\mathrm{Sym}^2((\wedge^2 H)/\mathbb{Q})$ is generated by $T_1 \cup T_2 \cup T_3 \cup T_4$ subject to appropriate versions of the relations in $R$. In the next three sections, we will construct sets $S_2, S_3, S_4 \subset \mathfrak{K}_g^s$ such that $\Phi$ takes $S_i$ bijectively to $T_i$, and we will prove that all relations in $R$ lift to relations between elements of $S = S_1 \cup \cdots \cup S_4$. The elements of $S_2$ and $S_3$ and $S_4$ are called $\Theta$-elements, $\Lambda$-elements, and $\Omega$-elements.

27.6. **Obvious blue elements.** Before we continue, we make a useful observation. Recall that we write elements of $\langle S_1 \rangle$ in blue. One easy way to recognize these is as follows. Consider a generator $[\![\kappa_1, \kappa_2]\!]_s$ such that there exists subsets $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{B}$ such that:

- $\kappa_1 \in \overline{\wedge^2 \langle \mathcal{B}_1 \rangle}$ and $\kappa_2 \in \overline{\wedge^2 \langle \mathcal{B}_2 \rangle}$; and
- $\omega(x,y) = 0$ for all $x \in \mathcal{B}_1$ and $y \in \mathcal{B}_2$.

Note that $\mathcal{B}_1$ and $\mathcal{B}_2$ need not be disjoint. We then have that $[\![\kappa_1, \kappa_2]\!]_s \in \langle S_1 \rangle$, so we can write $[\![\kappa_1, \kappa_2]\!]_s$. This is most easily seen by example:

$$\begin{aligned}[\![(a_1+3b_1)\wedge a_2, a_2 \wedge (a_3 - 2b_3)]\!]_s =& [\![(a_1+3b_1)\wedge a_2, a_2 \wedge a_3]\!]_s - 2[\![(a_1+3b_1)\wedge a_2, a_2 \wedge b_3]\!]_s \\ =& [\![a_1 \wedge a_2, a_2 \wedge a_3]\!]_s + 3[\![b_1 \wedge a_2, a_2 \wedge a_3]\!]_s \\ & - 2[\![a_1 \wedge a_2, a_2 \wedge b_3]\!]_s - 6[\![b_1 \wedge a_2, a_2 \wedge b_3]\!]_s.\end{aligned}$$

We thus have $[\![(a_1+3b_1)\wedge a_2, a_2 \wedge (a_3 - 2b_3)]\!]_s \in \langle S_1 \rangle$.

## 28. Symmetric kernel, symmetric version II: $S_2$ and the $\Theta$-elements

We continue using all the notation from §27. This section constructs the set $S_2$ that lifts $T_2$. It consists of what are called $\Theta$-elements of $\mathfrak{K}_g^s$, and the first part of this section constructs these in more generality than is needed for $S_2$ alone.

28.1. **Definition.** Let $a \wedge b$ be a symplectic pair in $\wedge^2 H_{\mathbb{Z}}$ and let $x, y \in \langle a, b \rangle^{\perp}$. The $\Theta$-elements $\Theta[a \wedge b, x \wedge b]_s$ and $\Theta[a \wedge b, a \wedge y]_s$ are elements of $\mathfrak{K}_g^s$ that are taken by $\Phi$ to

$$(a \wedge b)\cdot(x \wedge b) \in \mathrm{Sym}^2((\wedge^2 H)/\mathbb{Q}) \quad \text{and} \quad (a \wedge b)\cdot(a \wedge y) \in \mathrm{Sym}^2((\wedge^2 H)/\mathbb{Q}),$$

respectively. To find these elements, note that

$$((a+x)\wedge b)\cdot((a+x)\wedge b) = (a \wedge b)\cdot(a \wedge b) + 2(a \wedge b)\cdot(x \wedge b) + (x \wedge b)\cdot(x \wedge b),$$
$$(a \wedge (b+y))\cdot(a \wedge (b+y)) = (a \wedge b)\cdot(a \wedge b) + 2(a \wedge b)\cdot(a \wedge y) + (a \wedge y)\cdot(a \wedge y).$$

Since $\mathfrak{c}$ is alternating, for any $z \in (\wedge^2 H)/\mathbb{Q}$ we have $\mathfrak{c}(z,z) = 0$ and thus there exists a generator $[\![z, z]\!]_s$ of $\mathfrak{K}_g^s$. This suggests:

**Definition 28.1.** For a symplectic pair $a \wedge b$ in $\wedge^2 H_{\mathbb{Z}}$ and $x, y \in \langle a, b \rangle^{\perp}$, define

$$\Theta[a \wedge b, x \wedge b]_s = \frac{1}{2}\left(\llbracket (a+x) \wedge b, (a+x) \wedge b \rrbracket_s - \llbracket a \wedge b, a \wedge b \rrbracket_s - \llbracket x \wedge b, x \wedge b \rrbracket_s\right),$$

$$\Theta[a \wedge b, a \wedge y]_s = \frac{1}{2}\left(\llbracket a \wedge (b+y), a \wedge (b+y) \rrbracket_s - \llbracket a \wedge b, a \wedge b \rrbracket_s - \llbracket a \wedge y, a \wedge y \rrbracket_s\right). \quad \square$$

By construction, we have

$$\Phi(\Theta[a \wedge b, x \wedge b]_s) = (a \wedge b) \cdot (x \wedge b),$$
$$\Phi(\Theta[a \wedge b, a \wedge y]_s) = (a \wedge b) \cdot (a \wedge y).$$

*Remark* 28.2. Despite our notation $\Theta[a \wedge b, x \wedge b]_s$, this depends on the ordered tuple $(a, b, x)$, not on $a \wedge b$ and $x \wedge b$. A similar remark applies to $\Theta[a \wedge b, a \wedge y]_s$. We chose to abuse notation like this to emphasize that $\Theta[a \wedge b, x \wedge b]_s$ should be regarded as the "missing" element $\llbracket a \wedge b, x \wedge b \rrbracket_s$ of $\mathfrak{K}_g^s$ that should exist if $\mathfrak{K}_g^s$ is isomorphic to $\mathrm{Sym}^2((\wedge^2 H)/\mathbb{Q})$. Later we will prove that $\Theta[a \wedge b, x \wedge b]_s$ behaves as if it only depends on $a \wedge b$ and $x \wedge b$, e.g., Lemma 28.10 below says that $\Theta[(a + nb) \wedge b, x \wedge b]_s = \Theta[a \wedge b, x \wedge b]_s$ for all $n \in \mathbb{Z}$. $\square$

28.2. **Θ-expansion I.** If $x, y, z \in H_{\mathbb{Z}}$ are pairwise orthogonal elements such that $x \wedge z$ and $y \wedge z$ and $(x + y) \wedge z$ are isotropic pairs, then the relations in $\mathfrak{K}_g^s$ imply that

$$\llbracket (x+y) \wedge z, (x+y) \wedge z \rrbracket_s = \llbracket x \wedge z, x \wedge z \rrbracket_s + 2\llbracket x \wedge z, y \wedge z \rrbracket_s + \llbracket y \wedge z, y \wedge z \rrbracket_s.$$

Using Θ-elements, we can similarly expand out some other elements:

**Lemma 28.3** (Θ-expansion I)**.** *Let $a \wedge b$ be a symplectic pair and let $x, y \in \langle a, b \rangle^{\perp}$. Then*

$$\llbracket (a+x) \wedge b, (a+x) \wedge b \rrbracket_s = \llbracket a \wedge b, a \wedge b \rrbracket_s + 2\Theta[a \wedge b, x \wedge b]_s + \llbracket x \wedge b, x \wedge b \rrbracket_s,$$
$$\llbracket a \wedge (b+y), a \wedge (b+y) \rrbracket_s = \llbracket a \wedge b, a \wedge b \rrbracket_s + 2\Theta[a \wedge b, y \wedge a]_s + \llbracket y \wedge a, y \wedge a \rrbracket_s.$$

*Proof.* Immediate from Definition 28.1. $\square$

28.3. **Θ-linearity.** The following is a key property of the Θ-elements:

**Lemma 28.4** (Θ-linearity)**.** *Let $a \wedge b$ be a symplectic pair. Then for all $z_1, z_2 \in (a \wedge b)^{\perp}$ and $\lambda_1, \lambda_2 \in \mathbb{Z}$ we have*

$$\Theta[a \wedge b, (\lambda_1 z_1 + \lambda_2 z_2) \wedge b]_s = \lambda_1 \Theta[a \wedge b, z_1 \wedge b]_s + \lambda_2 \Theta[a \wedge b, z_2 \wedge b]_s,$$
$$\Theta[a \wedge b, a \wedge (\lambda_1 z_1 + \lambda_2 z_2)]_s = \lambda_1 \Theta[a \wedge b, a \wedge z_1]_s + \lambda_2 \Theta[a \wedge b, a \wedge z_2]_s.$$

*Proof.* Both formulas are proved similarly, so we will prove the first. The key calculation is the following special case of the lemma:

**Claim.** *For a partial basis $\{z_1, z_2\}$ of $(a \wedge b)^{\perp}$ with $\omega(z_1, z_2) = 0$ and $n_1, n_2 \in \mathbb{Z}$, we have*

$$\Theta[a \wedge b, (n_1 z_1 + n_2 z_2) \wedge b]_s = \Theta[a \wedge b, n_1 z_1 \wedge b]_s + \Theta[a \wedge b, n_2 z_2 \wedge b]_s.$$

*Proof of claim.* Whether or not the claim holds is invariant under the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathfrak{K}_g^s$. Recall that we have our fixed symplectic basis $\mathcal{B} = \{a_1, b_1, \ldots, a_g, b_g\}$ for $H_{\mathbb{Z}}$. Applying an appropriate element of $\mathrm{Sp}_{2g}(\mathbb{Z})$, we can assume that

$$a_1 = a, \;\; b_1 = b, \;\; a_2 = z_1, \;\; a_3 = z_2.$$

To make our notation easier to digest, replace $n_1, n_2 \in \mathbb{Z}$ by $n_2, n_3 \in \mathbb{Z}$. Since the restriction of $\Phi$ to $\langle S_1 \rangle$ is injective (Lemma 27.1) and $\Phi$ takes both

$$\Theta[a_1 \wedge b_1, (n_2 a_2 + n_3 a_3) \wedge b_1]_s \tag{28.1}$$

and

$$\Theta[a_1 \wedge b_1, n_2 a_2 \wedge b_1]_s + \Theta[a_1 \wedge b_1, n_3 a_3 \wedge b_1]_s \tag{28.2}$$

to the same element of $\text{Sym}^2((\wedge^2 H)/\mathbb{Q})$, it is enough to prove that (28.1) and (28.2) are equal modulo $\langle S_1 \rangle$. Let $\equiv$ denote equality in $\mathfrak{K}_g^s$ modulo $\langle S_1 \rangle$.

By definition, $2\Theta[a_1 \wedge b_1, (n_2 a_2 + n_3 a_3) \wedge b_1]_s$ equals[35]

$$\begin{aligned} &[\![(a_1 + n_2 a_2 + n_3 a_3) \wedge b_1, (a_1 + n_2 a_2 + n_3 a_3) \wedge b_1]\!]_s \\ &\quad - [\![a_1 \wedge b_1, a_1 \wedge b_1]\!]_s - [\![(n_2 a_2 + n_3 a_3) \wedge b_1, (n_2 a_2 + n_3 a_3) \wedge b_1]\!]_s \\ \equiv &[\![(a_1 + n_2 a_2 + n_3 a_3) \wedge b_1, (a_1 + n_2 a_2 + n_3 a_3) \wedge b_1]\!]_s \end{aligned}$$

and $2\Theta[a_1 \wedge b_1, n_2 a_2 \wedge b_1]_s + 2\Theta[a_1 \wedge b_1, n_3 a_3 \wedge b_1]_s$ equals

$$\begin{aligned} &[\![(a_1 + n_2 a_2) \wedge b_1, (a_1 + n_2 a_2) \wedge b_1]\!]_s - [\![a_1 \wedge b_1, a_1 \wedge b_1]\!]_s - [\![n_2 a_2 \wedge b_1, n_2 a_2 \wedge b_1]\!]_s, \\ &+ [\![(a_1 + n_3 a_3) \wedge b_1, (a_1 + n_3 a_3) \wedge b_1]\!]_s - [\![a_1 \wedge b_1, a_1 \wedge b_1]\!]_s - [\![n_3 a_3 \wedge b_1, n_3 a_3 \wedge b_1]\!]_s \\ \equiv &[\![(a_1 + n_2 a_2) \wedge b_1, (a_1 + n_2 a_2) \wedge b_1]\!]_s + [\![(a_1 + n_3 a_3) \wedge b_1, (a_1 + n_3 a_3) \wedge b_1]\!]_s. \end{aligned}$$

Our goal, therefore, is to prove that

$$\begin{aligned} &[\![(a_1 + n_2 a_2 + n_3 a_3) \wedge b_1, (a_1 + n_2 a_2 + n_3 a_3) \wedge b_1]\!]_s \\ \equiv &[\![(a_1 + n_2 a_2) \wedge b_1, (a_1 + n_2 a_2) \wedge b_1]\!]_s + [\![(a_1 + n_3 a_3) \wedge b_1, (a_1 + n_3 a_3) \wedge b_1]\!]_s \end{aligned} \tag{28.3}$$

In $(\wedge^2 H)/\mathbb{Q}$, we have the following identities. The colors are there to help the reader match up terms with later formulas.[36]

$$\begin{aligned} &(a_1 + n_2 a_2 + n_3 a_3) \wedge b_1 + a_2 \wedge (b_2 - n_2 b_1) + a_3 \wedge (b_3 - n_3 b_1) + \sum\nolimits_{i=4}^g a_i \wedge b_i = 0, \\ &(a_1 + n_2 a_2) \wedge b_1 + a_2 \wedge (b_2 - n_2 b_1) + a_3 \wedge b_3 + \sum\nolimits_{i=4}^g a_i \wedge b_i = 0, \\ &(a_1 + n_3 a_3) \wedge b_1 + a_2 \wedge b_2 + a_3 \wedge (b_3 - n_3 a_1) + \sum\nolimits_{i=4}^g a_i \wedge b_i = 0. \end{aligned}$$

We plug these into the terms of (28.3). Matching up terms of the same color, we see that $[\![(a_1 + n_2 a_2 + n_3 a_3) \wedge b_1, (a_1 + n_2 a_2 + n_3 a_3) \wedge b_1]\!]_s$ equals

$$\begin{aligned} &- [\![(a_1 + n_2 a_2 + n_3 a_3) \wedge b_1, a_2 \wedge (b_2 - n_2 b_1)]\!]_s - [\![(a_1 + n_2 a_2 + n_3 a_3) \wedge b_1, a_3 \wedge (b_3 - n_3 b_1)]\!]_s \\ &- \sum\nolimits_{i=4}^g [\![(a_1 + n_2 a_2 + n_3 a_3) \wedge b_1, a_i \wedge b_i]\!]_s \\ \equiv &- [\![(a_1 + n_2 a_2) \wedge b_1, a_2 \wedge (b_2 - n_2 b_1)]\!]_s - n_3 [\![a_3 \wedge b_1, a_2 \wedge (b_2 - n_2 b_1)]\!]_s \\ &- [\![(a_1 + n_3 a_3) \wedge b_1, a_3 \wedge (b_3 - n_3 b_1)]\!]_s - n_2 [\![a_2 \wedge b_1, a_3 \wedge (b_3 - n_3 b_1)]\!]_s \\ \equiv &- [\![(a_1 + n_2 a_2) \wedge b_1, a_2 \wedge (b_2 - n_2 b_1)]\!]_s - [\![(a_1 + n_3 a_3) \wedge b_1, a_3 \wedge (b_3 - n_3 b_1)]\!]_s \end{aligned}$$

and $[\![(a_1 + n_2 a_2) \wedge b_1, (a_1 + n_2 a_2) \wedge b_1]\!]_s$ equals

$$\begin{aligned} &- [\![(a_1 + n_2 a_2) \wedge b_1, a_2 \wedge (b_2 - n_2 b_1)]\!]_s - [\![(a_1 + n_2 a_2) \wedge b_1, a_3 \wedge b_3]\!]_s \\ &- \sum\nolimits_{i=4}^g [\![(a_1 + n_2 a_2) \wedge b_1, a_i, b_i]\!]_s \\ \equiv &- [\![(a_1 + n_2 a_2) \wedge b_1, a_2 \wedge (b_2 - n_2 b_1)]\!]_s \end{aligned}$$

and $[\![(a_1 + n_3 a_3) \wedge b_1, (a_1 + n_3 a_3) \wedge b_1]\!]_s$ equals

$$\begin{aligned} &- [\![(a_1 + n_3 a_3) \wedge b_1, a_2 \wedge b_2]\!]_s - [\![(a_1 + n_3 a_3) \wedge b_1, a_3 \wedge (b_3 - n_3 a_1)]\!]_s \\ &- \sum\nolimits_{i=4}^g [\![(a_1 + n_3 a_3) \wedge b_1, a_i \wedge b_i]\!]_s \\ \equiv &- [\![(a_1 + n_3 a_3) \wedge b_1, a_3 \wedge (b_3 - n_3 a_1)]\!]_s. \end{aligned}$$

Combining these three equalities gives (28.3). □

[35] Here and in future calculations we use §27.6 to identify and then delete blue terms lying in $\langle S_1 \rangle$.

[36] The only color which has a definite meaning right now is blue, which is used to indicate elements of $\langle S_1 \rangle$. In later sections we will give meanings to purple and orange and green terms, but currently these colors have no meaning and we are free to use them.

We now return to the proof of the lemma. Define $\mathfrak{K}_g^s[a \wedge b, - \wedge b]$ to be the subspace of $\mathfrak{K}_g^s$ spanned by the $\Theta[a \wedge b, x \wedge b]_s$ as $x$ ranges over all elements of $\langle a, b\rangle^{\perp}$. The linearization map $\Phi\colon \mathfrak{K}_g^s \to \mathrm{Sym}^2((\wedge^2 H)/\mathbb{Q})$ takes $\mathfrak{K}_g^s[a \wedge b, - \wedge b]$ to the subspace

$$\left\{(a \wedge b)\cdot(h \wedge b) \mid h \in \langle a, b\rangle_{\mathbb{Q}}^{\perp}\right\} \cong \langle a, b\rangle_{\mathbb{Q}}^{\perp}. \tag{28.4}$$

To prove the lemma, it is enough to prove that the restriction of $\Phi$ to $\mathfrak{K}_g^s[a \wedge b, - \wedge b]$ is an isomorphism.

Let $\mathfrak{K}_g^s[a \wedge b, - \wedge b]_{\text{prim}}$ be the subspace of $\mathfrak{K}_g^s[a \wedge b, - \wedge b]$ spanned by $\Theta[a \wedge b, x \wedge b]_s$ with $x$ a primitive element of $\langle a, b\rangle^{\perp}$. The case $n_1 = n_2 = 1$ of the above claim implies that we can use Theorem C to see that $\Phi$ takes $\mathfrak{K}_g^s[a \wedge b, - \wedge b]_{\text{prim}}$ isomorphically to (28.4).

To complete the proof, we must prove that every element of $\mathfrak{K}_g^s[a \wedge b, - \wedge b]$ equals an element of $\mathfrak{K}_g^s[a \wedge b, - \wedge b]_{\text{prim}}$. For this, consider a general $x \in \langle a, b\rangle^{\perp}$. Write $x = nx'$ with $x'$ primitive. Let $y$ be such that $\{x', y\}$ is a partial basis of $(a \wedge b)^{\perp}$ with $\omega(x', y) = 0$. The above claim then implies that

$$\Theta[a \wedge b, (x + y) \wedge b]_s = \Theta[a \wedge b, x \wedge b]_s + \Theta[a \wedge b, y \wedge b]_s.$$

On the other hand, since $x + y$ is primitive the fact that $\Phi$ takes $\mathfrak{K}_g^s[a \wedge b, - \wedge b]_{\text{prim}}$ isomorphically to (28.4) implies that

$$\Theta[a \wedge b, (x + y) \wedge b]_s = \Theta[a \wedge b, (nx' + y) \wedge b]_s = n\Theta[a \wedge b, x' \wedge b]_s + \Theta[a \wedge b, y \wedge b]_s.$$

Combining these two identities, we conclude that

$$\Theta[a \wedge b, x \wedge b]_s = n\Theta[a \wedge b, x' \wedge b]_s \in \mathfrak{K}_g^s[a \wedge b, - \wedge b]_{\text{prim}},$$

as desired. □

28.4. **Θ-symmetry.** It is inconvenient to require the entries of $\Theta[a \wedge b, x \wedge b]_s$ and $\Theta[a \wedge b, a \wedge y]_s$ to appear in a definite order. We therefore define that each of the following terms equals $\Theta[a \wedge b, x \wedge b]_s$:

$$\begin{array}{llll} \Theta[a \wedge b, x \wedge b]_s, & -\Theta[b \wedge a, x \wedge b]_s, & -\Theta[a \wedge b, b \wedge x]_s, & \Theta[b \wedge a, b \wedge x]_s, \\ \Theta[x \wedge b, a \wedge b]_s, & -\Theta[x \wedge b, b \wedge a]_s, & -\Theta[b \wedge x, a \wedge b]_s, & \Theta[b \wedge x, b \wedge a]_s. \end{array}$$

Similarly, we define that each of the following terms equals $\Theta[a \wedge b, a \wedge y]_s$:

$$\begin{array}{llll} \Theta[a \wedge b, a \wedge y]_s, & -\Theta[b \wedge a, a \wedge y]_s, & -\Theta[a \wedge b, y \wedge a]_s, & \Theta[b \wedge a, y \wedge a]_s, \\ \Theta[a \wedge y, a \wedge b]_s, & -\Theta[a \wedge y, b \wedge a]_s, & -\Theta[y \wedge a, a \wedge b]_s, & \Theta[y \wedge a, b \wedge a]_s. \end{array}$$

28.5. **Θ-signs.** Lemma 28.4 (Θ-linearity) implies that

$$\Theta[a \wedge b, (-x) \wedge b]_s = -\Theta[a \wedge b, x \wedge b]_s \quad \text{and} \quad \Theta[a \wedge b, a \wedge (-y)]_s = -\Theta[a \wedge b, a \wedge y]_s.$$

For a symplectic pair $a \wedge b$, there are three other symplectic pairs obtained by swapping $a$ and $b$ while multiplying them by $\pm 1$, namely $(-b) \wedge a$ and $b \wedge (-a)$ and $(-a) \wedge (-b)$. The following shows that changing $a \wedge b$ to one of these changes $\Theta[a \wedge b, x \wedge b]_s$ and $\Theta[a \wedge b, y \wedge a]_s$ in the obvious way. The statement uses the notation from §28.4:

**Lemma 28.5.** *Let $a \wedge b$ be a symplectic pair in $H_{\mathbb{Z}}$ and let $x, y \in \langle a, b\rangle^{\perp}$. Then*

$$\begin{array}{rlrl} \Theta[a \wedge (-b), x \wedge (-b)]_s = & \Theta[a \wedge b, x \wedge b]_s, & \Theta[a \wedge (-b), a \wedge y]_s = & -\Theta[a \wedge b, a \wedge y]_s, \\ \Theta[(-a) \wedge b, x \wedge b]_s = & -\Theta[a \wedge b, x \wedge b]_s, & \Theta[(-a) \wedge b, (-a) \wedge y]_s = & \Theta[a \wedge b, a \wedge y]_s, \\ \Theta[(-a) \wedge (-b), x \wedge (-b)]_s = & -\Theta[a \wedge b, x \wedge b]_s, & \Theta[(-a) \wedge (-b), (-a) \wedge y]_s = & -\Theta[a \wedge b, a \wedge y]_s. \end{array}$$

*Proof.* These are all proved the same way, so we will give the details for $\Theta[(-a)\wedge b, x\wedge b]_s = -\Theta[a\wedge b, x\wedge b]_s$ and leave the others to the reader.[37] Here $(-a)\wedge b$ is not a symplectic pair, but $b\wedge(-a)$ is a symplectic pair. Using the notation from §28.4, we interpret $\Theta[(-a)\wedge b, x\wedge b]_s$ as[38] $\Theta[b\wedge(-a), b\wedge x]_s$, so our goal is to prove that $\Theta[b\wedge(-a), b\wedge x]_s = -\Theta[a\wedge b, x\wedge b]_s$. By definition, $\Theta[b\wedge(-a), b\wedge x]_s$ equals

$$\begin{aligned} &[\![b\wedge(-a+x), b\wedge(-a+x)]\!]_s - [\![b\wedge(-a), b\wedge(-a)]\!]_s - [\![b\wedge x, b\wedge x]\!]_s \\ &= [\![(a-x)\wedge b, (a-x)\wedge b]\!]_s - [\![a\wedge b, a\wedge b]\!]_s - [\![(-x)\wedge b, (-x)\wedge b]\!]_s. \end{aligned}$$

This last expression equals $\Theta[a\wedge b, -x\wedge b]_s$, which by Lemma 28.4 ($\Theta$-linearity) equals $-\Theta[a\wedge b, x\wedge b]_s$. □

28.6. **The set $S_2$.** We now return to constructing $S_2$. Recall that

$$T_2 = \{(a_i\wedge b_i)\cdot(x\wedge b_i),\ (a_i\wedge b_i)\cdot(a_i\wedge y) \mid 1\le i\le g,\ x,y\in\mathcal{B}\setminus\{a_i,b_i\}\}.$$

Define

$$S_2 = \{\Theta[a_i\wedge b_i, x\wedge b_i]_s,\ \Theta[a_i\wedge b_i, a_i\wedge y]_s \mid 1\le i\le g,\ x,y\in\mathcal{B}\setminus\{a_i,b_i\}\}.$$

Like we did here, we will write elements of $\langle S_2\rangle$ in purple. For example, using Lemma 28.4 ($\Theta$-linearity), for $1\le i\le g$ and $z\in\langle a_i,b_i\rangle^\perp$ we have elements $\Theta[a_i\wedge b_i, z\in b_i]_s$ and $\Theta[a_i\wedge b_i, a_i\wedge z]_s$ in $\langle S_2\rangle$. By construction, the linearization map $\Phi$ takes $S_2$ bijectively to $T_2$. Even better:

**Lemma 28.6.** *The linearization map $\Phi$ takes $\langle S_1, S_2\rangle$ isomorphically to $\langle T_1, T_2\rangle$.*

*Proof.* Recall that in Lemma 27.1 we proved that $\Phi$ takes $\langle S_1\rangle$ isomorphically onto $\langle T_1\rangle$. Part of the proof of that lemma was that $\Phi$ takes $S_1$ bijectively to $T_1$. It follows that $\Phi$ takes $S_1\cup S_2$ bijectively to $T_1\cup T_2$. What we must prove is that all relations between elements of $T_1\cup T_2$ lift to relations between $S_1\cup S_2$.

We constructed all the relations between elements of $T = T_1\cup\cdots\cup T_4$ in §27.1, and in fact all of them only involve elements of $T_1\cup T_2$. Some only involve elements of $T_1$, and as we observed in the proof of Lemma 27.1 these all lift to relations between elements of $S_1$. The remaining relations are of the form

$$\sum\nolimits_{i=1}^{g}(a_i\wedge b_i)\cdot(x\wedge y) \text{ with } x,y\in\mathcal{B} \text{ with } x\prec y \text{ and } \omega(x,y)=0.$$

Lemma 28.7 below proves that these do indeed lift to relations between elements of $S_1\cup S_2$. □

The above proof used the following, which for later use we state in more generality than we need at the moment:

**Lemma 28.7** ($\Theta$-symplectic basis)**.** *Let $\{x_1,y_1,\ldots,x_g,y_g\}$ be a symplectic basis for $H_{\mathbb{Z}}$, let $1\le n<m\le g$, and let $z\in\{x_n,y_n\}$ and $w\in\{x_m,y_m\}$. Then*

$$\Theta[x_n\wedge y_n, z\wedge w]_s + \Theta[x_m\wedge y_m, z\wedge w]_s + \sum_{\substack{1\le i\le g\\ i\ne n,m}} [\![x_i\wedge y_i, z\wedge w]\!]_s = 0.$$

*Proof.* Whether this holds is invariant under the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathfrak{K}_g^s$, so applying an appropriate element of $\mathrm{Sp}_{2g}(\mathbb{Z})$ we can assume that the given symplectic basis is our fixed symplectic basis $\mathcal{B}=\{a_1,b_1,\ldots,a_g,b_g\}$. Moreover, recalling the subgroup $\mathrm{SymSp}_g$ from §7 we can apply an appropriate element of $\mathrm{SymSp}_g$ and ensure that $n=1$ and $m=2$. Our desired relation is thus

$$\Theta[a_1\wedge b_1, z\wedge w]_s + \Theta[a_2\wedge b_2, z\wedge w]_s + \sum\nolimits_{i=3}^{g} [\![a_i\wedge b_i, z\wedge 2]\!]_s = 0. \tag{28.5}$$

[37]We chose this one because it is slightly harder than the other cases.

[38]There is no sign change here since both $(-a)\wedge b$ and $x\wedge b$ are flipped and each flip causes a sign change.

Finally, since $\Phi$ restricted to $\langle S_1 \rangle$ is injective (Lemma 27.1) and $\Phi$ takes (28.5) to a true relation in $\text{Sym}^2((\wedge^2 H)/\mathbb{Q})$, it is enough to prove that (28.5) holds modulo $\langle S_1 \rangle$. In other words, letting $\equiv$ denote equality modulo $\langle S_1 \rangle$ we must prove that

$$\Theta[a_1 \wedge b_1, z \wedge w]_s + \Theta[a_2 \wedge b_2, z \wedge w]_s \equiv 0. \tag{28.6}$$

For concreteness, we will prove this for $z = a_1$ and $w = b_2$. The other cases are similar. Using Lemma 28.4 ($\Theta$-linearity), the relation (28.6) is equivalent to

$$\Theta[a_1 \wedge b_1, a_1 \wedge b_2]_s - \Theta[a_2 \wedge b_2, -a_1 \wedge b_2]_s \equiv 0. \tag{28.7}$$

By definition, we have

$$\begin{aligned} 2\Theta[a_1 \wedge b_1, a_1 \wedge b_2]_s =& [\![a_1 \wedge (b_1 + b_2), a_1 \wedge (b_1 + b_2)]\!]_s - [\![a_1 \wedge b_1, a_1 \wedge b_1]\!]_s \\ & - [\![a_1 \wedge b_2, a_1 \wedge b_2]\!]_s \equiv [\![a_1 \wedge (b_1 + b_2), a_1 \wedge (b_1 + b_2)]\!]_s, \\ 2\Theta[a_2 \wedge b_2, -a_1 \wedge b_2]_s =& [\![(a_2 - a_1) \wedge b_2, (a_2 - a_1) \wedge b_2]\!]_s - [\![a_2 \wedge b_2, a_2 \wedge b_2]\!]_s \\ & - [\![a_1 \wedge b_2, a_1 \wedge b_2]\!]_s \equiv [\![(a_2 - a_1) \wedge b_2, (a_2 - a_1) \wedge b_2]\!]_s. \end{aligned}$$

The relation (28.7) is thus equivalent to

$$[\![a_1 \wedge (b_1 + b_2), a_1 \wedge (b_1 + b_2)]\!]_s - [\![(a_2 - a_1) \wedge b_2, (a_2 - a_1) \wedge b_2]\!]_s \equiv 0. \tag{28.8}$$

In $(\wedge^2 H)/\mathbb{Q}$, we have

$$a_1 \wedge (b_1 + b_2) + (a_2 - a_1) \wedge b_2 + \sum_{i=3}^{g} a_i \wedge b_i = 0.$$

This implies that

$$\begin{aligned} [\![a_1 \wedge (b_1 + b_2), a_1 \wedge (b_1 + b_2)]\!]_s =& - [\![a_1 \wedge (b_1 + b_2), (a_2 - a_1) \wedge b_2]\!]_s \\ & - \sum_{i=3}^{g} [\![a_1 \wedge (b_1 + b_2), a_i \wedge b_i]\!]_s \\ \equiv & - [\![a_1 \wedge (b_1 + b_2), (a_2 - a_1) \wedge b_2]\!]_s. \end{aligned}$$

Plugging this into (28.8), we see that our desired relation is equivalent to showing that the following is equivalent to 0:

$$\begin{aligned} & [\![a_1 \wedge (b_1 + b_2), (a_2 - a_1) \wedge b_2]\!]_s + [\![(a_2 - a_1) \wedge b_2, (a_2 - a_1) \wedge b_2]\!]_s \\ =& [\![a_1 \wedge (b_1 + b_2) + (a_2 - a_1) \wedge b_2, (a_2 - a_1) \wedge b_2]\!]_s \\ =& [\![a_1 \wedge b_1 + a_2 \wedge b_2, (a_2 - a_1) \wedge b_2]\!]_s = - \sum_{i=3}^{g} [\![a_i \wedge b_i, (a_2 - a_1) \wedge b_2]\!]_s \equiv 0. \end{aligned}$$ □

28.7. **Additional bilinearity relations.** We close this section by proving some additional relations between the $\Theta$-elements.

**Lemma 28.8** ($\Theta$-bilinearity I)**.** *Let $a \wedge b$ be a symplectic pair in $H_{\mathbb{Z}}$ and $z \in \langle a, b \rangle^{\perp}$. Then:*

- *for $x \in \langle a, b, z \rangle^{\perp}$ we have $\Theta[(a + z) \wedge b, x \wedge b]_s = \Theta[a \wedge b, x \wedge b]_s + [\![z \wedge b, x \wedge b]\!]_s$.*
- *for $y \in \langle a, b, z \rangle^{\perp}$ we have $\Theta[a \wedge (b + z), a \wedge y]_s = \Theta[a \wedge b, a \wedge y]_s + [\![a \wedge z, a \wedge y]\!]_s$.*

*Proof.* Both are proved the same way, so we prove the first. By Lemma 28.3 ($\Theta$-expansion I), the element $[\![(a + z + x) \wedge b, (a + z + x) \wedge b]\!]_s$ equals

$$\begin{aligned} & [\![(a + z) \wedge b, (a + z) \wedge b]\!]_s + 2\Theta[(a + z) \wedge b, x \wedge b]_s + [\![x \wedge b, x \wedge b]\!]_s \\ =& [\![a \wedge b, a \wedge b]\!]_s + 2\Theta[a \wedge b, z \wedge b]_s + [\![z \wedge b, z \wedge b]\!]_s \\ & + 2\Theta[(a + z) \wedge b, x \wedge b]_s + [\![x \wedge b, x \wedge b]\!]_s. \end{aligned}$$

On the other hand, it also equals

$$\begin{aligned}&[\![a \wedge b, a \wedge b]\!]_s + 2\Theta[a \wedge b, (z+x) \wedge b]_s + [\![(z+x) \wedge b, (z+x) \wedge b]\!]_s \\ &= [\![a \wedge b, a \wedge b]\!]_s + 2\Theta[a \wedge b, z \wedge b]_s + 2\Theta[a \wedge b, x \wedge b]_s \\ &\quad + [\![z \wedge b, z \wedge b]\!]_s + 2[\![z \wedge b, x \wedge b]\!]_s + [\![x \wedge b, x \wedge b]\!]_s.\end{aligned}$$

Here the equality uses Lemma 28.4 ($\Theta$-linearity). The above two displays are thus equal, and the result follows. □

Lemma 28.8 allows many standard generators of $\mathfrak{K}_g^s$ to be written as the sum of two $\Theta$-elements:

**Lemma 28.9** ($\Theta$-expansion II)**.** *Let $a \wedge b$ and $a' \wedge b'$ be symplectic pairs in $H_{\mathbb{Z}}$ such that $\langle a, b\rangle$ and $\langle a', b'\rangle$ are orthogonal. Then*

$$\begin{aligned}
[\![(a+a') \wedge (b-b'), x \wedge (b-b')]\!]_s &= \Theta[a \wedge (b-b'), x \wedge (b-b')]_s + \Theta[a' \wedge (b-b'), x \wedge (b-b')]_s, \\
[\![(a+a') \wedge (b-b'), (a+a') \wedge y]\!]_s &= \Theta[(a+a') \wedge b, (a+a') \wedge y]_s - \Theta[(a+a') \wedge b', (a+a') \wedge y]_s, \\
[\![(a+b') \wedge (b+a'), x' \wedge (b+a')]\!]_s &= \Theta[a \wedge (b+a'), x' \wedge (b+a')]_s + \Theta[b' \wedge (b+a'), x' \wedge (b+a')]_s, \\
[\![(a+b') \wedge (b+a'), (a+b') \wedge y']\!]_s &= \Theta[(a+b') \wedge b, (a+b') \wedge y']_s + \Theta[(a+b') \wedge a', (a+b') \wedge y']_s.
\end{aligned}$$

*for $x \in \langle a, a', b-b'\rangle^{\perp}$ and $y \in \langle b, b', a+a'\rangle^{\perp}$ and $x' \in \langle a, b', b+a'\rangle$ and $y' \in \langle b, a', a+b'\rangle$.*

*Proof.* All are proved the same way, so we will prove the first. Lemma 28.8 ($\Theta$-bilinearity I) implies that

$$\Theta[(-a') \wedge (b-b'), x \wedge (b-b')]_s + [\![(a+a') \wedge (b-b'), x \wedge (b-b')]\!]_s$$

equals

$$\Theta[(-a' + (a+a')) \wedge (b-b'), x \wedge (b-b')]_s = \Theta[a \wedge (b-b'), x \wedge (b-b')]_s.$$

Rearranging this, we see that

$$[\![(a+a') \wedge (b-b'), x \wedge (b-b')]\!]_s = \Theta[a \wedge (b-b'), x \wedge (b-b')]_s + \Theta[a' \wedge (b-b'), x \wedge (b-b')]_s. \quad \square$$

**Lemma 28.10** ($\Theta$-bilinearity II)**.** *Let $a \wedge b$ be a symplectic pair in $H_{\mathbb{Z}}$ and $n \in \mathbb{Z}$. Then:*

- *for $x \in \langle a, b\rangle^{\perp}$ we have*

$$\begin{aligned}\Theta[(a+nb) \wedge b, x \wedge b]_s &= \Theta[a \wedge b, x \wedge b]_s, \\ \Theta[a \wedge (b+na), x \wedge (b+na)]_s &= \Theta[a \wedge b, x \wedge b]_s + n\Theta[a \wedge b, x \wedge a]_s.\end{aligned}$$

- *for $y \in \langle a, b\rangle^{\perp}$ we have*

$$\begin{aligned}\Theta[a \wedge (b+na), a \wedge y]_s &= \Theta[a \wedge b, a \wedge y]_s, \\ \Theta[(a+nb) \wedge b, (a+nb) \wedge y]_s &= \Theta[a \wedge b, a \wedge y]_s + n\Theta[a \wedge b, b \wedge y]_s.\end{aligned}$$

*Proof.* The two bullet points are proved the same way, so we will prove the second. Observe first that $2\Theta[a \wedge (b+na), a \wedge y]_s$ equals

$$\begin{aligned}&[\![a \wedge (b+na+y), a \wedge (b+na+y)]\!]_s - [\![a \wedge (b+na), a \wedge (b+na)]\!]_s - [\![a \wedge y, a \wedge y]\!]_s \\ &= [\![a \wedge (b+y), a \wedge (b+y)]\!]_s - [\![a \wedge b, a \wedge b]\!]_s - [\![a \wedge y, a \wedge y]\!]_s,\end{aligned}$$

which equals $2\Theta[a \wedge b, a \wedge y]_s$. This gives the first equation. For the second, by Lemma 28.4 ($\Theta$-linearity) it is enough to prove it for $y$ primitive. We can then find a symplectic basis $\{x_1, y_1, \ldots, x_g, y_g\}$ for $H_{\mathbb{Z}}$ such that $x_1 = a$ and $y_1 = b$ and $y_2 = y$. Our goal is to prove that

$$\Theta[(x_1 + ny_1) \wedge y_1, (x_1 + ny_1) \wedge y_2]_s = \Theta[x_1 \wedge y_1, x_1 \wedge y_2]_s + n\Theta[x_1 \wedge y_1, y_1 \wedge y_2]_s.$$

Applying Lemma 28.7 ($\Theta$-symplectic basis) to the alternate symplectic basis $\mathcal{B}' = \{x_1 + ny_1, y_1, x_2, y_2, \ldots, x_g, y_g\}$, we see that

$$\begin{aligned} 0 =& \Theta[(x_1 + ny_1) \wedge y_1, (x_1 + ny_1) \wedge y_2]_s \\ &+ \Theta[x_2 \wedge y_2, (x_1 + ny_1) \wedge y_2]_s + \sum\nolimits_{i=3}^{g} [\![x_i \wedge y_i, (x_1 + ny_1) \wedge y_2]\!]_s \end{aligned}$$

From this, we see that $\Theta[(x_1 + ny_1) \wedge y_1, (x_1 + ny_1) \wedge y_2]_s$ equals

$$\begin{aligned} &- \Theta[x_2 \wedge y_2, (x_1 + ny_1) \wedge y_2]_s - \sum\nolimits_{i=3}^{g} [\![x_i \wedge y_i, (x_1 + ny_1) \wedge y_2]\!]_s \\ &= - \left(\Theta[x_2 \wedge y_2, x_1 \wedge y_2]_s - \sum\nolimits_{i=3}^{g} [\![x_i \wedge y_i, x_1 \wedge y_2]\!]_s\right) \\ &\quad - n \left(\Theta[x_2 \wedge y_2, y_1 \wedge y_2]_s - \sum\nolimits_{i=3}^{g} [\![x_i \wedge y_i, y_1 \wedge y_2]\!]_s\right). \end{aligned}$$

The equality here uses Lemma 28.4 ($\Theta$-linearity). Again using Lemma 28.7 ($\Theta$-symplectic basis), we recognize this as being $\Theta[x_1 \wedge y_1, x_1 \wedge y_2]_s + n\Theta[x_1 \wedge y_1, y_1 \wedge y_2]_s$, as desired. □

## 29. Symmetric kernel, symmetric version III: $S_3$ and the $\Lambda$-elements

We continue using all the notation from §27 – §28. This section constructs the set $S_3$ that lifts $T_3$. It consists of what are called $\Lambda$-elements of $\mathfrak{K}_g^s$, and the first part of this section constructs these in more generality than is needed for $S_3$ alone.

29.1. **Definition.** Let $a \wedge b$ be a symplectic pair in $\wedge^2 H_{\mathbb{Z}}$ and let $x, y \in \langle a, b \rangle^{\perp}$ satisfy $\omega(x, y) = 0$. The $\Lambda$-element $\Lambda[a \wedge y, x \wedge b]_s$ is an element of $\mathfrak{K}_g^s$ that is taken by $\Phi$ to

$$(a \wedge y) \cdot (x \wedge b) \in \mathrm{Sym}^2((\wedge^2 H)/\mathbb{Q}).$$

To find it, note that

$$\begin{aligned} (a \wedge (b + y)) \cdot (x \wedge (b + y)) &= (a \wedge (b + y)) \cdot (x \wedge y) + (a \wedge b) \cdot (x \wedge b) + (a \wedge y) \cdot (x \wedge b), \\ ((a + x) \wedge b) \cdot ((a + x) \wedge y) &= ((a + x) \wedge b) \cdot (x \wedge y) + (a \wedge b) \cdot (a \wedge y) + (x \wedge b) \cdot (a \wedge y). \end{aligned}$$

This suggests two possible elements of $\mathfrak{K}_g^s$ projecting to $(a \wedge y) \cdot (x \wedge b)$:

**Definition 29.1.** For a symplectic pair $a \wedge b$ in $H_{\mathbb{Z}}$ and $x, y \in \langle a, b \rangle^{\perp}$ with $\omega(x, y) = 0$, define:

$$\begin{aligned} \Lambda_1[a \wedge y, x \wedge b]_s &= \Theta[a \wedge (b + y), x \wedge (b + y)]_s - \Theta[a \wedge b, x \wedge b]_s - [\![a \wedge (b + y), x \wedge y]\!]_s, \\ \Lambda_2[a \wedge y, x \wedge b]_s &= \Theta[(a + x) \wedge b, (a + x) \wedge y]_s - \Theta[a \wedge b, a \wedge y]_s - [\![(a + x) \wedge b, x \wedge y]\!]_s. \end{aligned}$$ □

See Remark 29.3 below for a caveat about this notation. By construction, we have

$$\Phi(\Lambda_1[a \wedge y, x \wedge b]_s) = \Phi(\Lambda_2[a \wedge y, x \wedge b]_s) = (a \wedge y) \cdot (x \wedge b).$$

Since we are trying to prove that $\Phi$ is an isomorphism, we must prove that these are actually the same element:

**Lemma 29.2.** *Let $a \wedge b$ be a symplectic pair in $H_{\mathbb{Z}}$ and let $x, y \in \langle a, b \rangle^{\perp}$ satisfy $\omega(x, y) = 0$. Then $\Lambda_1[a \wedge y, x \wedge b]_s = \Lambda_2[a \wedge y, x \wedge b]_s$.*

*Proof.* We must prove that

$$\begin{aligned} &\Theta[a \wedge (b + y), x \wedge (b + y)]_s - \Theta[a \wedge b, x \wedge b]_s - [\![a \wedge (b + y), x \wedge y]\!]_s, \\ =&\Theta[(a + x) \wedge b, (a + x) \wedge y]_s - \Theta[a \wedge b, a \wedge y]_s - [\![(a + x) \wedge b, x \wedge y]\!]_s. \end{aligned}$$

Since

$$\begin{aligned} [\![a \wedge (b + y), x \wedge y]\!]_s &= [\![a \wedge b, x \wedge y]\!]_s + [\![a \wedge y, x \wedge y]\!]_s, \\ [\![(a + x) \wedge b, x \wedge y]\!]_s &= [\![a \wedge b, x \wedge y]\!]_s + [\![x \wedge b, x \wedge y]\!]_s, \end{aligned}$$

we can rearrange this and see that it is equivalent to prove that

$$\Theta[a \wedge (b+y), x \wedge (b+y)]_s - \Theta[(a+x) \wedge b, (a+x) \wedge y]_s \tag{29.1}$$

equals

$$\Theta[a \wedge b, x \wedge b]_s - \Theta[a \wedge b, a \wedge y]_s + [\![a \wedge y, x \wedge y]\!]_s - [\![x \wedge b, x \wedge y]\!]_s. \tag{29.2}$$

Applying Lemma 28.3 ($\Theta$-expansion I) twice, the element $[\![(a+x) \wedge (b+y), (a+x) \wedge (b+y)]\!]_s$ equals

$$\begin{aligned}
&[\![a \wedge (b+y), a \wedge (b+y)]\!]_s + 2\Theta[a \wedge (b+y), x \wedge (b+y)]_s + [\![x \wedge (b+y), x \wedge (b+y)]\!]_s \\
&= [\![a \wedge b, a \wedge b]\!]_s + 2\Theta[a \wedge b, a \wedge y]_s + [\![a \wedge y, a \wedge y]\!]_s \\
&\quad + 2\Theta[a \wedge (b+y), x \wedge (b+y)]_s + [\![x \wedge (b+y), x \wedge (b+y)]\!]_s.
\end{aligned}$$

Applying Lemma 28.3 ($\Theta$-expansion I) twice again but in a different order, the same element $[\![(a+x) \wedge (b+y), (a+x) \wedge (b+y)]\!]_s$ also equals

$$\begin{aligned}
&[\![(a+x) \wedge b, (a+x) \wedge b]\!]_s + 2\Theta[(a+x) \wedge b, (a+x) \wedge y]_s + [\![(a+x) \wedge y, (a+x) \wedge y)]\!]_s \\
&= [\![a \wedge b, a \wedge b]\!]_s + 2\Theta[a \wedge b, x \wedge b]_s + [\![x \wedge b, x \wedge b]\!]_s \\
&\quad + 2\Theta[(a+x) \wedge b, (a+x) \wedge y]_s + [\![(a+x) \wedge y, (a+x) \wedge y)]\!]_s.
\end{aligned}$$

Equating the previous two displays and rearranging terms, we deduce that 2 times (29.1) equals

$$\begin{aligned}
&2\Theta[a \wedge b, x \wedge b]_s + [\![x \wedge b, x \wedge b]\!]_s + [\![(a+x) \wedge y, (a+x) \wedge y)]\!]_s \\
&\quad - 2\Theta[a \wedge b, a \wedge y]_s - [\![a \wedge y, a \wedge y]\!]_s - [\![x \wedge (b+y), x \wedge (b+y)]\!]_s,
\end{aligned}$$

which using the usual bilinearity relations in $\mathfrak{K}_g^s$ equals

$$\begin{aligned}
&2\Theta[a \wedge b, x \wedge b]_s - 2\Theta[a \wedge b, a \wedge y]_s + [\![x \wedge b, x \wedge b]\!]_s - [\![a \wedge y, a \wedge y]\!]_s \\
&\quad + ([\![a \wedge y, a \wedge y]\!]_s + 2[\![a \wedge y, x \wedge y]\!]_s + [\![x \wedge y, x \wedge y]\!]_s) \\
&\quad - ([\![x \wedge b, x \wedge b]\!]_s + 2[\![x \wedge b, x \wedge y]\!]_s + [\![x \wedge y, x \wedge y]\!]_s),
\end{aligned}$$

which after canceling terms equals 2 times (29.2). □

In light of this lemma, we will denote the common value of $\Lambda_1[a \wedge y, x \wedge b]_s$ and $\Lambda_2[a \wedge y, x \wedge b]_s$ by $\Lambda[a \wedge y, x \wedge b]_s$.

*Remark* 29.3. Just like for the $\Theta$-elements (cf. Remark 28.2), the elements $\Lambda_1[a \wedge y, x \wedge b]_s$ and $\Lambda_2[a \wedge y, x \wedge b]_s$ and $\Lambda[a \wedge y, x \wedge b]_s$ depend on the ordered tuple $(a, y, x, b)$, not on $a \wedge y$ and $x \wedge b$. □

29.2. **$\Lambda$-expansion I.** The following is an important way that $\Lambda$-elements appear in our calculations:

**Lemma 29.4** ($\Lambda$-expansion I)**.** *Let $a \wedge b$ be a symplectic pair and let $x, y \in \langle a, b \rangle^{\perp}$ satisfy $\omega(x, y) = 0$. Then*

$$\begin{aligned}
\Theta[a \wedge (b+y), x \wedge (b+y)]_s &= \Lambda[a \wedge y, x \wedge b]_s + \Theta[a \wedge b, x \wedge b]_s + [\![a \wedge (b+y), x \wedge y]\!]_s, \\
\Theta[(a+x) \wedge b, (a+x) \wedge y]_s &= \Lambda[a \wedge y, x \wedge b]_s + \Theta[a \wedge b, a \wedge y]_s + [\![(a+x) \wedge b, x \wedge y]\!]_s.
\end{aligned}$$

*Proof.* Immediate from Definition 29.1. □

29.3. **Λ-linearity.** The following says that $\Lambda[a \wedge y, x \wedge b]_s$ is linear in both $x$ and $y$:

**Lemma 29.5** (Λ-linearity)**.** *Let $a \wedge b$ be a symplectic pair in $H_{\mathbb{Z}}$. Then:*

- *For $x, y_1, y_2 \in \langle a, b\rangle^{\perp}$ with $\omega(x, y_1) = \omega(x, y_2) = 0$ and $\lambda_1, \lambda_2 \in \mathbb{Z}$, we have*
$$\Lambda[a \wedge (\lambda_1 y_1 + \lambda_2 y_2), x \wedge b]_s = \lambda_1 \Lambda[a \wedge y_1, x \wedge b]_s + \lambda_2 \Lambda[a \wedge y_2, x \wedge b]_s.$$
- *For $x_1, x_2, y \in \langle a, b\rangle^{\perp}$ with $\omega(x_1, y) = \omega(x_2, y) = 0$ and $\lambda_1, \lambda_2 \in \mathbb{Z}$, we have*
$$\Lambda[a \wedge y, (\lambda_1 x_1 + \lambda_2 x_2) \wedge b]_s = \lambda_1 \Lambda[a \wedge y, x_1 \wedge b]_s + \lambda_2 \Lambda[a \wedge y, x_2 \wedge b]_s.$$

In fact, we will prove something more general. Let $a \wedge b$ be a symplectic pair in $H_{\mathbb{Z}}$. Define $\mathfrak{K}_g^{s,\Lambda}[a \wedge -, - \wedge b]$ to be the subspace of $\mathfrak{K}_g^s$ spanned by $\Lambda[a \wedge y, x \wedge b]_s$ as $x$ and $y$ range over elements of $\langle a, b\rangle^{\perp}$ satisfying $\omega(x, y) = 0$. The linearization map $\Phi\colon \mathfrak{K}_g^s \to \mathrm{Sym}^2((\wedge^2 H)/\mathbb{Q})$ takes $\mathfrak{K}_g^{s,\Lambda}[a \wedge -, - \wedge b]$ into
$$\langle (a \wedge y)\cdot(x \wedge b) \mid x, y \in \langle a, b\rangle^{\perp}\rangle \cong \left(\langle a, b\rangle_{\mathbb{Q}}^{\perp}\right)^{\otimes 2}.$$
This isomorphism takes $(a \wedge y)\cdot(x \wedge b)$ to $y \otimes x$. The image is in the kernel of map
$$\left(\langle a, b\rangle_{\mathbb{Q}}^{\perp}\right)^{\otimes 2} \longrightarrow \mathbb{Q}$$
induced by $\omega$, which we denote $\mathcal{Z}(\langle a, b\rangle_{\mathbb{Q}}^{\perp})$ (c.f. §9.2). We will prove the following, which strengthens Lemma 29.5:

**Lemma 29.6** (strong Λ-linearity)**.** *Let $a \wedge b$ and $\mathcal{Z}(\langle a, b\rangle_{\mathbb{Q}}^{\perp})$ be as above. Then the linearization map*
$$\Phi\colon \mathfrak{K}_g^{s,\Lambda}[a \wedge -, - \wedge b] \longrightarrow \mathcal{Z}(\langle a, b\rangle_{\mathbb{Q}}^{\perp}).$$
*is an isomorphism.*

*Proof.* Theorem 9.3 gives a presentation for $\mathcal{Z}(\langle a, b\rangle_{\mathbb{Q}}^{\perp})$. In light of this presentation, it is enough to prove the following two special cases of Lemma 29.5:

- For $x \in \langle a, b\rangle^{\perp}$ and a partial basis $\{y_1, y_2\}$ of $\langle a, b, x\rangle^{\perp}$, we have
$$\Lambda[a \wedge (y_1 + y_2), x \wedge b]_s = \Lambda[a \wedge y_1, x \wedge b]_s + \Lambda[a \wedge y_2, x \wedge b]_s.$$
- For $y \in \langle a, b\rangle^{\perp}$ and a partial basis $\{x_1, x_2\}$ of $\langle a, b, y\rangle^{\perp}$, we have
$$\Lambda[a \wedge y, (x_1 + x_2) \wedge b]_s = \Lambda[a \wedge y, x_1 \wedge b]_s + \Lambda[a \wedge y, x_2 \wedge b]_s.$$

For the first bullet point, $\Lambda[a \wedge (y_1 + y_2), x \wedge b]_s = \Lambda_2[a \wedge (y_1 + y_2), x \wedge b]_s$ equals
$$\Theta[(a + x) \wedge b, (a + x) \wedge (y_1 + y_2)]_s - \Theta[a \wedge b, a \wedge (y_1 + y_2)]_s - [\![(a + x) \wedge b, x \wedge (y_1 + y_2)]\!]_s.$$
Using Lemma 28.4 (Θ-linearity), all three terms are linear in the $y_i$:
$$\begin{aligned}
\Theta[(a + x) \wedge b, (a + x) \wedge (y_1 + y_2)]_s =& \Theta[(a + x) \wedge b, (a + x) \wedge y_1]_s \\
&+ \Theta[(a + x) \wedge b, (a + x) \wedge y_2]_s, \\
\Theta[a \wedge b, a \wedge (y_1 + y_2)]_s =& \Theta[a \wedge b, a \wedge y_1]_s + \Theta[a \wedge b, a \wedge y_2]_s, \\
[\![(a + x) \wedge b, x \wedge (y_1 + y_2)]\!]_s =& [\![(a + x) \wedge b, x \wedge y_1]\!]_s + [\![(a + x) \wedge b, x \wedge y_2]\!]_s.
\end{aligned}$$
The first bullet point follows. The second bullet point is proved the same way, but using $\Lambda_1[a \wedge -, - \wedge b]_s$ instead of $\Lambda_2[a \wedge -, - \wedge b]_s$. □

29.4. **Λ-symmetry.** It is inconvenient to require the entries of $\Lambda[a \wedge y, x \wedge b]_s$ to appear in a definite order. We therefore define that each of the following terms equals $\Lambda[a \wedge y, x \wedge b]_s$:
$$\Lambda[a \wedge y, x \wedge b]_s, \quad -\Lambda[y \wedge a, x \wedge b]_s, \quad -\Lambda[a \wedge y, b \wedge x]_s, \quad \Lambda[y \wedge a, b \wedge x]_s,$$
$$\Lambda[x \wedge b, a \wedge y]_s, \quad -\Lambda[x \wedge b, y \wedge a]_s, \quad -\Lambda[b \wedge x, a \wedge y]_s, \quad \Lambda[b \wedge x, y \wedge a]_s.$$

29.5. **Λ-signs.** Lemma 29.5 (Λ-linearity) implies that

$$\Lambda[a \wedge (-y), x \wedge b]_s = -\Lambda[a \wedge y, x \wedge b]_s \quad \text{and} \quad \Lambda[a \wedge y, (-x) \wedge b]_s = -\Lambda[a \wedge y, x \wedge b]_s.$$

For a symplectic pair $a \wedge b$, there are three other symplectic pairs obtained by swapping $a$ and $b$ while multiplying them by $\pm 1$, namely $(-b) \wedge a$ and $b \wedge (-a)$ and $(-a) \wedge (-b)$. The following shows that changing $a \wedge b$ to one of these changes $\Lambda[a \wedge y, x \wedge b]_s$ in the obvious way. The statement uses the notation from §29.4:

**Lemma 29.7.** *Let $a \wedge b$ be a symplectic pair in $H_{\mathbb{Z}}$ and let $x, y \in \langle a, b\rangle^{\perp}$ satisfy $\omega(x, y) = 0$. Then*

$$\begin{aligned} \Lambda[a \wedge y, x \wedge (-b)]_s &= -\Lambda[a \wedge y, x \wedge b]_s, \\ \Lambda[(-a) \wedge y, x \wedge b]_s &= -\Lambda[a \wedge y, x \wedge b]_s, \\ \Lambda[(-a) \wedge y, x \wedge (-b)]_s &= \ \ \Lambda[a \wedge y, x \wedge b]_s. \end{aligned}$$

*Proof.* These are all proved the same way, so we will give the details for $\Lambda[a \wedge y, x \wedge (-b)]_s = -\Lambda[a \wedge y, x \wedge b]_s$ and leave the others to the reader. Using the notation from §29.4, we interpret $\Lambda[a \wedge y, x \wedge (-b)]_s$ as $\Lambda[(-b) \wedge x, y \wedge a]_s$. Our goal is to prove that $\Lambda[(-b) \wedge x, y \wedge a]_s = -\Lambda[a \wedge y, x \wedge b]_s$ By the definition of Λ-elements (Definition 29.1), $\Lambda[(-b) \wedge x, y \wedge a]_s = \Lambda_1[(-b) \wedge x, y \wedge a]_s$ equals

$$\begin{aligned} &\Theta[(-b) \wedge (a + x), y \wedge (a + x)]_s - \Theta[(-b) \wedge a, y \wedge a]_s - [\![(-b) \wedge (a + x), y \wedge x]\!]_s \\ &= -\Theta[b \wedge (a + x), y \wedge (a + x)]_s + \Theta[b \wedge a, y \wedge a]_s + [\![b \wedge (a + x), y \wedge x]\!]_s \\ &= -\Theta[(a + x) \wedge b, (a + x) \wedge y]_s + \Theta[a \wedge b, a \wedge y]_s + [\![(a + x) \wedge b, x \wedge y]\!]_s. \end{aligned}$$

This last expression equals $-\Lambda_2[a \wedge y, x \wedge b]_s = -\Lambda[a \wedge y, x \wedge b]_s$. □

29.6. **The set $S_3$.** We now return to constructing $S_3$. Recall that

$$T_3 = \{(a_i \wedge y) \cdot (x \wedge b_i) \mid 1 \leq i \leq g,\ x, y \in \mathcal{B} \setminus \{a_i, b_i\},\ \omega(x, y) = 0\}.$$

Define

$$S_3 = \{\Lambda[a_i \wedge y, x \wedge b_i]_s \mid 1 \leq i \leq g,\ x, y \in \mathcal{B} \setminus \{a_i, b_i\},\ \omega(x, y) = 0\}.$$

Like we did here, we will write elements of $\langle S_3 \rangle$ in orange. For example, using Lemma 29.5 (Λ-linearity) the following holds for $1 \leq i \leq g$. Consider $x, y \in \langle a_i, b_i\rangle^{\perp}$ with $\omega(x, y) = 0$. Assume there exist $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{B} \setminus \{a_i, b_i\}$ (not necessarily disjoint) with $\omega(z_1, z_2) = 0$ for all $z_1 \in \mathcal{B}_1$ and $z_2 \in \mathcal{B}_2$ such that $x \in \langle \mathcal{B}_1 \rangle$ and $y \in \langle \mathcal{B}_2 \rangle$. Then $\Lambda[a_i \wedge y, x \wedge b_i]_s \in \langle S_3 \rangle$.

*Remark* 29.8. It is not true, however, that in general elements of the form $\Lambda[a_i \wedge y, x \wedge b_i]_s$ with $x, y \in \langle a_i, b_i\rangle^{\perp}$ lie in $\langle S_3 \rangle$. See §30.2 below. □

By construction, the linearization map $\Phi$ takes $S_3$ bijectively to $T_3$. Even better:

**Lemma 29.9.** *The linearization map $\Phi$ takes $\langle S_1, S_2, S_3 \rangle$ isomorphically to $\langle T_1, T_2, T_3 \rangle$.*

*Proof.* Recall that in Lemma 28.6 we proved that $\Phi$ takes $\langle S_1, S_2 \rangle$ isomorphically to $\langle T_1, T_2 \rangle$. Part of the proof of that lemma was that $\Phi$ takes $S_1 \cup S_2$ bijectively to $T_1 \cup T_2$. It follows that $\Phi$ takes $S_1 \cup S_2 \cup S_3$ bijectively to $T_1 \cup T_2 \cup T_3$. What is more, in the proof of Lemma 28.6 we proved that all relations between elements of $T_1 \cup \cdots \cup T_4$ are actually relations between elements of $T_1 \cup T_2$ and lift to relations between elements of $S_1 \cup S_2$. The lemma follows. □

29.7. **Additional bilinearity relations.** We close this section by proving some additional relations between the $\Lambda$-elements.

**Lemma 29.10** ($\Lambda$-bilinearity I)**.** *Let $a \wedge b$ be a symplectic pair in $H_{\mathbb{Z}}$, let $x, y \in \langle a, b\rangle^{\perp}$ satisfy $\omega(x, y) = 0$, and let $z \in \langle a, b, x, y\rangle^{\perp}$. Then:*

$$\Lambda[(a+z) \wedge y, x \wedge b]_s = \Lambda[a \wedge y, x \wedge b]_s + [\![z \wedge y, x \wedge b]\!]_s,$$
$$\Lambda[a \wedge y, x \wedge (b+z)]_s = \Lambda[a \wedge y, x \wedge b]_s + [\![a \wedge y, x \wedge z]\!]_s.$$

*Proof.* Both formulas are proved the same way, so we will prove the first. By definition, $\Lambda[(a+z) \wedge y, x \wedge b]_s = \Lambda_1[(a+z) \wedge y, x \wedge b]_s$ equals

$$\Theta[(a+z) \wedge (b+y), x \wedge (b+y)]_s - \Theta[(a+z) \wedge b, x \wedge b]_s - [\![(a+z) \wedge (b+y), x \wedge y]\!]_s.$$

By Lemma 28.8 ($\Theta$-bilinearity I), this equals

$$\begin{aligned} &\Theta[a \wedge (b+y), x \wedge (b+y)]_s + [\![z \wedge (b+y), x \wedge (b+y)]\!]_s - \Theta[a \wedge b, x \wedge b]_s - [\![z \wedge b, x \wedge b]\!]_s \\ &\quad - [\![(a+z) \wedge (b+y), x \wedge y]\!]_s \\ =&\Lambda[a \wedge y, x \wedge b]_s + [\![a \wedge (b+y), x \wedge y]\!]_s + [\![z \wedge (b+y), x \wedge (b+y)]\!]_s - [\![z \wedge b, x \wedge b]\!]_s \\ &\quad - [\![a \wedge (b+y), x \wedge y]\!]_s - [\![z \wedge (b+y), x \wedge y]\!]_s. \end{aligned}$$

All the terms after the first in this can be expanded out and many of the resulting terms cancel, leaving $\Lambda[a \wedge y, x \wedge b]_s + [\![z \wedge y, x \wedge b]\!]_s$. □

Lemma 29.10 allows some standard generators of $\mathfrak{K}_g^s$ to be written as the sum of two $\Lambda$-elements:[39]

**Lemma 29.11** ($\Lambda$-expansion II)**.** *Let $a \wedge b$ and $a' \wedge b'$ be symplectic pairs in $H_{\mathbb{Z}}$ such that $\langle a, b\rangle$ and $\langle a', b'\rangle$ are orthogonal and let $y, w \in \langle a, b', b + a'\rangle^{\perp}$ satisfy $\omega(y, w) = 0$. Then*

$$[\![(a+b') \wedge y, (b+a') \wedge w]\!]_s = \Lambda[a \wedge y, (b+a') \wedge w]_s + \Lambda[b' \wedge y, (b+a') \wedge w]_s.$$

*Proof.* Lemma 29.10 implies that

$$\Lambda[(-b') \wedge y, w \wedge (b+a')]_s + [\![(a+b') \wedge y, w \wedge (b+a')]\!]_s$$

equals

$$\Lambda[(-b' + (a+b')) \wedge y, w \wedge (b+a')]_s = \Lambda[a \wedge y, w \wedge (b+a')]_s.$$

Rearranging this, we see that

$$[\![(a+b') \wedge y, (b+a') \wedge w]\!]_s = \Lambda[a \wedge y, (b+a') \wedge w]_s + \Lambda[b' \wedge y, (b+a') \wedge w]_s. \qquad \square$$

**Lemma 29.12** ($\Lambda$-bilinearity II)**.** *For a symplectic pair $a \wedge b$ in $H_{\mathbb{Z}}$ and $x, y \in \langle a, b\rangle^{\perp}$ with $\omega(x, y) = 0$ and $n \in \mathbb{Z}$, we have*

$$\Lambda[(a+nb) \wedge y, x \wedge b]_s = \Lambda[a \wedge y, x \wedge b]_s + n[\![b \wedge y, x \wedge b]\!]_s,$$
$$\Lambda[a \wedge y, x \wedge (b+na)]_s = \Lambda[a \wedge y, x \wedge b]_s + n[\![a \wedge y, x \wedge a]\!]_s.$$

*Proof.* These two relations are proved in the same way, so we will give the details for the first. The element $\Lambda[(a+nb) \wedge y, x \wedge b]_s = \Lambda_1[(a+nb) \wedge y, x \wedge b]_s$ equals

$$\begin{aligned} &\Theta[(a+nb) \wedge (b+y), x \wedge (b+y)]_s - \Theta[(a+nb) \wedge b, x \wedge b]_s \\ &\quad - [\![(a+nb) \wedge b, x \wedge y]\!]_s - [\![(a+nb) \wedge y, x \wedge y]\!]_s \\ &= \Theta[(a+nb) \wedge (b+y), x \wedge (b+y)]_s - \Theta[a \wedge b, x \wedge b]_s \\ &\quad - [\![a \wedge b, x \wedge y]\!]_s - [\![a \wedge y, x \wedge y]\!]_s - n[\![b \wedge y, x \wedge y]\!]_s. \end{aligned} \tag{29.3}$$

[39] There are many variants of this in the style of Lemma 28.9, but we give the only one we use.

Here we use Lemma 28.10 (Θ-bilinearity II). That lemma also implies that $\Theta[(a+nb)\wedge(b+y), x\wedge(b+y)]_s$ equals

$$\begin{aligned}&\Theta[(a-ny)\wedge(b+y), x\wedge(b+y)]_s\\&=\Theta[a\wedge(b+y), x\wedge(b+y)]_s - n[\![y\wedge(b+y), x\wedge(b+y)]\!]_s\\&=\Theta[a\wedge(b+y), x\wedge(b+y)]_s - n[\![y\wedge b, x\wedge b]\!]_s - n[\![y\wedge b, x\wedge y]\!]_s,\end{aligned}$$

where we are also using Lemma 28.8 (Θ-bilinearity I). Plugging this into (29.3) and canceling terms gives

$$\Theta[a\wedge(b+y), x\wedge(b+y)]_s - \Theta[a\wedge b, x\wedge b]_s - [\![a\wedge b, x\wedge y]\!]_s - [\![a\wedge y, x\wedge y]\!]_s - n[\![y\wedge b, x\wedge b]\!]_s,$$

which equals $\Lambda[a\wedge y, x\wedge b]_s + n[\![b\wedge y, x\wedge b]\!]_s$. □

Again, Lemma 29.12 allows some standard generators of $\mathfrak{K}_g^s$ to be written as the sum of two Λ-elements:

**Lemma 29.13** (Λ-expansion III)**.** *Let $a\wedge b$ be a symplectic pair in $H_{\mathbb{Z}}$ and let $y, w \in \langle a, b\rangle^{\perp}$ satisfy $\omega(y,w)=0$. Then for $\epsilon\in\{\pm 1\}$ we have*

$$[\![(a+\epsilon b)\wedge y, (a+\epsilon b)\wedge w]\!]_s = \Lambda[a\wedge y, (a+\epsilon b)\wedge w]_s + \epsilon\Lambda[b\wedge y, (a+\epsilon b)\wedge w]_s.$$

*Proof.* Lemma 29.10 implies that

$$\Lambda[(-\epsilon b)\wedge y, w\wedge(a+\epsilon b)]_s + [\![(a+\epsilon b)\wedge y, w\wedge(a+\epsilon b)]\!]_s$$

equals

$$\Lambda[(-\epsilon b+(a+\epsilon b))\wedge y, w\wedge(a+\epsilon b)]_s = \Lambda[a\wedge y, w\wedge(a+\epsilon b)]_s.$$

Rearranging this, we see that

$$[\![(a+\epsilon b)\wedge y, (a+\epsilon b)\wedge w]\!]_s = \Lambda[a\wedge y, (a+\epsilon b)\wedge w]_s + \epsilon\Lambda[b\wedge y, (a+\epsilon b)\wedge w]_s \qquad \square.$$

## 30. Symmetric kernel, symmetric version IV: $S_4$ and the Ω-elements

We continue using all the notation from §27 – §29. This section constructs the set $S_4$ that lifts $T_4$. It consists of what are called Ω-elements of $\mathfrak{K}_g^s$, and the first part of this section constructs these in more generality than is needed for $S_4$ alone.

30.1. **Definition.** Let $a\wedge b$ and $a'\wedge b'$ be symplectic pairs in $\wedge^2 H_{\mathbb{Z}}$ such that $\langle a,b\rangle$ and $\langle a',b'\rangle$ are orthogonal. The Ω-element $\Omega[a\wedge a', b'\wedge b]_s$ is an element of $\mathfrak{K}_g^s$ that is taken by Φ to

$$(a\wedge a')\cdot(b'\wedge b)\in \mathrm{Sym}^2((\wedge^2 H)/\mathbb{Q}).$$

To find it, note that $(a\wedge(b+a'))\cdot((a+b')\wedge(b+a'))$ equals

$$\begin{aligned}&\big(a\wedge(b+a')\big)\cdot\big(a\wedge(b+a')\big) + (a\wedge b)\cdot(b'\wedge b)\\&\quad + (a\wedge b)\cdot(b'\wedge a') + (a\wedge a')\cdot(b'\wedge b) + (a\wedge a')\cdot(b'\wedge a').\end{aligned}$$

There are similar formulas involving

$$\begin{aligned}&\big(a'\wedge(a+b')\big)\cdot\big((b+a')\wedge(a+b')\big), \text{ and}\\&\big((a+b')\wedge b\big)\cdot\big((a+b')\wedge(b+a')\big), \text{ and}\\&\big((b+a')\wedge b'\big)\cdot\big((b+a')\wedge(a+b')\big).\end{aligned}$$

This suggests four possible elements of $\mathfrak{K}_g^s$ projecting to $(a\wedge a')\cdot(b'\wedge b)$:

**Definition 30.1.** For symplectic pairs $a \wedge b$ and $a' \wedge b'$ in $H_{\mathbb{Z}}$ with $\langle a, b\rangle$ and $\langle a', b'\rangle$ orthogonal, define

$$\begin{aligned}\Omega_1[a \wedge a', b' \wedge b]_s =& \Theta[a \wedge (b+a'), (a+b') \wedge (b+a')]_s - [\![a \wedge (b+a'), a \wedge (b+a')]\!]_s \\ &- \Theta[a \wedge b, b' \wedge b]_s - [\![a \wedge b, b' \wedge a']\!]_s - \Theta[a \wedge a', b' \wedge a']_s, \\ \Omega_2[a \wedge a', b' \wedge b]_s =& \Theta[a' \wedge (a+b'), (b+a') \wedge (a+b')]_s - [\![a' \wedge (a+b'), a' \wedge (a+b')]\!]_s \\ &- \Theta[a' \wedge b', b \wedge b']_s - [\![a' \wedge b', b \wedge a]\!]_s - \Theta[a' \wedge a, b \wedge a]_s, \\ \Omega_3[a \wedge a', b' \wedge b]_s =& \Theta[(a+b') \wedge b, (a+b') \wedge (b+a')]_s - [\![(a+b') \wedge b, (a+b') \wedge b]\!]_s \\ &- \Theta[a \wedge b, a \wedge a']_s - [\![a \wedge b, b' \wedge a']\!]_s - \Theta[b' \wedge b, b' \wedge a']_s, \\ \Omega_4[a \wedge a', b' \wedge b]_s =& \Theta[(b+a') \wedge b', (b+a') \wedge (a+b')]_s - [\![(b+a') \wedge b', (b+a') \wedge b']\!]_s \\ &- \Theta[a' \wedge b', a' \wedge a]_s - [\![a' \wedge b', b \wedge a]\!]_s - \Theta[b \wedge b', b \wedge a]_s. \end{aligned}$$ □

See Remark 30.3 below for a caveat about this notation. By construction, we have

$$\Phi(\Omega_i[a \wedge a', b' \wedge b]_s) = (a \wedge a') \cdot (b' \wedge b) \quad \text{for } 1 \le i \le 4.$$

Since we are trying to prove that $\Phi$ is an isomorphism, we must prove that these are actually the same element:

**Lemma 30.2.** *Let $a \wedge b$ and $a' \wedge b'$ be symplectic pairs in $H_{\mathbb{Z}}$ such that $\langle a, b\rangle$ and $\langle a', b'\rangle$ are orthogonal. Then the $\Omega_i[a \wedge a', b' \wedge b]_s$ for $1 \le i \le 4$ are all equal.*

*Proof.* Whether or not they are equal is invariant under the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$. Recall that we have our fixed symplectic basis $\mathcal{B} = \{a_1, b_1, \ldots, a_g, b_g\}$ for $H_{\mathbb{Z}}$. Applying an appropriate element of $\mathrm{Sp}_{2g}(\mathbb{Z})$, we can assume that

$$a_1 = a, \ \ b_1 = b, \ \ a_2 = a', \ \ b_2 = b'.$$

Since $\Phi$ takes the $\Omega_i[a_1 \wedge a_2, b_2 \wedge b_1]_s$ to the same thing and the restriction of $\Phi$ to $\langle S_1, S_2, S_3\rangle$ is injective (Lemma 29.9), it is enough to prove that the $\Omega_i[a_1 \wedge a_2, b_2 \wedge b_1]_s$ are equal modulo $\langle S_1, S_2, S_3\rangle$. Let $\equiv$ denote equality modulo $\langle S_1, S_2, S_3\rangle$. We have

$$\begin{aligned}\Omega_1[a_1 \wedge a_2, b_2 \wedge b_1]_s =& \Theta[a_1 \wedge (b_1+a_2), (a_1+b_2) \wedge (b_1+a_2)]_s - [\![a_1 \wedge (b_1+a_2), a_1 \wedge (b_1+a_2)]\!]_s \\ &- \Theta[a_1 \wedge b_1, b_2 \wedge b_1]_s - [\![a_1 \wedge b_1, b_2 \wedge a_2]\!]_s - \Theta[a_1 \wedge a_2, b_2 \wedge a_2]_s \\ \equiv& \Theta[a_1 \wedge (b_1+a_2), (a_1+b_2) \wedge (b_1+a_2)]_s \\ &- ([\![a_1 \wedge b_1, a_1 \wedge b_1]\!]_s + 2\Theta[a_1 \wedge b_1, a_1 \wedge a_2]_s + [\![a_1 \wedge a_2, a_1 \wedge a_2]\!]_s) \\ \equiv& \Theta[a_1 \wedge (b_1+a_2), (a_1+b_2) \wedge (b_1+a_2)]_s.\end{aligned}$$

Similarly,

$$\begin{aligned}\Omega_2[a_1 \wedge a_2, b_2 \wedge b_1]_s &\equiv \Theta[a_2 \wedge (a_1+b_2), (b_1+a_2) \wedge (a_1+b_2)]_s, \\ \Omega_3[a_1 \wedge a_2, b_2 \wedge b_1]_s &\equiv \Theta[(a_1+b_2) \wedge b_1, (a_1+b_2) \wedge (b_1+a_2)]_s, \\ \Omega_4[a_1 \wedge a_2, b_2 \wedge b_1]_s &\equiv \Theta[(b_1+a_2) \wedge b_2, (b_1+a_2) \wedge (a_1+b_2)]_s.\end{aligned}$$

It is therefore enough to prove the following two claims:

**Claim 1.** *We have $\Theta[a_1 \wedge (b_1+a_2), (a_1+b_2) \wedge (b_1+a_2)]_s \equiv \Theta[a_2 \wedge (a_1+b_2), (b_1+a_2) \wedge (a_1+b_2)]_s$ and $\Theta[(a_1+b_2) \wedge b_1, (a_1+b_2) \wedge (b_1+a_2)]_s \equiv \Theta[(b_1+a_2) \wedge b_2, (b_1+a_2) \wedge (a_1+b_2)]_s$.*

Both equalities are proved the same way, so we will give details for the first. Consider the symplectic basis $\{a_1, b_1+a_2, a_2, a_1+b_2, a_3, b_3, \ldots, a_g, b_g\}$ for $H_{\mathbb{Z}}$. Lemma 28.7 ($\Theta$-symplectic basis) implies that

$$\begin{aligned}0 =& \Theta[a_1 \wedge (b_1+a_2), (a_1+b_2) \wedge (b_1+a_2)]_s + \Theta[a_2 \wedge (a_1+b_2), (a_1+b_2) \wedge (b_1+a_2)]_s \\ &+ \sum\nolimits_{i=3}^{g} \Theta[a_i \wedge b_i, (a_1+b_2) \wedge (b_1+a_2)]_s.\end{aligned}$$

We conclude that

$$\begin{aligned}\Theta[a_1 \wedge (b_1+a_2), (a_1+b_2) \wedge (b_1+a_2)]_s &\equiv -\Theta[a_2 \wedge (a_1+b_2), (a_1+b_2) \wedge (b_1+a_2)]_s \\ &= \Theta[a_2 \wedge (a_1+b_2), (b_1+a_2) \wedge (a_1+b_2)]_s.\end{aligned}$$

**Claim 2.** *We have* $\Theta[a_1 \wedge (b_1+a_2), (a_1+b_2) \wedge (b_1+a_2)]_s \equiv \Theta[(b_1+a_2) \wedge b_2, (b_1+a_2) \wedge (a_1+b_2)]_s$.

Using Lemma 28.4 ($\Theta$-linearity), we have

$$-\Theta[a_1 \wedge (b_1+a_2), (a_1+b_2) \wedge (b_1+a_2)]_s = \Theta[a_1 \wedge (b_1+a_2), (-a_1-b_2) \wedge (b_1+a_2)]_s.$$

By definition (Definition 28.1), this equals

$$\begin{aligned}&\frac{1}{2}(\llbracket(-b_2) \wedge (b_1+a_2), (-b_2) \wedge (b_1+a_2)\rrbracket_s - \llbracket a_1 \wedge (b_1+a_2), a_1 \wedge (b_1+a_2)\rrbracket_s \\ &\quad - \llbracket(a_1+b_2) \wedge (b_1+a_2), (a_1+b_2) \wedge (b_1+a_2)\rrbracket_s).\end{aligned}$$

The term $\llbracket(-b_2) \wedge (b_1+a_2), (-b_2) \wedge (b_1+a_2)\rrbracket_s$ equals

$$\llbracket b_2 \wedge b_1, b_2 \wedge b_1\rrbracket_s + 2\Theta[b_2 \wedge b_1, b_2 \wedge a_2]_s + \llbracket b_2 \wedge a_2, b_2 \wedge a_2\rrbracket_s \equiv 0$$

and the term $\llbracket a_1 \wedge (b_1+a_2), a_1 \wedge (b_1+a_2)\rrbracket_s$ equals

$$\llbracket a_1 \wedge b_1, a_1 \wedge b_1\rrbracket_s + 2\Theta[a_1 \wedge b_1, a_1 \wedge a_2]_s + \llbracket a_1 \wedge a_2, a_1 \wedge a_2\rrbracket_s \equiv 0.$$

We deduce that

$$\Theta[a_1 \wedge (b_1+a_2), (a_1+b_2) \wedge (b_1+a_2)]_s \equiv \frac{1}{2}\llbracket(a_1+b_2) \wedge (b_1+a_2), (a_1+b_2) \wedge (b_1+a_2)\rrbracket_s.$$

Lemma 28.9 ($\Theta$-expansion II) implies that

$$\begin{aligned}\llbracket(a_1+b_2) \wedge (b_1+a_2), (a_1+b_2) \wedge (b_1+a_2)\rrbracket_s =&\Theta[a_1 \wedge (b_1+a_2), (a_1+b_2) \wedge (b_1+a_2)]_s \\ &+ \Theta[b_2 \wedge (b_1+a_2), (a_1+b_2) \wedge (b_1+a_2)]_s.\end{aligned}$$

Combining this with our previous formula, we conclude that

$$\begin{aligned}\Theta[a_1 \wedge (b_1+a_2), (a_1+b_2) \wedge (b_1+a_2)]_s \equiv&\Theta[b_2 \wedge (b_1+a_2), (a_1+b_2) \wedge (b_1+a_2)]_s \\ =&\Theta[(b_1+a_2) \wedge b_2, (b_1+a_2) \wedge (a_1+b_2)]_s. \quad \square\end{aligned}$$

In light of this lemma, we will denote the common value of $\Omega_i[a \wedge a', b' \wedge b]_s$ by $\Omega[a \wedge a', b' \wedge b]_s$.

*Remark* 30.3. Just like for the $\Theta$- and $\Lambda$-elements (cf. Remarks 28.2 and 29.3), the elements $\Omega_i[a \wedge a', b' \wedge b]_s$ and $\Omega[a \wedge a', b' \wedge b]_s$ depend on the ordered tuple $(a, a', b', b)$, not on $a \wedge a'$ and $b \wedge b'$. $\square$

30.2. **Relation to $\Lambda$-elements.** The following lemma shows that the $\Omega$-elements can almost (but not quite) be written in terms of the $\Lambda$-elements:

**Lemma 30.4** ($\Lambda$ to $\Omega$)**.** *Let $a \wedge b$ and $a' \wedge b'$ and $a'' \wedge b''$ be symplectic pairs in $H_{\mathbb{Z}}$ such that $\langle a, b\rangle$ and $\langle a', b'\rangle$ and $\langle a'', b''\rangle$ are all orthogonal. Then*

$$\Lambda[a \wedge (a'+a''), (b'-b'') \wedge b]_s - \Lambda[a \wedge a'', b' \wedge b]_s + \Lambda[a \wedge a', b'' \wedge b]_s$$

*equals* $\Omega[a \wedge a', b' \wedge b]_s - \Omega[a \wedge a'', b'' \wedge b]_s$.

*Proof.* Whether or not they are equal is invariant under the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$. Recall that we have our fixed symplectic basis $\mathcal{B} = \{a_1, b_1, \ldots, a_g, b_g\}$ for $H_{\mathbb{Z}}$. Applying an appropriate element of $\mathrm{Sp}_{2g}(\mathbb{Z})$, we can assume that

$$a_1 = a, \;\; b_1 = b, \;\; a_2 = a', \;\; b_2 = b', \;\; a_3 = a'', \;\; b_3 = b''.$$

Since $\Phi$ takes our two elements to the same thing and the restriction of $\Phi$ to $\langle S_1, S_2, S_3\rangle$ is injective (Lemma 29.9), it is enough to prove that they are equal modulo $\langle S_1, S_2, S_3\rangle$. Let $\equiv$ denote equality modulo $\langle S_1, S_2, S_3\rangle$. The element

$$\Lambda[a_1 \wedge (a_2 + a_3), (b_2 - b_3) \wedge b_1]_s - \Lambda[a_1 \wedge a_2, b_2 \wedge b_1]_s + \Lambda[a_1 \wedge a_3, b_3 \wedge b_1]_s$$

is equivalent to $\Lambda[a_1 \wedge (a_2 + a_3), (b_2 - b_3) \wedge b_1]_s$, and we must prove that this is equivalent to $\Omega[a_1 \wedge a_2, b_2 \wedge b_1]_s - \Omega[a_1 \wedge a_3, b_3 \wedge b_1]_s$. Below we will prove the following three facts:

$$\begin{aligned}
\Lambda[a_1 \wedge (a_2 + a_3), (b_2 - b_3) \wedge b_1]_s &\equiv \Theta[a_1 \wedge (b_1 + a_2 + a_3), (b_2 - b_3) \wedge (b_1 + a_2 + a_3)]_s, \\
\Omega[a_1 \wedge a_2, b_2 \wedge b_1]_s &\equiv \Theta[a_1 \wedge (b_1 + a_2 + a_3), (a_1 + b_2) \wedge (b_1 + a_2 + a_3)]_s, \\
\Omega[a_1 \wedge a_3, b_3 \wedge b_1]_s &\equiv \Theta[a_1 \wedge (b_1 + a_2 + a_3), (a_1 + b_3) \wedge (b_1 + a_2 + a_3)]_s.
\end{aligned}$$

These will imply the lemma; indeed, the linearity of $\Theta$-elements will imply that

$$\begin{aligned}
&\Omega[a_1 \wedge a_2, b_2 \wedge b_1]_s - \Omega[a_1 \wedge a_3, b_3 \wedge b_1]_s \\
\equiv&\Theta[a_1 \wedge (b_1 + a_2 + a_3), (a_1 + b_2) \wedge (b_1 + a_2 + a_3)]_s \\
&\quad - \Theta[a_1 \wedge (b_1 + a_2 + a_3), (a_1 + b_3) \wedge (b_1 + a_2 + a_3)]_s \\
=&\Theta[a_1 \wedge (b_1 + a_2 + a_3), (b_2 - b_3) \wedge (b_1 + a_2 + a_3)]_s \\
\equiv&\Lambda[a_1 \wedge (a_2 + a_3), (b_2 - b_3) \wedge b_1]_s.
\end{aligned}$$

It remains to prove the above three facts:

**Claim 1.** *We have*

$$\Lambda[a_1 \wedge (a_2 + a_3), (b_2 - b_3) \wedge b_1]_s \equiv \Theta[a_1 \wedge (b_1 + a_2 + a_3), (b_2 - b_3) \wedge (b_1 + a_2 + a_3)]_s.$$

By definition, $\Lambda[a_1 \wedge (a_2 + a_3), (b_2 - b_3) \wedge b_1]_s = \Lambda_1[a_1 \wedge (a_2 + a_3), (b_2 - b_3) \wedge b_1]_s$ equals

$$\begin{aligned}
&\Theta[a_1 \wedge (b_1 + a_2 + a_3), (b_2 - b_3) \wedge (b_1 + a_2 + a_3)]_s \\
&- \Theta[a_1 \wedge b_1, (b_2 - b_3) \wedge b_1]_s - [\![a_1 \wedge (b_1 + a_2 + a_3), (b_2 - b_3) \wedge (a_2 + a_3)]\!]_s \\
\equiv&\Theta[a_1 \wedge (b_1 + a_2 + a_3), (b_2 - b_3) \wedge (b_1 + a_2 + a_3)]_s \\
&- [\![a_1 \wedge b_1, (b_2 - b_3) \wedge (a_2 + a_3)]\!]_s - [\![a_1 \wedge (a_2 + a_3), (b_2 - b_3) \wedge (a_2 + a_3)]\!]_s \\
\equiv&\Theta[a_1 \wedge (b_1 + a_2 + a_3), (b_2 - b_3) \wedge (b_1 + a_2 + a_3)]_s \\
&- [\![(a_2 + a_3) \wedge (b_2 - b_3), (a_2 + a_3) \wedge a_1]\!]_s
\end{aligned}$$

To prove the claim, we must show that $[\![(a_2 + a_3) \wedge (b_2 - b_3), (a_2 + a_3) \wedge a_1]\!]_s$ is equivalent to 0. By Lemma 28.9 ($\Theta$-expansion II), this element equals

$$\Theta[(a_2 + a_3) \wedge b_2, (a_2 + a_3) \wedge a_1]_s - \Theta[(a_2 + a_3) \wedge b_3, (a_2 + a_3) \wedge a_1]_s.$$

By Lemma 29.4 ($\Lambda$-expansion I), the element $\Theta[(a_2 + a_3) \wedge b_2, (a_2 + a_3) \wedge a_1]_s$ equals

$$\Lambda[a_2 \wedge a_1, a_3 \wedge b_2]_s + \Theta[a_2 \wedge b_2, a_2 \wedge a_1]_s + [\![(a_2 + a_3) \wedge b_2, a_3 \wedge a_1]\!]_s \equiv 0.$$

Similarly, $\Theta[(a_2 + a_3) \wedge b_3, (a_2 + a_3) \wedge a_1]_s \equiv 0$. The claim follows.

**Claim 2.** *We have*

$$\begin{aligned}
\Omega[a_1 \wedge a_2, b_2 \wedge b_1]_s &\equiv \Theta[a_1 \wedge (b_1 + a_2 + a_3), (a_1 + b_2) \wedge (b_1 + a_2 + a_3)]_s, \\
\Omega[a_1 \wedge a_3, b_3 \wedge b_1]_s &\equiv \Theta[a_1 \wedge (b_1 + a_2 + a_3), (a_1 + b_3) \wedge (b_1 + a_2 + a_3)]_s.
\end{aligned}$$

Both equalities are proved in the same way, so we will give the details for the first. Consider the symplectic basis

$$\{a_1, b_1 + a_2 + a_3, a_2, a_1 + b_2, a_3, a_1 + b_3, a_4, b_4, \ldots, a_g, b_g\}$$

for $H_{\mathbb{Z}}$. Applying Lemma 28.7 ($\Theta$-symplectic basis), we deduce that

$$\Theta[a_1 \wedge (b_1 + a_2 + a_3), (a_1 + b_2) \wedge (b_1 + a_2 + a_3)]_s - \Theta[a_2 \wedge (a_1 + b_2), (b_1 + a_2 + a_3) \wedge (a_1 + b_2)]_s$$
$$+ [\![a_3 \wedge (a_1 + b_3), (a_1 + b_2) \wedge (b_1 + a_2 + a_3)]\!]_s + \sum\nolimits_{i=4}^{g} [\![a_i \wedge b_i, (a_1 + b_2) \wedge (b_1 + a_2 + a_3)]\!]_s$$

equals 0. Throwing away terms that are equivalent modulo $\langle S_1, S_2, S_3 \rangle$ to 0, we deduce that $\Theta[a_1 \wedge (b_1 + a_2 + a_3), (a_1 + b_2) \wedge (b_1 + a_2 + a_3)]_s$ is equivalent to

$$\begin{aligned} &\Theta[a_2 \wedge (a_1 + b_2), (b_1 + a_2 + a_3) \wedge (a_1 + b_2)]_s - [\![a_3 \wedge (a_1 + b_3), (a_1 + b_2) \wedge (b_1 + a_2 + a_3)]\!]_s \\ =&\Theta[a_2 \wedge (a_1 + b_2), (b_1 + a_2) \wedge (a_1 + b_2)]_s + \Theta[a_2 \wedge (a_1 + b_2), a_3 \wedge (a_1 + b_2)]_s \\ &- [\![a_3 \wedge (a_1 + b_3), (a_1 + b_2) \wedge (b_1 + a_2 + a_3)]\!]_s. \end{aligned}$$

Just like at the beginning of the proof of Lemma 30.2, we have

$$\Theta[a_2 \wedge (a_1 + b_2), (b_1 + a_2) \wedge (a_1 + b_2)]_s \equiv \Omega_2[a_1 \wedge a_2, b_2 \wedge b_1]_s = \Omega[a_1 \wedge a_2, b_2 \wedge b_1]_s.$$

To prove the claim, we must therefore prove that the other two terms in the above sum are equivalent to 0.

For $\Theta[a_2 \wedge (a_1 + b_2), a_3 \wedge (a_1 + b_2)]_s$, it follows from Lemma 29.4 ($\Lambda$-expansion I) that it equals

$$\Lambda[a_2 \wedge a_1, a_3 \wedge b_2]_s + \Theta[a_2 \wedge b_2, a_3 \wedge b_2]_s + [\![a_2 \wedge (b_2 + a_1), a_3 \wedge a_1]\!]_s \equiv 0.$$

For $[\![a_3 \wedge (a_1 + b_3), (a_1 + b_2) \wedge (b_1 + a_2 + a_3)]\!]_s$, it equals

$$\begin{aligned} &[\![a_3 \wedge (a_1 + b_3), (a_1 + b_2) \wedge a_2]\!]_s + [\![a_3 \wedge (a_1 + b_3), (a_1 + b_2) \wedge (b_1 + a_3)]\!]_s \\ \equiv&[\![a_3 \wedge (a_1 + b_3), a_1 \wedge (b_1 + a_3)]\!]_s + [\![a_3 \wedge (a_1 + b_3), b_2 \wedge (b_1 + a_3)]\!]_s. \end{aligned}$$

We must show that both of these terms vanish. For $[\![a_3 \wedge (a_1 + b_3), a_1 \wedge (b_1 + a_3)]\!]_s$, in $(\wedge^2 H)/\mathbb{Q}$ we have

$$a_3 \wedge (a_1 + b_3) + a_1 \wedge (b_1 + a_3) + \sum_{\substack{2 \leq i \leq g \\ i \neq 3}} a_i \wedge b_i = 0.$$

This implies that $[\![a_3 \wedge (a_1 + b_3), a_1 \wedge (b_1 + a_3)]\!]_s$ equals

$$\begin{aligned} &- [\![a_1 \wedge (b_1 + a_3), a_1 \wedge (b_1 + a_3)]\!]_s - \sum_{\substack{2 \leq i \leq g \\ i \neq 3}} [\![a_i \wedge b_i, a_1 \wedge (b_1 + a_3)]\!]_s \\ \equiv& - [\![a_1 \wedge b_1, a_1 \wedge b_1]\!]_s - 2\Theta[a_1 \wedge b_1, a_1 \wedge a_3]_s - [\![a_1 \wedge a_3, a_1 \wedge a_3]\!]_s \equiv 0. \end{aligned}$$

For $[\![a_3 \wedge (a_1 + b_3), b_2 \wedge (b_1 + a_3)]\!]_s$, we use the same symplectic basis

$$\{a_3, a_1 + b_3, a_1, b_1 + a_3, a_2, b_2, a_4, b_4, \ldots, a_g, b_g\}$$

for $H_{\mathbb{Z}}$, but this time we use Lemma 28.7 ($\Theta$-symplectic basis) to see that

$$\begin{aligned} 0 =&[\![a_3 \wedge (a_1 + b_3), b_2 \wedge (b_1 + a_3)]\!]_s + \Theta[a_1 \wedge (b_1 + a_3), b_2 \wedge (b_1 + a_3)]_s \\ &- \Theta[a_2 \wedge b_2, (b_1 + a_3) \wedge b_2]_s + \sum_{\substack{2 \leq i \leq g \\ i \neq 3}} [\![a_i \wedge b_i, b_2 \wedge (b_1 + a_3)]\!]_s. \end{aligned}$$

Throwing away terms that are equivalent to 0, we deduce that

$$[\![a_3 \wedge (a_1 + b_3), b_2 \wedge (b_1 + a_3)]\!]_s \equiv -\Theta[a_1 \wedge (b_1 + a_3), b_2 \wedge (b_1 + a_3)]_s.$$

By Lemma 29.4 ($\Lambda$-expansion I), this equals $-1$ times

$$\Lambda[a_1 \wedge a_3, b_2 \wedge b_1]_s + \Theta[a_1 \wedge b_1, b_2 \wedge b_1]_s + [\![a_1 \wedge (b_1 + a_3), b_2 \wedge a_3]\!]_s \equiv 0,$$

as desired. $\square$

30.3. **$\Omega$-symmetry and signs.** It is inconvenient to require the entries of $\Omega[a \wedge a', b' \wedge b]_s$ to appear in a definite order. We therefore would like to define that each of the following terms equals $\Omega[a \wedge a', b' \wedge b]_s$:

$$\Omega[a \wedge a', b' \wedge b]_s, \ -\Omega[a' \wedge a, b' \wedge b]_s, \ -\Omega[a \wedge a', b \wedge b']_s, \ \Omega[a' \wedge a, b \wedge b']_s,$$
$$\Omega[b' \wedge b, a \wedge a']_s, \ -\Omega[b' \wedge b, a' \wedge a]_s, \ -\Omega[b \wedge b', a \wedge a']_s, \ \Omega[b \wedge b', a' \wedge a]_s.$$

Similarly, we would like to by able to multiply terms by $-1$ in the usual way and define that each of the following terms equals $\Omega[a \wedge a', b' \wedge b]_s$:

$$\begin{array}{ll}
\Omega[a \wedge a', b' \wedge b]_s, & -\Omega[(-a) \wedge a', b' \wedge b]_s, \\
-\Omega[a \wedge (-a'), b' \wedge b]_s, & \Omega[(-a) \wedge (-a'), b' \wedge b]_s, \\
-\Omega[a \wedge a', (-b') \wedge b]_s, & \Omega[(-a) \wedge a', (-b') \wedge b]_s, \\
\Omega[a \wedge (-a'), (-b') \wedge b]_s, & -\Omega[(-a) \wedge (-a'), (-b') \wedge b]_s, \\
-\Omega[a \wedge a', b' \wedge (-b)]_s, & \Omega[(-a) \wedge a', b' \wedge (-b)]_s, \\
\Omega[a \wedge (-a'), b' \wedge (-b)]_s, & -\Omega[(-a) \wedge (-a'), b' \wedge (-b)]_s, \\
\Omega[a \wedge a', (-b') \wedge (-b)]_s, & -\Omega[(-a) \wedge a', (-b') \wedge (-b)]_s, \\
-\Omega[a \wedge (-a'), (-b') \wedge (-b)]_s, & \Omega[(-a) \wedge (-a'), (-b') \wedge (-b)]_s.
\end{array}$$

The problem is that these definitions introduce ambiguity into our notation since some of these are other $\Omega$-elements. For instance, $\Omega[a \wedge (-a'), (-b') \wedge b]_s$ is another $\Omega$-element associated to the symplectic pairs $a \wedge b$ and $(-a') \wedge (-b')$. The following lemma says that all the possible other $\Omega$-elements obtained in this way are actually the same:

**Lemma 30.5.** *Let $a \wedge b$ and $a' \wedge b'$ be symplectic pairs in $H_{\mathbb{Z}}$ such that $\langle a, b\rangle$ and $\langle a', b'\rangle$ are orthogonal. Then the following are all equal to $\Omega[a \wedge a', b' \wedge b]_s$:*

$$\begin{array}{lll}
\Omega[(-a) \wedge a', b' \wedge (-b)]_s & \Omega[a' \wedge a, b \wedge b']_s & \Omega[(-b) \wedge (-b'), a' \wedge a]_s \\
\Omega[a \wedge (-a'), (-b') \wedge b]_s & \Omega[(-a') \wedge a, b \wedge (-b')]_s & \Omega[b \wedge (-b'), a' \wedge (-a)]_s \\
\Omega[(-a) \wedge (-a'), (-b') \wedge (-b)]_s & \Omega[a' \wedge (-a), (-b) \wedge b']_s & \Omega[(-b) \wedge b', (-a') \wedge a]_s \\
\Omega[(-b') \wedge (-b), a \wedge a']_s & \Omega[(-a') \wedge (-a), (-b) \wedge (-b')]_s & \Omega[b \wedge b', (-a') \wedge (-a)]_s \\
\Omega[b' \wedge (-b), a \wedge (-a')]_s & \Omega[(-b') \wedge b, (-a) \wedge a']_s & \Omega[b' \wedge b, (-a) \wedge (-a')]_s
\end{array}$$

*Proof.* Below we will prove the following three special cases:

(i) $\Omega[a \wedge a', b' \wedge b]_s = \Omega[a' \wedge a, b \wedge b']_s$
(ii) $\Omega[a \wedge a', b' \wedge b]_s = \Omega[a \wedge (-a'), (-b') \wedge b]_s$
(iiii) $\Omega[a \wedge a', b' \wedge b]_s = \Omega[(-b') \wedge (-b), a \wedge a']_s$

As is easily verified, all the other equalities we are trying to prove can be obtained by composing these three. For instance, to see that $\Omega[a \wedge a', b' \wedge b]_s = \Omega[(-a) \wedge a', b' \wedge (-b)]_s$ we compose (i) and (ii) and (i):

$$\Omega[a \wedge a', b' \wedge b]_s = \Omega[a' \wedge a, b \wedge b']_s = \Omega[a' \wedge (-a), (-b) \wedge b']_s = \Omega[(-a) \wedge a', b' \wedge (-b)]_s.$$

We separate the proofs of (i) and (ii) and (iii) into the following three claims:

**Claim 1.** $\Omega[a \wedge a', b' \wedge b]_s = \Omega[a' \wedge a, b \wedge b']_s$.

We can calculate these using any of the $\Omega_i$-formulas from Definition 30.1, so the claim follows from the fact that the following are equal: $\Omega_1[a \wedge a', b' \wedge b]_s$, which is

$$\Theta[a \wedge (b + a'), (a + b') \wedge (b + a')]_s - [\![a \wedge (b + a'), a \wedge (b + a')]\!]_s$$
$$- \Theta[a \wedge b, b' \wedge b]_s - [\![a \wedge b, b' \wedge a']\!]_s - \Theta[a \wedge a', b' \wedge a']_s.$$

and $\Omega_2[a' \wedge a, b \wedge b']_s$, which is

$$\begin{aligned}&\Theta[a \wedge (a'+b), (b'+a) \wedge (a'+b)]_s - [\![a \wedge (a'+b), a \wedge (a'+b)]\!]_s \\ &- \Theta[a \wedge b, b' \wedge b]_s - [\![a \wedge b, b' \wedge a']\!]_s - \Theta[a \wedge a', b' \wedge a']_s\end{aligned}$$

**Claim 2.** $\Omega[a \wedge a', b' \wedge b]_s = \Omega[a \wedge (-a'), (-b') \wedge b]_s$.

Since $g \geq 4$ (Assumption 25.1), we can find a symplectic pair $a'' \wedge b''$ such that $\langle a'', b'' \rangle$ is orthogonal to both $\langle a, b \rangle$ and $\langle a', b' \rangle$. Lemma 30.4 ($\Lambda$ to $\Omega$) says that

$$\Lambda[a \wedge (a'+a''), (b'-b'') \wedge b]_s - \Lambda[a \wedge a'', b' \wedge b]_s + \Lambda[a \wedge a', b'' \wedge b]_s \tag{30.1}$$

equals $\Omega[a \wedge a', b' \wedge b]_s - \Omega[a \wedge a'', b'' \wedge b]_s$ and that

$$\Lambda[a \wedge (-a'+a''), (-b'-b'') \wedge b]_s - \Lambda[a \wedge a'', (-b') \wedge b]_s + \Lambda[a \wedge (-a'), b'' \wedge b]_s \tag{30.2}$$

equals $\Omega[a \wedge (-a'), (-b') \wedge b]_s - \Omega[a \wedge a'', b'' \wedge b]_s$. To prove the claim, it is thus enough to prove that (30.1) equals (30.2). By Lemma 29.6 (strong $\Lambda$-linearity), this is equivalent to the following identity in $(\langle a, b \rangle^{\perp})^{\otimes 2}$:

$$(a'+a'') \otimes (b'-b'') - a'' \otimes b' + a' \otimes b'' = (-a'+a'') \otimes (-b'-b'') - a'' \otimes (-b') + (-a') \otimes b''.$$

In fact, both sides of this equal $a' \otimes b' - a'' \otimes b''$.

**Claim 3.** $\Omega[a \wedge a', b' \wedge b]_s = \Omega[(-b') \wedge (-b), a \wedge a']_s$.

By Claim 2, it is enough to prove instead that $\Omega[a \wedge (-a'), (-b') \wedge b]_s = \Omega[(-b') \wedge (-b), a \wedge a']_s$. We can calculate these using any of the $\Omega_i$-formulas from Definition 30.1, so the claim follows from the fact that the following are equal: $\Omega_1[a \wedge (-a'), (-b') \wedge b]_s$, which is

$$\begin{aligned}&\Theta[a \wedge (b-a'), (a-b') \wedge (b-a')]_s - [\![a \wedge (b-a'), a \wedge (b-a')]\!]_s \\ &- \Theta[a \wedge b, (-b') \wedge b]_s - [\![a \wedge b, (-b') \wedge (-a')]\!]_s - \Theta[a \wedge (-a'), (-b') \wedge (-a')]_s,\end{aligned}$$

and $\Omega_4[(-b') \wedge (-b), a \wedge a']_s$, which is

$$\begin{aligned}&\Theta[(a'-b) \wedge a, (a'-b) \wedge (-b'+a)]_s - [\![(a'-b) \wedge a, (a'-b) \wedge a]\!]_s \\ &- \Theta[(-b) \wedge a, (-b) \wedge (-b')]_s - [\![(-b) \wedge a, a' \wedge (-b')]\!]_s - \Theta[a' \wedge a, a' \wedge (-b')]_s. \qquad \square\end{aligned}$$

In light of this lemma, we can permute terms and multiply them by $-1$ as described before the lemma. For instance, if $a \wedge b$ and $a' \wedge b'$ are symplectic pairs such that $\langle a, b \rangle$ and $\langle a', b' \rangle$ are orthogonal, then we can write things in the natural order and discuss $\Omega[a \wedge a', b \wedge b']_s$, which equals $-\Omega[a \wedge a', b' \wedge b]_s$. We only used the unnatural order to avoid signs in our formulas. We can also now talk about $\Omega[a \wedge b', b \wedge a']_s$, which equals $\Omega[a \wedge (-b'), a' \wedge b]_s$ or $\Omega[a \wedge b', (-a') \wedge b]_s$.

30.4. **The set $S_4$.** We now return to constructing $S_4$. Recall that

$$T_4 = \{(a_i \wedge a_j) \cdot (b_i \wedge b_j),\ (a_i \wedge b_j) \cdot (b_i \wedge a_j) \mid 1 \leq i < j \leq g\}.$$

Define

$$S_4 = \{\Omega[a_i \wedge a_j, b_i \wedge b_j]_s,\ \Omega[a_i \wedge b_j, b_i \wedge a_j]_s \mid 1 \leq i < j \leq g\}.$$

Like we did here, we will write elements of $\langle S_4 \rangle$ in green. By construction, the linearization map $\Phi$ takes $S_4$ bijectively to $T_4$. Even better:

**Lemma 30.6.** *The linearization map $\Phi$ takes $\langle S_1, \ldots, S_4 \rangle$ isomorphically to* $\mathrm{Sym}^2((\wedge^2 H)/\mathbb{Q})$.

*Proof.* Recall that in Lemma 29.9 we proved that $\Phi$ takes $\langle S_1, S_2, S_3\rangle$ isomorphically to $\langle T_1, T_2, T_3\rangle$. Part of the proof of that lemma was that $\Phi$ takes $S_1 \cup S_2 \cup S_3$ bijectively to $T_1 \cup T_2 \cup T_3$. It follows that $\Phi$ takes $S_1 \cup \cdots \cup S_4$ bijectively to $T_1 \cup \cdots \cup T_4$. As we discussed in §27, the set $T_1 \cup \cdots \cup T_4$ generates $\mathrm{Sym}^2((\wedge^2 H)/\mathbb{Q})$. In the proof of Lemma 29.9 we proved that all relations between elements of $T_1 \cup \cdots \cup T_4$ are actually relations between elements of $T_1 \cup T_2$ and lift to relations between elements of $S_1 \cup S_2$. The lemma follows. □

30.5. **Additional relations.** For later use, we record the following. Its statement uses our fixed symplectic basis $\mathcal{B} = \{a_1, b_1, \ldots, a_g, b_g\}$.

**Lemma 30.7.** *Let $\equiv$ denote equality modulo $\langle S_1, S_2, S_3\rangle$. For distinct $1 \leq i, j \leq g$, we have*

$$\begin{aligned}\llbracket (a_i + b_j) \wedge (b_i + a_j), (a_i + b_j) \wedge (b_i + a_j)\rrbracket_s &\equiv \llbracket (a_i - b_j) \wedge (b_i - a_j), (a_i - b_j) \wedge (b_i - a_j)\rrbracket_s \\ &\equiv -2\Omega[a_i \wedge a_j, b_i \wedge b_j]_s\end{aligned}$$

*and*

$$\begin{aligned}\llbracket (a_i + a_j) \wedge (b_i - b_j), (a_i + a_j) \wedge (b_i - b_j)\rrbracket_s &\equiv \llbracket (a_i - a_j) \wedge (b_i + b_j), (a_i - a_j) \wedge (b_i + b_j)\rrbracket_s \\ &\equiv 2\Omega[a_i \wedge b_j, b_i \wedge a_j]_s.\end{aligned}$$

*Proof.* Both equalities are proved the same way, so we will give the details for the first. Lemma 28.9 ($\Theta$-expansion II) implies that $\llbracket (a_i + b_j) \wedge (b_i + a_j), (a_i + b_j) \wedge (b_i + a_j)\rrbracket_s$ equals

$$\Theta[a_i \wedge (b_i + a_j), (a_i + b_j) \wedge (b_i + a_j)]_s + \Theta[b_j \wedge (b_i + a_j), (a_i + b_j) \wedge (b_i + a_j)]_s \tag{30.3}$$

and $\llbracket (a_i - b_j) \wedge (b_i - a_j), (a_i - b_j) \wedge (b_i - a_j)\rrbracket_s$ equals

$$\Theta[a_i \wedge (b_i - a_j), (a_i - b_j) \wedge (b_i - a_j)]_s + \llbracket (-b_j) \wedge (b_i - a_j), (a_i - b_j) \wedge (b_i - a_j)\rrbracket_s. \tag{30.4}$$

It is enough to prove that each term in (30.3) and (30.4) is equivalent to $-\Omega[a_i \wedge a_j, b_i \wedge b_j]_s = \Omega[a_i \wedge a_j, b_j \wedge b_i]_s$. For (30.3), we proved in the beginning of the proof of Lemma 30.2 that[40]

$$\begin{aligned}\Omega_1[a_i \wedge a_j, b_j \wedge b_i]_s &\equiv \Theta[a_i \wedge (b_i + a_j), (a_i + b_j) \wedge (b_i + a_j)]_s, \\ \Omega_4[a_i \wedge a_j, b_j \wedge b_i]_s &\equiv \Theta[(b_i + a_j) \wedge b_j, (b_i + a_j) \wedge (a_i + b_j)]_s \\ &= \Theta[b_j \wedge (b_i + a_j), (a_i + b_j) \wedge (b_i + a_j)]_s.\end{aligned}$$

For (30.4), that same argument shows that

$$\begin{aligned}\Omega_1[a_i \wedge (-a_j), (-b_j) \wedge b_i]_s &\equiv \Theta[a_i \wedge (b_i - a_j), (a_i - b_j) \wedge (b_i - a_j)]_s, \\ \Omega_4[a_i \wedge (-a_j), (-b_j) \wedge b_i]_s &\equiv \Theta[(b_i - a_j) \wedge (-b_j), (b_i - a_j) \wedge (a_i - b_j)]_s \\ &= \Theta[(-b_j) \wedge (b_i - a_j), (a_i - b_j) \wedge (b_i - a_j)]_s.\end{aligned}$$

Lemma 30.5 implies that $\Omega[a_i \wedge (-a_j), (-b_j) \wedge b_i]_s = \Omega[a_i \wedge a_j, b_j \wedge b_i]_s$, so this is enough. □

## 31. Symmetric kernel, symmetric version V: skeleton of rest of proof

We continue using all the notation from §27 – §30. Recall that our goal in this part of the paper is to prove:

**Theorem G.** *For[41] $g \geq 4$, the linearization map $\Phi\colon \mathfrak{K}_g^s \to \mathrm{Sym}^2((\wedge^2 H)/\mathbb{Q})$ is an isomorphism.*

We prove this using the three step proof technique outlined in §3.

[40] That lemma only dealt with $i = 1$ and $j = 2$, but the proof works in general.

[41] Note that this is our standing assumption in this part; see Assumption 25.1.

31.1. **Step 1.** In the previous four sections, we took the first step towards proving Theorem G. We accomplished the following, which is a restatement with more details of Lemma 30.6. Recall that $\mathcal{B} = \{a_1, b_1, \ldots, a_g, b_g\}$ is our fixed symplectic basis for $H_{\mathbb{Z}}$, which is endowed with the total order $\prec$ in which the indicated list is strictly increasing.

**Lemma 31.1** (Step 1)**.** *Let $S = S_1 \cup \cdots \cup S_4$, where the $S_i$ are:*

$$S_1 = \{[\![x \wedge y, z \wedge w]\!]_s \mid x, y, z, w \in \mathcal{B},\ x \prec y,\ z \prec w,\ \mathfrak{c}(x \wedge y, z \wedge w) = 0\},$$
$$S_2 = \{\Theta[a_i \wedge b_i, x \wedge b_i]_s,\ \Theta[a_i \wedge b_i, a_i \wedge y]_s \mid 1 \leq i \leq g,\ x, y \in \mathcal{B} \setminus \{a_i, b_i\}\},$$
$$S_3 = \{\Lambda[a_i \wedge y, x \wedge b_i]_s \mid 1 \leq i \leq g,\ x, y \in \mathcal{B} \setminus \{a_i, b_i\},\ \omega(x, y) = 0\},$$
$$S_4 = \{\Omega[a_i \wedge a_j, b_i \wedge b_j]_s,\ \Omega[a_i \wedge b_j, b_i \wedge a_j]_s \mid 1 \leq i < j \leq g\}.$$

*Then the restriction of $\Phi$ to $\langle S \rangle$ is an isomorphism.*

31.2. **Step 2.** The next step is:

**Lemma 31.2** (Step 2)**.** *The $\mathrm{Sp}_{2g}(\mathbb{Z})$-orbit of the set $S$ from Lemma 31.1 spans $\mathfrak{K}_g^s$.*

*Proof.* Using our original generating set for $\mathfrak{K}_g^s$ from Definition 1.17 together with Lemma 11.2, we see that $\mathfrak{K}_g^s$ is generated by elements of the form $[\![a \wedge b, a' \wedge b']\!]_s$, where $a \wedge b$ and $a' \wedge b'$ are symplectic pairs such that $\langle a, b \rangle$ and $\langle a', b' \rangle$ are orthogonal. The group $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts transitively on such elements. The set $S$ contains many such elements; for instance, it contains $[\![a_1 \wedge b_1, a_2 \wedge b_2]\!]_s$. It follows that $\mathrm{Sp}_{2g}(\mathbb{Z})$-orbit of $S$ spans $\mathfrak{K}_g^s$. □

31.3. **Step 3.** The following lemma completes the proof of Theorem G.

**Lemma 31.3** (Step 3)**.** *Let $S \subset \mathfrak{K}_g^s$ be the set from Lemma 31.1. Then the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathfrak{K}_g^s$ takes $\langle S \rangle$ to itself. By Lemma 31.2 this implies that $\langle S \rangle = \mathfrak{K}_g^s$, and thus by Lemma 31.1 that $\Phi$ is an isomorphism.*

We begin the proof of Lemma 31.3, with the main steps postponed to the next 4 sections.

*Beginning of proof of Lemma 31.3.* Corollary 7.3 says that $\mathrm{Sp}_{2g}(\mathbb{Z})$ is generated as a monoid by $\mathrm{SymSp}_g \cup \{X_1, X_1^{-1}, Y_{12}\}$. Let $f \in \mathrm{SymSp}_g \cup \{X_1, X_1^{-1}, Y_{12}\}$ and let $s \in S$. It is enough to prove that $f(s) \in \langle S \rangle$.

The first case is $f \in \mathrm{SymSp}_g$. Assume first that $s \in S_1$. Write $s = [\![x \wedge y, z \wedge w]\!]_s$ with $x, y, z, w \in \mathcal{B}$ satisfying $x \prec y$ and $z \prec w$ and $\mathfrak{c}(x \wedge y, z \wedge w) = 0$. There exist $x', y', z', w' \in \mathcal{B}$ and signs $\epsilon_1, \ldots, \epsilon_4 \in \{\pm 1\}$ such that

$$f(x) = \epsilon_1 x',\ f(y) = \epsilon_2 y',\ f(z) = \epsilon_3 z',\ f(w) = \epsilon_4 w'.$$

We then have that $f(s)$ equals

$$[\![(\epsilon_1 x') \wedge (\epsilon_2 y'), (\epsilon_3 z') \wedge (\epsilon_4 w')]\!]_s = \epsilon_1 \cdots \epsilon_4 [\![x' \wedge y', z' \wedge w']\!]_s \in \langle S_1 \rangle.$$

We remark that it is possible that either $y' \prec x'$ or $w' \prec z'$, so $[\![x' \wedge y', z' \wedge w']\!]_s$ itself might not lie in $S_1$; however, either it or $(-1)$ times it lies in $S_1$. The cases where $s \in S_2$ or $s \in S_3$ or $s \in S_4$ are handled the same way, using the sign rules for $\Theta$- and $\Lambda$- and $\Omega$-elements discussed in §28.5 and §29.5 and §30.3.

We now must deal with the cases where $f \in \{X_1, X_1^{-1}, Y_{12}\}$. These calculations are lengthy, so we postpone them. We deal with $s \in S_1$ in §32, with $s \in S_2$ in §33, with $s \in S_3$ in §34, and finally with $s \in S_4$ in §35. □

## 32. Symmetric kernel, symmetric version VI: closure of $S_1$

We continue using all the notation from §27 – §31. In this section, we continue the proof of Lemma 31.3 by proving that for $f \in \{X_1, X_1^{-1}, Y_{12}\}$ and $s \in S_1$, we have $f(s) \in \langle S \rangle$.

32.1. **X-closure.** Recall that $X_1 \in \mathrm{Sp}_{2g}(\mathbb{Z})$ takes $a_1$ to $a_1 + b_1$ and fixes all other generators in $\mathcal{B}$. We start with:

**Lemma 32.1.** *Let*

$$s \in S_1 = \{[\![x \wedge y, z \wedge w]\!]_s \mid x, y, z, w \in \mathcal{B},\ x \prec y,\ z \prec w,\ \mathfrak{c}(x \wedge y, z \wedge w) = 0\}.$$

*Then* $X_1^{\epsilon}(s) \in \langle S \rangle$ *for* $\epsilon \in \{\pm 1\}$.

*Proof.* Write $s = [\![x \wedge y, z \wedge w]\!]_s$. There is nothing to prove if $a_1 \notin \{x, y, z, w\}$. What is more, the result is obvious if one of $\{x, y\}$ and $\{z, w\}$ contains $a_1$ and the other does not. For instance, we have (c.f. §27.6)

$$X_1^{\epsilon}([\![a_1 \wedge a_2, a_3 \wedge b_3]\!]_s) = [\![(a_1 + \epsilon b_1) \wedge a_2, a_3 \wedge b_3]\!]_s \in \langle S_1 \rangle.$$

Since $x \prec y$ and $z \prec w$, we have reduced ourselves to $s = [\![a_1 \wedge y, a_1 \wedge w]\!]_s$ with $y, w \in \mathcal{B} \setminus \{a_1\}$ satisfying $\mathfrak{c}(a_1 \wedge y, a_1 \wedge w) = 0$. The condition $\mathfrak{c}(a_1 \wedge y, a_1 \wedge w) = 0$ implies that one of the following holds:

- $y, w \in \mathcal{B} \setminus \{a_1, b_1\}$ and $\omega(y, w) = 0$; or
- $y = w = b_1$.

For instance, if $w \in \mathcal{B} \setminus \{a_1, b_1\}$ then

$$\mathfrak{c}(a_1 \wedge b_1, a_1 \wedge w) = a_1 \cdot w \neq 0.$$

We deal with the above cases separately. Let $\equiv$ denote equality modulo $\langle S \rangle$, so our goal is to prove that $X_1^{\epsilon}(s) \equiv 0$.

**Case 1.** $s = [\![a_1 \wedge b_1, a_1 \wedge b_1]\!]_s$.

We have

$$X_1^{\epsilon}([\![a_1 \wedge b_1, a_1 \wedge b_1]\!]_s) = [\![(a_1 + \epsilon b_1) \wedge b_1, (a_1 + \epsilon b_1) \wedge b_1]\!]_s = [\![a_1 \wedge b_1, a_1 \wedge b_1]\!]_s \equiv 0.$$

**Case 2.** $s = [\![a_1 \wedge y, a_1 \wedge w]\!]_s$ *with* $y, w \in \mathcal{B} \setminus \{a_1, b_1\}$ *satisfying* $\omega(y, w) = 0$.

By Lemma 29.13 ($\Lambda$-expansion III), we have that $X_1^{\epsilon}([\![a_1 \wedge y, a_1 \wedge w]\!]_s)$ equals

$$[\![(a_1 + \epsilon b_1) \wedge y, (a_1 + \epsilon b_1) \wedge w]\!]_s = \Lambda[a_1 \wedge y, (a_1 + \epsilon b_1) \wedge w]_s + \epsilon\Lambda[b_1 \wedge y, (a_1 + \epsilon b_1) \wedge w]_s.$$

By Lemma 29.12 ($\Lambda$-bilinearity II), this equals

$$[\![a_1 \wedge y, a_1 \wedge w]\!]_s + \epsilon\Lambda[a_1 \wedge y, b_1 \wedge w]_s + \epsilon\Lambda[b_1 \wedge y, a_1 \wedge w]_s + \epsilon^2[\![b_1 \wedge y, b_1 \wedge w]\!]_s \equiv 0. \qquad \square$$

32.2. **1-2 swaps.** Recall that $Y_{12} \in \mathrm{Sp}_{2g}(\mathbb{Z})$ takes $a_1$ to $a_1 + b_2$ and $a_2$ to $a_2 + b_1$ and fixes all other generators in $\mathcal{B}$. For this element, the indices 1 and 2 are special. The 1-2 *swap* is the element $\sigma \in \mathrm{SymSp}_g$ such that

$$\sigma(a_1) = a_2,\ \ \sigma(b_1) = b_2,\ \ \sigma(a_2) = a_1,\ \ \sigma(b_2) = b_1$$

and such that $\sigma$ fixes all other elements of $\mathcal{B}$. It satisfies the following:

**Lemma 32.2.** *Let* $\sigma \in \mathrm{SymSp}_g$ *be the* 1-2 *swap and let* $z \in \mathfrak{K}_g^s$. *Then* $Y_{12}(z) \in \langle S \rangle$ *if and only if* $Y_{12}(\sigma(z)) \in \langle S \rangle$.

*Proof.* The element $\sigma$ commutes with $Y_{12}$, so

$$Y_{12}(\sigma(z)) = \sigma(Y_{12}(z)). \tag{32.1}$$

We already proved at the end of §31 that $\mathrm{SymSp}_g$ takes $\langle S \rangle$ to itself. This holds in particular for $\sigma$. In light of (32.1), the lemma follows. $\square$

32.3. **Y-closure.** We now prove:

**Lemma 32.3.** *Let*

$$s \in S_1 = \{[\![x \wedge y, z \wedge w]\!]_s \mid x, y, z, w \in \mathcal{B},\ x \prec y,\ z \prec w,\ \mathfrak{c}(x \wedge y, z \wedge w) = 0\}.$$

*Then* $Y_{12}(s) \in \langle S \rangle$.

*Proof.* Write $s = [\![x \wedge y, z \wedge w]\!]_s$. To avoid having the deal with even more special cases, we relax the conditions $x \prec y$ and $z \prec w$ to $x \neq y$ and $z \neq w$.

There are a large number of cases to consider: each of $x$ and $y$ and $z$ and $w$ can either lie in $\{a_1, b_1, a_2, b_2\}$, or they can be some other element of $\mathcal{B}$. The condition $\mathfrak{c}(x \wedge y, z \wedge w) = 0$ eliminates some of these, but there are still far too many cases. To cut this down to something reasonable, we use the following three symmetries, which we will call the $Y_{12}$-*symmetries*:

- Flipping $x \wedge y$ and $z \wedge w$ does not change $s$.
- Flipping $x$ and $y$ multiplies $s$ by $-1$, and thus does not change the truth of the lemma. Similarly, we can flip $z$ and $w$.
- Finally, by Lemma 32.2 we can apply a 1-2 swap to $s$ without changing whether or not it lies in $\langle S \rangle$.

Using these, we will reduce ourselves to six cases as follows.

If $a_1, a_2 \notin \{x, y, z, w\}$, then there is nothing to prove. Otherwise, after performing a sequence of $Y_{12}$-symmetries we can assume that $x$ is either $a_1$ or $a_2$. Applying a 1-2 swap if necessary, we can assume that $x = a_1$, so $s = [\![a_1 \wedge y, z \wedge w]\!]_s$.

If $a_2 \notin \{y, z, w\}$, then in most cases we have $Y_{12}([\![a_1 \wedge y, z \wedge w]\!]_s) \in \langle S_1 \rangle$. For instance, we have (c.f. §27.6)

$$Y_{12}([\![a_1 \wedge a_3, a_1 \wedge b_2]\!]_s) = [\![(a_1 + b_2) \wedge a_3, (a_1 + b_2) \wedge b_2]\!]_s \in \langle S_1 \rangle.$$

Up to flipping $z$ and $w$, the only case where $a_2 \notin \{y, z, w\}$ and $Y_{12}([\![a_1 \wedge y, z \wedge w]\!]_s) \notin \langle S_1 \rangle$ is $[\![a_1 \wedge b_1, a_1 \wedge b_1]\!]_s$, which is Case 1 below.

It remains to enumerate the cases where $a_2 \in \{y, z, w\}$. We start by enumerating the cases where $y = a_2$, so $s = [\![a_1 \wedge a_2, z \wedge w]\!]_s$. If $a_1, a_2 \notin \{z, w\}$, then $Y_{12}([\![a_1 \wedge a_2, z \wedge w]\!]_s) \in \langle S_1 \rangle$. For instance,

$$Y_{12}([\![a_1 \wedge a_2, a_3 \wedge b_3]\!]_s) = [\![(a_1 + b_2) \wedge (b_1 + a_2), a_3 \wedge b_3]\!]_s \in \langle S_1 \rangle.$$

If instead either $a_1$ or $a_2$ lie in $\{z, w\}$, then up to $Y_{12}$-symmetries (including possibly a 1-2 swap) we can assume that $z = a_1$. The condition $\mathfrak{c}(a_1 \wedge a_2, a_1 \wedge w) = 0$ implies that $\omega(a_1, w) = \omega(a_2, w) = 0$, so there are two cases: $s = [\![a_1 \wedge a_2, a_1 \wedge a_2]\!]_s$, and $s = [\![a_1 \wedge a_2, a_1 \wedge w]\!]_s$ with $w \in \mathcal{B} \setminus \{a_1, b_1, a_2, b_2\}$. These are Cases 2 and 3 below. This completes our enumeration of the cases where $y = a_2$.

The remaining cases are $s = [\![a_1 \wedge y, z \wedge w]\!]_s$ with $y \neq a_2$ but $a_2 \in \{z, w\}$. After possibly flipping $z$ and $w$, we can assume that $z = a_2$. In other words, we have reduced ourselves to enumerating the cases where $s = [\![a_1 \wedge y, a_2 \wedge w]\!]_s$ with $y \neq a_2$. Up to $Y_{12}$-symmetries, we have already handled the case where $w = a_1$, so we can also assume that $w \neq a_1$. The condition $\mathfrak{c}(a_1 \wedge y, a_2 \wedge w) = 0$ implies that $y \neq b_2$ and $w \neq b_1$. There are now four cases:

- $[\![a_1 \wedge b_1, a_2 \wedge b_2]\!]_s$, which is Case 4 below.
- $[\![a_1 \wedge b_1, a_2 \wedge w]\!]_s$ with $w \in \mathcal{B} \setminus \{a_1, b_1, a_2, b_2\}$ and $[\![a_1 \wedge y, a_2 \wedge b_2]\!]_s$ with $w \in \mathcal{B} \setminus \{a_1, b_1, a_2, b_2\}$. These differ by $Y_{12}$-symmetries, so we only need to deal with the first. This is Case 5 below.
- $[\![a_1 \wedge y, a_2 \wedge w]\!]_s$ with $y, w \in \mathcal{B} \setminus \{a_1, b_1, a_2, b_2\}$ satisfying $\omega(y, w) = 0$. This is Case 6 below.

It remains to deal with all these cases. Let $\equiv$ denote equality modulo $\langle S \rangle$, so our goal is to prove that $Y_{12}(s) \equiv 0$.

**Case 1.** $[\![a_1 \wedge b_1, a_1 \wedge b_1]\!]_s$.

By Lemma 28.3 ($\Theta$-expansion I), the element $Y_{12}([\![a_1 \wedge b_1, a_2 \wedge b_1]\!]_s) = [\![(a_1+b_2) \wedge b_1, (a_1+b_2) \wedge b_1]\!]_s$ equals

$$[\![a_1 \wedge b_1, a_1 \wedge b_1]\!]_s + 2\Theta[a_1 \wedge b_1, b_2 \wedge b_1]_s + [\![b_2 \wedge b_1, b_2 \wedge b_1]\!]_s \equiv 0.$$

**Case 2.** $[\![a_1 \wedge a_2, a_1 \wedge a_2]\!]_s$.

Lemma 30.7 implies that $Y_{12}([\![a_1 \wedge a_2, a_1 \wedge a_2]\!]_s) = [\![(a_1+b_2)\wedge(b_1+a_2), (a_1+b_2)\wedge(b_1+a_2)]\!]_s$ is equivalent to $-2\Omega[a_1 \wedge a_2, b_1 \wedge b_2]_s \equiv 0$.

**Case 3.** $[\![a_1 \wedge a_2, a_1 \wedge w]\!]_s$ *with* $w \in \mathcal{B} \setminus \{a_1, b_1, a_2, b_2\}$.

Lemma 28.9 ($\Theta$-expansion II) implies that

$$Y_{12}([\![a_1 \wedge a_2, a_1 \wedge w]\!]_s) = [\![(a_1+b_2) \wedge (b_1+a_2), (a_1+b_2) \wedge w]\!]_s$$

equals

$$\Theta[(a_1+b_2) \wedge b_1, (a_1+b_2) \wedge w]_s + \Theta[(a_1+b_2) \wedge a_2, (a_1+b_2) \wedge w]_s.$$

Lemma 29.4 ($\Lambda$-expansion I) implies that $\Theta[(a_1+b_2) \wedge b_1, (a_1+b_2) \wedge w]_s$ equals

$$\Theta[a_1 \wedge b_1, a_1 \wedge w]_s + [\![a_1 \wedge b_1, b_2 \wedge w]\!]_s + \Lambda[b_2 \wedge b_1, a_1 \wedge w]_s + [\![b_2 \wedge b_1, b_2 \wedge w]\!]_s \equiv 0.$$

Similarly, $\Theta[(a_1+b_2) \wedge a_2, (a_1+b_2) \wedge w]_s \equiv 0$. The case follows.

**Case 4.** $[\![a_1 \wedge b_1, a_2 \wedge b_2]\!]_s$.

In $(\wedge^2 H)/\mathbb{Q}$, we have

$$a_2 \wedge b_2 = -a_1 \wedge b_1 - \sum\nolimits_{i=3}^{g} a_i \wedge b_i.$$

Plugging this into $[\![a_1 \wedge b_1, a_2 \wedge b_2]\!]_s$, we see that

$$[\![a_1 \wedge b_1, a_2 \wedge b_2]\!]_s = -[\![a_1 \wedge b_1, a_1 \wedge b_1]\!]_s - \sum\nolimits_{i=3}^{g} [\![a_1 \wedge b_1, a_i \wedge b_i]\!]_s.$$

It follows that $Y_{12}([\![a_1 \wedge b_1, a_2 \wedge b_2]\!]_s)$ equals

$$-Y_{12}([\![a_1 \wedge b_1, a_1 \wedge b_1]\!]_s) - \sum\nolimits_{i=3}^{g} Y_{12}([\![a_1 \wedge b_1, a_i \wedge b_i]\!]_s).$$

We proved that $Y_{12}([\![a_1 \wedge b_1, a_1 \wedge b_1]\!]_s) \equiv 0$ in Case 1, and for $3 \leq i \leq g$ we have

$$Y_{12}([\![a_1 \wedge b_1, a_i \wedge b_i]\!]_s) = [\![(a_1+b_2) \wedge b_1, a_i \wedge b_i]\!]_s \equiv 0;$$

see §27.6. The case follows.

**Case 5.** $[\![a_1 \wedge b_1, a_2 \wedge w]\!]_s$ *with* $w \in \mathcal{B} \setminus \{a_1, b_1, a_2, b_2\}$.

To simplify our notation, we will explain how to deal with $w = a_3$. The other cases are similar. Lemma 28.7 ($\Theta$-symplectic basis) implies that

$$[\![a_1 \wedge b_1, a_2 \wedge a_3]\!]_s + \Theta[a_2 \wedge b_2, a_2 \wedge a_3]_s + \Theta[a_3 \wedge b_3, a_2 \wedge a_3]_s + \sum\nolimits_{i=4}^{g} [\![a_i \wedge b_i, a_2 \wedge a_3]\!]_s = 0.$$

It follows that $Y_{12}([\![a_1 \wedge b_1, a_2 \wedge a_3]\!]_s)$ equals

$$\begin{aligned} &- Y_{12}(\Theta[a_2 \wedge b_2, a_2 \wedge a_3]_s) - Y_{12}(\Theta[a_3 \wedge b_3, a_2 \wedge a_3]_s) - \sum\nolimits_{i=4}^{g} Y_{12}([\![a_i \wedge b_i, a_2 \wedge a_3]\!]_s) \\ = &- \Theta[(a_2+b_1) \wedge b_2, (a_2+b_1) \wedge a_3]_s \\ &- \Theta[a_3 \wedge b_3, (a_2+b_1) \wedge a_3]_s - \sum\nolimits_{i=4}^{g} [\![a_i \wedge b_i, (a_2+b_1) \wedge a_3]\!]_s \\ \equiv &- \Theta[(a_2+b_1) \wedge b_2, (a_2+b_1) \wedge a_3]_s. \end{aligned}$$

The last $\equiv$ uses Lemma 28.4 ($\Theta$-linearity) to show that

$$\Theta[a_3 \wedge b_3, (a_2+b_1) \wedge a_3]_s \in \langle S_2 \rangle.$$

This reduces us to proving that $\Theta[(a_2+b_1)\wedge b_2, (a_2+b_1)\wedge a_3]_s \equiv 0$. For this, Lemma 29.4 ($\Lambda$-expansion I) implies that $\Theta[(a_2+b_1)\wedge b_2, (a_2+b_1)\wedge a_3]_s$ equals

$$\Theta[a_2\wedge b_2, a_2\wedge a_3]_s + [\![a_2\wedge b_2, b_1\wedge a_3]\!]_s + \Lambda[b_1\wedge b_2, a_2\wedge a_3]_s + [\![b_1\wedge b_2, b_1\wedge a_3]\!]_s \equiv 0.$$

**Case 6.** *$[\![a_1\wedge y, a_2\wedge w]\!]_s$ with $y,w\in\mathcal{B}\setminus\{a_1,b_1,a_2,b_2\}$ satisfying $\omega(y,w)=0$.*

Lemma 29.11 ($\Lambda$-expansion II) implies that $Y_{12}([\![a_1\wedge y, a_2\wedge w]\!]_s) = [\![(a_1+b_2)\wedge y, (b_1+a_2)\wedge w]\!]_s$ equals

$$\Lambda[a_1\wedge y, (b_1+a_2)\wedge w]_s + \Lambda[b_2\wedge y, (b_1+a_2)\wedge w]_s.$$

Lemma 29.10 ($\Lambda$-bilinearity I) shows that

$$\Lambda[a_1\wedge y, (b_1+a_2)\wedge w]_s = \Lambda[a_1\wedge y, b_1\wedge w]_s + [\![a_1\wedge y, a_2\wedge w]\!]_s \equiv 0.$$

Similarly, $\Lambda[b_2\wedge y, (b_1+a_2)\wedge w]_s \equiv 0$. The case follows. □

## 33. Symmetric kernel, symmetric version VII: closure of $S_2$

We continue using all the notation from §27 – §31. In this section, we continue the proof of Lemma 31.3 by proving that for $f\in\{X_1, X_1^{-1}, Y_{12}\}$ and $s\in S_2$, we have $f(s)\in\langle S\rangle$.

33.1. **X-closure.** Recall that $X_1\in \mathrm{Sp}_{2g}(\mathbb{Z})$ takes $a_1$ to $a_1+b_1$ and fixes all other generators in $\mathcal{B}$. We start with:

**Lemma 33.1.** *Let*

$$s\in S_2 = \{\Theta[a_i\wedge b_i, x\wedge b_i]_s,\ \Theta[a_i\wedge b_i, a_i\wedge y]_s \mid 1\le i\le g,\ x,y\in\mathcal{B}\setminus\{a_i,b_i\}\}.$$

*Then $X_1^{\epsilon}(s)\in\langle S\rangle$ for $\epsilon\in\{\pm 1\}$.*

*Proof.* The lemma is trivial if $X_1$ fixes $s$. The remaining cases are as follows. Let $\equiv$ denote equality modulo $\langle S\rangle$, so our goal is to prove that $X_1^{\epsilon}(s)\equiv 0$.

**Case 1.** *$s=\Theta[a_1\wedge b_1, x\wedge b_1]_s$ with $x\in\mathcal{B}\setminus\{a_1,b_1\}$.*

Lemma 28.10 ($\Theta$-bilinearity II) implies that

$$X_1^{\epsilon}(\Theta[a_1\wedge b_1, x\wedge b_1]_s) = \Theta[(a_1+\epsilon b_1)\wedge b_1, x\wedge b_1]_s = \Theta[a_1\wedge b_1, x\wedge b_1]_s \equiv 0.$$

**Case 2.** *$s=\Theta[a_1\wedge b_1, a_1\wedge y]_s$ with $y\in\mathcal{B}\setminus\{a_1,b_1\}$.*

Lemma 28.10 ($\Theta$-bilinearity II) implies that

$$\begin{aligned} X_1^{\epsilon}(\Theta[a_1\wedge b_1, a_1\wedge y]_s) &= \Theta[(a_1+\epsilon b_1)\wedge b_1, (a_1+\epsilon b_1)\wedge y]_s \\ &= \Theta[a_1\wedge b_1, a_1\wedge y]_s + \epsilon\Theta[a_1\wedge b_1, b_1\wedge y]_s \equiv 0. \end{aligned}$$

**Case 3.** *$s=\Theta[a_i\wedge b_i, a_1\wedge b_i]_s$ or $s=\Theta[a_i\wedge b_i, a_i\wedge a_1]_s$ for some $2\le i\le g$.*

Both cases are handled identically, so we will give the details for $s=\Theta[a_i\wedge b_i, a_1\wedge b_i]_s$. By Lemma 28.4 ($\Theta$-linearity), we have that $X_1^{\epsilon}(\Theta[a_i\wedge b_i, a_1\wedge b_i]_s) = \Theta[a_i\wedge b_i, (a_1+\epsilon b_1)\wedge b_i]_s$ equals

$$\Theta[a_i\wedge b_i, a_1\wedge b_i]_s + \epsilon\Theta[a_i\wedge b_i, b_1\wedge b_i]_s \equiv 0.$$ □

33.2. **Y-closure.** Recall that $Y_{12} \in \mathrm{Sp}_{2g}(\mathbb{Z})$ takes $a_1$ to $a_1 + b_2$ and $a_2$ to $a_2 + b_1$ and fixes all other generators in $\mathcal{B}$. We next prove:

**Lemma 33.2.** *Let*

$$s \in S_2 = \{\Theta[a_i \wedge b_i, x \wedge b_i]_s,\ \Theta[a_i \wedge b_i, a_i \wedge y]_s \mid 1 \leq i \leq g,\ x, y \in \mathcal{B} \setminus \{a_i, b_i\}\}.$$

*Then* $Y_{12}(s) \in \langle S \rangle$.

*Proof.* For $i \geq 3$, Lemma 28.4 (Θ-linearity) implies that this holds for $s = \Theta[a_i \wedge b_i, x \wedge b_i]_s$ and $s = \Theta[a_i \wedge b_i, a_i \wedge y]_s$ with $x, y \in \mathcal{B} \setminus \{a_i, b_i\}$. For instance,

$$\begin{aligned} Y_{12}(\Theta[a_3 \wedge b_3, a_1 \wedge b_3]_s) &= \Theta[a_3 \wedge b_3, (a_1 + b_2) \wedge b_3]_s \\ &= \Theta[a_3 \wedge b_3, a_1 \wedge b_3]_s + \Theta[a_3 \wedge b_3, b_2 \wedge b_3]_s \in \langle S_2 \rangle. \end{aligned}$$

The remaining cases are when $i = 1$ and $i = 2$. Applying a 1-2 swap as described in Lemma 32.2, it is enough to deal with the case $i = 1$. We divide this into the following cases. Let $\equiv$ denote equality modulo $\langle S \rangle$, so our goal is to prove that $Y_{12}(s) \equiv 0$.

**Case 1.** $s = \Theta[a_1 \wedge b_1, x \wedge b_1]_s$ *with* $x \in \mathcal{B} \setminus \{a_1, b_1, a_2\}$.

Lemma 28.8 (Θ-bilinearity I) implies that $Y_{12}(\Theta[a_1 \wedge b_1, x \wedge b_1]_s) = \Theta[(a_1 + b_2) \wedge b_1, x \wedge b_1]_s)$ equals

$$\Theta[a_1 \wedge b_1, x \wedge b_1]_s + [\![b_2 \wedge b_1, x \wedge b_1]\!]_s \equiv 0.$$

**Case 2.** $s = \Theta[a_1 \wedge b_1, a_1 \wedge y]_s$ *with* $y \in \mathcal{B} \setminus \{a_1, b_1, a_2\}$.

By Lemma 29.4 (Λ-expansion I), we have that

$$Y_{12}(\Theta[a_1 \wedge b_1, a_1 \wedge y]_s) = \Theta[(a_1 + b_2) \wedge b_1, (a_1 + b_2) \wedge y]_s$$

equals

$$\Theta[a_1 \wedge b_1, a_1 \wedge y]_s + [\![a_1 \wedge b_1, b_2 \wedge y]\!]_s + \Lambda[b_2 \wedge b_1, a_1 \wedge y]_s + [\![b_2 \wedge b_1, b_2 \wedge y]\!]_s \equiv 0.$$

**Case 3.** $s = \Theta[a_1 \wedge b_1, a_2 \wedge b_1]_s$.

By the definition of Θ-elements (Definition 28.1), we have that

$$Y_{12}(\Theta[a_1 \wedge b_1, a_2 \wedge b_1]_s) = \Theta[(a_1 + b_2) \wedge b_1, (b_1 + a_2) \wedge b_1]_s$$

equals 1/2 times

$$\begin{aligned} &[\![(a_1 + b_2 + b_1 + a_2) \wedge b_1, (a_1 + b_2 + b_1 + a_2) \wedge b_1]\!]_s - [\![(a_1 + b_2) \wedge b_1, (a_1 + b_2) \wedge b_1]\!]_s \\ &\quad - [\![(b_1 + a_2) \wedge b_1, (b_1 + a_2) \wedge b_1]\!]_s \\ \equiv&[\![(a_1 + b_2 + a_2) \wedge b_1, (a_1 + b_2 + a_2) \wedge b_1]\!]_s - Y_{12}([\![a_1 \wedge b_1, a_1 \wedge b_1]\!]_s) \\ \equiv&[\![(a_1 + b_2 + a_2) \wedge b_1, (a_1 + b_2 + a_2) \wedge b_1]\!]_s. \end{aligned}$$

The last $\equiv$ uses the fact that we have already proved that $Y_{12}(t) \equiv 0$ for $t \in S_1$ (Lemma 32.3). Lemma 28.3 (Θ-expansion I) along with Lemma 28.4 (Θ-linearity) implies that this equals

$$\begin{aligned} &[\![a_1 \wedge b_1, a_1 \wedge b_1]\!]_s + 2\Theta[a_1 \wedge b_1, (b_2 + a_2) \wedge b_1]_s + [\![(b_2 + a_2) \wedge b_1, (b_2 + a_2) \wedge b_1]\!]_s \\ \equiv&2\Theta[a_1 \wedge b_1, b_2 \wedge b_1]_s + 2\Theta[a_1 \wedge b_1, a_2 \wedge b_1]_s + [\![(b_2 + a_2) \wedge b_1, (b_2 + a_2) \wedge b_1]\!]_s \\ \equiv&[\![(a_2 + b_2) \wedge b_1, (a_2 + b_2) \wedge b_1]\!]_s. \end{aligned}$$

Applying Lemma 29.13 (Λ-expansion III) and then Lemma 29.12 (Λ-bilinearity II), this equals

$$\begin{aligned} &\Lambda[a_2 \wedge b_1, (a_2 + b_2) \wedge b_1]_s + \Lambda[b_2 \wedge b_1, (a_2 + b_2) \wedge b_1]_s \\ =&[\![a_2 \wedge b_1, a_2 \wedge b_1]\!]_s + \Lambda[a_2 \wedge b_1, b_2 \wedge b_1]_s + \Lambda[b_2 \wedge b_1, a_2 \wedge b_1]_s + [\![b_2 \wedge b_1, b_2 \wedge b_1]\!]_s \equiv 0. \end{aligned}$$

**Case 4.** $s = \Theta[a_1 \wedge b_1, a_1 \wedge a_2]_s$.

We have

$$Y_{12}(\Theta[a_1 \wedge b_1, a_1 \wedge a_2]_s) = \Theta[(a_1 + b_2) \wedge b_1, (a_1 + b_2) \wedge (b_1 + a_2)]_s.$$

By Lemma 28.4 ($\Theta$-linearity), it is enough to prove that

$$\Theta[(a_1 + b_2) \wedge b_1, (a_1 + b_2) \wedge (-b_1 - a_2)]_s \equiv 0.$$

By the definition of $\Theta$-elements (Definition 28.1), this equals

$$\begin{aligned}
&[\![(a_1 + b_2) \wedge (-a_2), (a_1 + b_2) \wedge (-a_2)]\!]_s - [\![(a_1 + b_2) \wedge b_1, (a_1 + b_2) \wedge b_1]\!]_s \\
&- [\![(a_1 + b_2) \wedge (-b_1 - a_2), (a_1 + b_2) \wedge (-b_1 - a_2)]\!]_s \\
=&[\![(a_1 + b_2) \wedge a_2, (a_1 + b_2) \wedge a_2]\!]_s - Y_{12}([\![a_1 \wedge b_1, a_1 \wedge b_1]\!]_s) - Y_{12}([\![a_1 \wedge a_2, a_1 \wedge b_2]\!]_s) \\
\equiv&[\![(a_1 + b_2) \wedge a_2, (a_1 + b_2) \wedge a_2]\!]_s.
\end{aligned}$$

The last $\equiv$ uses the fact that we have already proved that $Y_{12}(t) \equiv 0$ for $t \in S_1$ (Lemma 32.3). Lemma 28.3 ($\Theta$-expansion I) says that this equals

$$[\![a_1 \wedge a_2, a_1 \wedge a_2]\!]_s + 2\Theta[a_1 \wedge a_2, b_2 \wedge a_2]_s + [\![b_2 \wedge a_2, b_2 \wedge a_2]\!]_s \equiv 0. \qquad \square$$

## 34. Symmetric kernel, symmetric version VIII: closure of $S_3$

We continue using all the notation from §27 – §31. In this section, we continue the proof of Lemma 31.3 by proving that for $f \in \{X_1, X_1^{-1}, Y_{12}\}$ and $s \in S_3$, we have $f(s) \in \langle S \rangle$.

34.1. **More general Lambda-elements.** Recall that

$$S_3 = \{\Lambda[a_i \wedge y, x \wedge b_i]_s \mid 1 \leq i \leq g,\ x, y \in \mathcal{B} \setminus \{a_i, b_i\},\ \omega(x, y) = 0\}.$$

Before we prove our main results, we prove:

**Lemma 34.1.** *Let $1 \leq i \leq g$ and let $x, y \in \langle a_i, b_i \rangle^{\perp}$ satisfy $\omega(x, y) = 0$. Then $\Lambda[a_i \wedge y, x \wedge b_i]_s \in \langle S \rangle$.*

*Proof.* Recall that $\mathfrak{K}_g^{s,\Lambda}[a_i \wedge -, - \wedge b_i]$ is the subspace of $\mathfrak{K}_g^s$ spanned by $\Lambda$-elements as in the statement of the lemma. It follows from Lemma 29.6 (strong $\Lambda$-linearity) that $\mathfrak{K}_g^{s,\Lambda}[a_i \wedge -, - \wedge b_i]$ is spanned by three kinds of elements:

- Elements of the form $\Lambda[a_i \wedge y, x \wedge b_i]_s$ with $x, y \in \mathcal{B} \setminus \{a_i, b_i\}$ satisfying $\omega(x, y) = 0$. These are elements of $S_3$.
- Elements of the form $\Lambda[a_i \wedge (a_j + a_k), (b_j - b_k) \wedge b_i]_s$ for distinct $1 \leq j, k \leq g$ with $j, k \neq i$. By Lemma 30.4 ($\Lambda$ to $\Omega$), these equal

$$\Omega[a_i \wedge a_j, b_j \wedge a_i]_s - \Omega[a_i \wedge a_k, b_k \wedge b_i]_s + \Lambda[a_i \wedge a_k, b_j \wedge b_i]_s - \Lambda[a_i \wedge a_j, b_k \wedge b_k]_s \in \langle S \rangle.$$

- Elements of the form $\Lambda[a_i \wedge (a_j + b_k), (b_j + a_k) \wedge b_i]_s$ for distinct $1 \leq j, k \leq g$ with $j, k \neq i$. Again, Lemma 30.4 ($\Lambda$ to $\Omega$) implies that these lie in $\langle S \rangle$. $\square$

34.2. **X-closure.** Recall that $X_1 \in \mathrm{Sp}_{2g}(\mathbb{Z})$ takes $a_1$ to $a_1 + b_1$ and fixes all other generators in $\mathcal{B}$. We now prove:

**Lemma 34.2.** *Let*

$$s \in S_3 = \{\Lambda[a_i \wedge y, x \wedge b_i]_s \mid 1 \leq i \leq g,\ x, y \in \mathcal{B} \setminus \{a_i, b_i\},\ \omega(x, y) = 0\}.$$

*Then $X_1^{\epsilon}(s) \in \langle S \rangle$ for $\epsilon \in \{\pm 1\}$.*

*Proof.* There are two cases. Let $\equiv$ denote equality modulo $\langle S \rangle$, so our goal is to prove that $X_1^{\epsilon}(s) \equiv 0$.

**Case 1.** $s = \Lambda[a_1 \wedge y, x \wedge b_1]_s$ *with $x, y \in \mathcal{B} \setminus \{a_1, b_1\}$ satisfying $\omega(x, y) = 0$.*

Lemma 29.12 ($\Lambda$-bilinearity II) says that $X_1^{\epsilon}(\Lambda[a_1 \wedge y, x \wedge b_1]_s) = \Lambda[(a_1 + \epsilon b_1) \wedge y, x \wedge b_1]_s$ equals

$$\Lambda[a_1 \wedge y, x \wedge b_1]_s + \epsilon [\![b_1 \wedge y, x \wedge b_1]\!]_s \equiv 0.$$

**Case 2.** *$s = \Lambda[a_i \wedge y, x \wedge b_i]_s$ with $x, y \in \mathcal{B} \setminus \{a_i, b_i\}$ satisfying $\omega(x, y) = 0$.*

Lemma 34.1 implies that

$$X_1^{\epsilon}(\Lambda[a_i \wedge y, x \wedge b_i]_s) = \Lambda[a_i \wedge X_1^{\epsilon}(y), X_1^{\epsilon}(x) \wedge b_i]_s \in \langle S \rangle. \qquad \square$$

34.3. **Y-closure.** Recall that $Y_{12} \in \mathrm{Sp}_{2g}(\mathbb{Z})$ takes $a_1$ to $a_1 + b_2$ and $a_2$ to $a_2 + b_1$ and fixes all other generators in $\mathcal{B}$. We next prove:

**Lemma 34.3.** *Let*

$$s \in S_3 = \{\Lambda[a_i \wedge y, x \wedge b_i]_s \mid 1 \leq i \leq g,\ x, y \in \mathcal{B} \setminus \{a_i, b_i\},\ \omega(x, y) = 0\}.$$

*Then $Y_{12}(s) \in \langle S \rangle$.*

*Proof.* Recall that $Y_{12} \in \mathrm{Sp}_{2g}(\mathbb{Z})$ takes $a_1$ to $a_1 + b_2$ and $a_2$ to $a_2 + b_1$ and fixes all other generators in $\mathcal{B}$. Write $s = \Lambda[a_i \wedge y, x \wedge b_i]_s$. The proof is different when $i = 1$, when $i = 2$, and when $3 \leq i \leq g$. However, applying a 1-2 swap as described in Lemma 32.2 we can reduce the proof for $i = 2$ to the proof for $i = 1$. We divide the cases $i = 1$ and $3 \leq i \leq g$ into the following cases. Let $\equiv$ denote equality modulo $\langle S \rangle$, so our goal is to prove that $Y_{12}(s) \equiv 0$.

**Case 1.** *$s = \Lambda[a_i \wedge y, x \wedge b_i]_s$ with $3 \leq i \leq g$ and $y, x \in \mathcal{B} \setminus \{a_i, b_i\}$ satisfying $\omega(x, y) = 0$.*

Lemma 34.1 implies that

$$Y_{12}(\Lambda[a_i \wedge y, x \wedge b_i]_s) = \Lambda[a_i \wedge Y_{12}(y), Y_{12}(x) \wedge b_i]_s \in \langle S \rangle.$$

**Case 2.** *$s = \Lambda[a_1 \wedge y, x \wedge b_1]_s$ with $x, y \in \mathcal{B} \setminus \{a_1, b_1, a_2\}$ such that $\omega(x, y) = 0$.*

Lemma 29.10 ($\Lambda$-bilinearity I) implies that $Y_{12}(\Lambda[a_1 \wedge y, x \wedge b_1]_s = \Lambda[(a_1 + b_2) \wedge y, x \wedge b_1]_s$ equals

$$\Lambda[a_1 \wedge y, x \wedge b_1]_s + [\![b_2 \wedge y, x \wedge b_1]\!]_s \equiv 0.$$

**Case 3.** *$s = \Lambda[a_1 \wedge a_2, x \wedge b_1]_s$ with $x \in \mathcal{B} \setminus \{a_1, b_1, a_2, b_2\}$.*

We have $Y_{12}(\Lambda[a_1 \wedge a_2, x \wedge b_1]_s) = \Lambda[(a_1+b_2)\wedge(b_1+a_2), x\wedge b_1]_s$. By Lemma 29.5 ($\Lambda$-linearity), it is enough to prove that

$$\Lambda[(a_1 + b_2) \wedge (-b_1 - a_2), x \wedge b_1]_s \equiv 0.$$

By the definition of $\Lambda$-elements (Definition 29.1), the element $\Lambda[(a_1+b_2)\wedge(-b_1-a_2), x\wedge b_1]_s = \Lambda_1[(a_1 + b_2) \wedge (-b_1 - a_2), x \wedge b_1]_s$ equals

$$\begin{aligned}
&\Theta[(a_1 + b_2) \wedge (-a_2), x \wedge (-a_2)]_s - \Theta[(a_1 + b_2) \wedge b_1, x \wedge b_1]_s \\
&\quad - [\![(a_1 + b_2) \wedge (-a_2), x \wedge (-b_1 - a_2)]\!]_s \\
=&\Theta[(a_1 + b_2) \wedge a_2, x \wedge a_2]_s - Y_{12}(\Theta[a_1 \wedge b_1, x \wedge b_1]_s) - [\![(a_1 + b_2) \wedge a_2, x \wedge (b_1 + a_2)]\!]_s \\
\equiv&\Theta[(a_1 + b_2) \wedge a_2, x \wedge a_2]_s - [\![(a_1 + b_2) \wedge a_2, x \wedge (b_1 + a_2)]\!]_s.
\end{aligned}$$

The last $\equiv$ uses the fact that we have already proved that $Y_{12}(t) \equiv 0$ for $t \in S_2$ (Lemma 33.2). We must prove that both of these terms are equivalent to 0. Lemma 28.8 ($\Theta$-bilinearity I) implies that the first term $\Theta[(a_1 + b_2) \wedge a_2, x \wedge a_2]_s$ equals

$$[\![a_1 \wedge a_2, x \wedge a_2]\!]_s + \Theta[b_2 \wedge a_2, x \wedge a_2]_s \equiv 0.$$

The second term $[\![(a_1+b_2)\wedge a_2, x\wedge(b_1+a_2)]\!]_s$ equals

$$\begin{aligned}&[\![(a_1+b_2)\wedge(b_1+a_2), x\wedge(b_1+a_2)]\!]_s - [\![(a_1+b_2)\wedge b_1, x\wedge(b_1+a_2)]\!]_s\\ =&Y_{12}([\![a_1\wedge a_2, x\wedge a_2]\!]_s) - Y_{12}([\![a_1\wedge b_1, x\wedge a_2]\!]_s)\equiv 0.\end{aligned}$$

The $\equiv$ uses the fact that we have already proved that $Y_{12}(t)\equiv 0$ for all $t\in S_1$ (Lemma 32.1).

**Case 4.** *$s=\Lambda[a_1\wedge y, a_2\wedge b_1]_s$ with $y\in\mathcal{B}\setminus\{a_1,b_1,a_2,b_2\}$.*

By the definition of $\Lambda$-elements (Definition 29.1), $Y_{12}(\Lambda[a_1\wedge y, a_2\wedge b_1]_s)=\Lambda_2[(a_1+b_2)\wedge y,(b_1+a_2)\wedge b_1]_s$ equals

$$\begin{aligned}&\Theta[(a_1+b_2+b_1+a_2)\wedge b_1,(a_1+b_2+b_1+a_2)\wedge y]_s\\ &-\Theta[(a_1+b_2)\wedge b_1,(a_1+b_2)\wedge y]_s - [\![(a_1+b_2+b_1+a_2)\wedge b_1,(b_1+a_2)\wedge y]\!]_s.\end{aligned}\tag{34.1}$$

We must prove that each term in (34.1) is equivalent to 0. The first is the most difficult, so we save it for last. The second term $\Theta[(a_1+b_2)\wedge b_1,(a_1+b_2)\wedge y]_s$ equals

$$Y_{12}(\Theta[a_1\wedge b_1, a_1\wedge y]_s)\equiv 0$$

since we have already proved that $Y_{12}(t)\equiv 0$ for all $t\in S_2$ (Lemma 33.1). The third term in (34.1) is $[\![(a_1+b_2+b_1+a_2)\wedge b_1,(b_1+a_2)\wedge y]\!]_s$, which equals

$$\begin{aligned}&[\![(a_1+b_2)\wedge b_1,(b_1+a_2)\wedge y]\!]_s + [\![(b_1+a_2)\wedge b_1,(b_1+a_2)\wedge y]\!]_s\\ \equiv&Y_{12}([\![a_1\wedge b_1, a_2\wedge y]\!]_s)\equiv 0.\end{aligned}$$

Here again we used the fact that $Y_{12}(t)\equiv 0$ for all $t\in S_2$ (Lemma 33.1).

It remains to deal with the first term $\Theta[(a_1+b_2+b_1+a_2)\wedge b_1,(a_1+b_2+b_1+a_2)\wedge y]_s$ in (34.1). By Lemma 28.10 ($\Theta$-bilinearity II), it equals

$$\Theta[(a_1+b_2+a_2)\wedge b_1,(a_1+b_2+a_2)\wedge y]_s+\Theta[(a_1+b_2+a_2)\wedge b_1, b_1\wedge y]_s.\tag{34.2}$$

We must show that both terms of (34.2) are equivalent to 0. Using Lemma 29.4 ($\Lambda$-expansion I) along with Lemma 29.5 ($\Lambda$-linearity), the first term of (34.2) equals

$$\begin{aligned}&\Lambda[a_1\wedge y,(b_2+a_2)\wedge b_1]_s+\Theta[a_1\wedge b_1, a_1\wedge y]_s+[\![(a_1+b_2+a_2)\wedge b_1,(b_2+a_2)\wedge y]\!]_s\\ \equiv&\Lambda[a_1\wedge y, b_2\wedge b_1]_s+\Lambda[a_1\wedge y, a_2\wedge b_1]_s\equiv 0.\end{aligned}$$

For the second term $\Theta[(a_1+b_2+a_2)\wedge b_1, b_1\wedge y]_s$ of (34.2), Lemma 28.8 ($\Theta$-bilinearity I) implies that it equals

$$\Theta[a_1\wedge b_1, b_1\wedge y]_s+[\![(b_2+a_2)\wedge b_1, b_1\wedge y]\!]_s\equiv 0.$$

**Case 5.** *$s=\Lambda[a_1\wedge a_2, a_2\wedge b_1]_s$.*

By Lemma 29.5 ($\Lambda$-linearity), to prove that

$$Y_{12}(\Lambda[a_1\wedge a_2, a_2\wedge b_1]_s)=\Lambda[(a_1+b_2)\wedge(b_1+a_2),(b_1+a_2)\wedge b_1]_s$$

is equivalent to 0 it is enough to prove that

$$\Lambda[(a_1+b_2)\wedge(-b_1-a_2),(b_1+a_2)\wedge b_1]_s$$

is equivalent to 0. By the definition of $\Lambda$-elements (Definition 29.1), this equals

$$\begin{aligned}&\Theta[(a_1+b_2)\wedge(-a_2),(b_1+a_2)\wedge(-a_2)]_s-\Theta[(a_1+b_2)\wedge b_1,(b_1+a_2)\wedge b_1]_s\\ &-[\![(a_1+b_2)\wedge(-a_2),(b_1+a_2)\wedge(-b_1-a_2)]\!]_s.\\ =&\Theta[(a_1+b_2)\wedge a_2,(b_1+a_2)\wedge a_2]_s-Y_{12}(\Theta[a_1\wedge b_1, a_2\wedge b_1]_s)\\ \equiv&\Theta[a_2\wedge(a_1+b_2), a_2\wedge(b_1+a_2)]_s.\end{aligned}$$

Here we used the fact that $Y_{12}(t) \equiv 0$ for all $t \in S_2$ (Lemma 33.1). Expanding this using the definition of $\Theta$-elements (Definition 28.1), we get 1/2 times

$$\begin{aligned}
&[\![a_2 \wedge (a_1 + b_2 + b_1 + a_2), a_2 \wedge (a_1 + b_2 + b_1 + a_2)]\!]_s - [\![a_2 \wedge (a_1 + b_2), a_2 \wedge (a_1 + b_2)]\!]_s \\
&- [\![a_2 \wedge (b_1 + a_2), a_2 \wedge (b_1 + a_2)]\!]_s \\
\equiv&[\![a_2 \wedge (b_2 + a_1 + b_1), a_2 \wedge (b_2 + a_1 + b_1)]\!]_s - [\![a_2 \wedge (b_2 + a_1), a_2 \wedge (b_2 + a_1)]\!]_s
\end{aligned}$$

We must prove that both of these terms are equivalent to 0. For the first term $[\![a_2 \wedge (b_2 + a_1 + b_1), a_2 \wedge (b_2 + a_1 + b_1)]\!]_s$, Lemma 28.3 ($\Theta$-expansion I) along with Lemma 28.4 ($\Theta$-linearity) implies that it equals

$$\begin{aligned}
&[\![a_2 \wedge b_2, a_2 \wedge b_2]\!]_s + 2\Theta[a_2 \wedge b_2, a_2 \wedge (a_1 + b_1)]_s + [\![a_2 \wedge (a_1 + b_1), a_2 \wedge (a_1 + b_1)]\!]_s \\
\equiv&2\Theta[a_2 \wedge b_2, a_2 \wedge a_1]_s + 2\Theta[a_2 \wedge b_2, a_2 \wedge b_1]_s + X_1([\![a_2 \wedge a_1, a_2 \wedge a_1]\!]_s) \equiv 0.
\end{aligned}$$

The last $\equiv$ uses the fact that we have already proved that $X_1(t) \equiv 0$ for $t \in S_1$ (Lemma 32.1). For the second term $[\![a_2 \wedge (b_2 + a_1), a_2 \wedge (b_2 + a_1)]\!]_s$, Lemma 28.3 ($\Theta$-expansion I) implies that it equals

$$[\![a_2 \wedge b_2]\!]_s + 2\Theta[a_2 \wedge b_2, a_2 \wedge a_1]_s + [\![a_2 \wedge a_1, a_2 \wedge a_1]\!]_s \equiv 0. \qquad \square$$

## 35. Symmetric kernel, symmetric version IX: closure of $S_4$

We continue using all the notation from §27 – §31. In this section, we complete the proof of Lemma 31.3 (and hence also of Theorem G) by proving that for $f \in \{X_1, X_1^{-1}, Y_{12}\}$ and $s \in S_4$, we have $f(s) \in \langle S \rangle$.

35.1. **X-closure.** Recall that $X_1 \in \mathrm{Sp}_{2g}(\mathbb{Z})$ takes $a_1$ to $a_1 + b_1$ and fixes all other generators in $\mathcal{B}$. We start with:

**Lemma 35.1.** *Let*

$$s \in S_4 = \{\Omega[a_i \wedge a_j, b_i \wedge b_j]_s,\ \Omega[a_i \wedge b_j, b_i \wedge a_j]_s \mid 1 \leq i < j \leq g\}$$

*Then $X_1^{\epsilon}(s) \in \langle S \rangle$ for $\epsilon \in \{\pm 1\}$.*

*Proof.* The lemma is trivial if $X_1$ fixes $s$. The remaining cases are as follows. Let $\equiv$ denote equality modulo $\langle S \rangle$, so our goal is to prove that $X_1^{\epsilon}(s) \equiv 0$.

**Case 1.** $s = \Omega[a_1 \wedge a_j, b_1 \wedge b_j]_s$ *with* $2 \leq j \leq g$.

Since $g \geq 4$ (Assumption 25.1), we can pick $2 \leq k \leq g$ with $k \neq j$. Lemma 30.4 ($\Lambda$ to $\Omega$) implies that $\Omega[a_1 \wedge a_j, b_1 \wedge b_j]_s = -\Omega[a_j \wedge a_1, b_1 \wedge b_j]_s$ equals

$$\begin{aligned}
&\Lambda[a_j \wedge (a_k + a_1), (b_k - b_1) \wedge b_j]_s - \Lambda[a_j \wedge a_1, b_k \wedge b_j]_s \\
&+ \Lambda[a_j \wedge a_k, b_1 \wedge b_j]_s - \Omega[a_j \wedge a_k, b_k \wedge b_j]_s
\end{aligned}$$

Applying $X_1^{\epsilon}$, since $X_1$ fixes $\{a_j, b_j, a_k, b_k\}$ we get

$$\begin{aligned}
&\Lambda[a_j \wedge (a_k + a_1 + \epsilon b_1), (b_k - b_1) \wedge b_j]_s - X_1^{\epsilon}(\Lambda[a_j \wedge a_1, b_k \wedge b_j]_s) \\
&+ X_1^{\epsilon}(\Lambda[a_j \wedge a_k, b_1 \wedge b_j]_s) - \Omega[a_j \wedge a_k, b_k \wedge b_j]_s \\
\equiv&\Lambda[a_j \wedge (a_k + a_1 + \epsilon b_1), (b_k - b_1) \wedge b_j]_s.
\end{aligned}$$

Here we are using the fact that $X_1^{\epsilon}(t) \equiv 0$ for all $t \in S_3$ (Lemma 34.2). Lemma 34.1 implies that $\Lambda[a_j \wedge (a_k + a_1 + \epsilon b_1), (b_k - b_1) \wedge b_j]_s \equiv 0$, and we are done.

**Case 2.** $s = \Omega[a_1 \wedge b_j, b_1 \wedge a_j]_s$ *with* $2 \leq j \leq g$.

Since $g \geq 4$ (Assumption 25.1), we can pick $2 \leq k \leq g$ with $k \neq j$. Lemma 30.4 ($\Lambda$ to $\Omega$) implies that $\Omega[a_1 \wedge b_j, b_1 \wedge a_j]_s = \Omega[(-b_j) \wedge a_1, b_1 \wedge a_j]_s$ equals

$$\begin{aligned} &\Lambda[(-b_j) \wedge (a_1 + a_k), (b_1 - b_k) \wedge a_j]_s - \Lambda[(-b_j) \wedge a_k, b_1 \wedge (a_j)]_s \\ &+ \Lambda[(-b_j) \wedge a_1, b_k \wedge a_j]_s + \Omega[(-b_j) \wedge a_k, b_k \wedge a_j]_s \\ = &- \Lambda[b_j \wedge (a_1 + a_k), (b_1 - b_k) \wedge a_j]_s + \Lambda[b_j \wedge a_k, b_1 \wedge (a_j)]_s \\ &- \Lambda[b_j \wedge a_1, b_k \wedge a_j]_s + \Omega[a_k \wedge b_j, b_k \wedge a_j]_s. \end{aligned}$$

Now proceed as in Case 1. □

35.2. **Y-closure.** Recall that $Y_{12} \in \mathrm{Sp}_{2g}(\mathbb{Z})$ takes $a_1$ to $a_1 + b_2$ and $a_2$ to $a_2 + b_1$ and fixes all other generators in $\mathcal{B}$. We next prove the following, which completes the proof of Lemma 31.3 and hence of Theorem G:

**Lemma 35.2.** *Let*

$$s \in S_4 = \{\Omega[a_i \wedge a_j, b_i \wedge b_j]_s,\ \Omega[a_i \wedge b_j, b_i \wedge a_j]_s \mid 1 \leq i < j \leq g\}$$

*Then* $Y_{12}(s) \in \langle S \rangle$.

*Proof.* Write $S_4 = S_4(1) \cup S_4(2) \cup S_4(3) \cup S_4(4)$ with

$$\begin{aligned} S_4(1) &= \{\Omega[a_i \wedge a_j, b_i \wedge b_j]_s,\ \Omega[a_i \wedge b_j, b_i \wedge a_j]_s \mid 3 \leq i < j \leq g\}, \\ S_4(2) &= \{\Omega[a_1 \wedge a_j, b_1 \wedge b_j]_s,\ \Omega[a_1 \wedge b_j, b_1 \wedge a_j]_s \mid 3 \leq j \leq g\}, \\ S_4(3) &= \{\Omega[a_2 \wedge a_j, b_2 \wedge b_j]_s,\ \Omega[a_2 \wedge b_j, b_2 \wedge a_j]_s \mid 3 \leq j \leq g\}, \\ S_4(4) &= \{\Omega[a_1 \wedge a_2, b_1 \wedge b_2]_s, \Omega[a_1 \wedge b_2, b_1 \wedge a_2]_s\}. \end{aligned}$$

The lemma is trivial for $s \in S_4(1)$ since in that case $Y_{12}(s) = s$. For $s \in S_4(2)$, the lemma can be proved exactly like Case 1 of the proof of Lemma 35.1. The only necessary change is that the $k$ in that proof should be chosen such that $3 \leq k \leq g$ and $k \neq j$, which is possible since $g \geq 4$ (Assumption 25.1; note that in Case 1 of the proof of Lemma 35.1 we really only used $g \geq 3$). The same argument works for $s \in S_4(3)$.

It remains to deal with $S_4(4)$, which we divide into two cases. Let $\equiv$ denote equality modulo $\langle S \rangle$, so our goal is to prove that $Y_{12}(s) \equiv 0$.

**Case 1.** $s = \Omega[a_1 \wedge a_2, b_1 \wedge b_2]_s$.

Set $s' = [\![(a_1 - b_2) \wedge (b_1 - a_2), (a_1 - b_2) \wedge (b_1 - a_2)]\!]_s$. In Lemma 30.7, we proved that $-2s$ equals $s'$ modulo $\langle S_1, S_2, S_3 \rangle$. We have already proved that $Y_{12}(t) \equiv 0$ for $t \in S_1 \cup S_2 \cup S_3$; see Lemmas 32.3 and 33.2 and 34.3. It is therefore enough to prove that $Y_{12}(s') \equiv 0$:

$$Y_{12}([\![(a_1 - b_2) \wedge (b_1 - a_2), (a_1 - b_2) \wedge (b_1 - a_2)]\!]_s) = [\![a_1 \wedge (-a_2), a_1 \wedge (-a_2)]\!]_s \equiv 0.$$

**Case 2.** $s = \Omega[a_1 \wedge b_2, b_1 \wedge a_2]_s$.

Set $s' = [\![(a_1 + a_2) \wedge (b_1 - b_2), (a_1 + a_2) \wedge (b_1 - b_2)]\!]_s$. In Lemma 30.7, we proved that $s$ equals $s'$ modulo $\langle S_1, S_2, S_3 \rangle$. Just like in the previous case, this implies that it is enough to prove that $Y_{12}(s') \equiv 0$. We calculate:

$$\begin{aligned} &Y_{12}([\![(a_1 + a_2) \wedge (b_1 - b_2), (a_1 + a_2) \wedge (b_1 - b_2)]\!]_s) \\ =&[\![(a_1 + b_2 + a_2 + b_1) \wedge (b_1 - b_2), (a_1 + b_2 + a_2 + b_1) \wedge (b_1 - b_2)]\!]_s \\ =&[\![(a_1 + a_2 + 2b_1) \wedge (b_1 - b_2), (a_1 + a_2 + 2b_1) \wedge (b_1 - b_2)]\!]_s = X_1^2(s'). \end{aligned}$$

In Lemmas 32.1 and 33.1 and 34.2 and 35.1, we proved that $X_1^{\epsilon}(t) \in \langle S \rangle$ for $\epsilon \in \{\pm 1\}$ and $s \in S_1 \cup \cdots \cup S_4 = S$. This implies that the cyclic group generated by $X_1$ takes $\langle S \rangle$ to $\langle S \rangle$. Lemma 30.7 implies that $s' \in S$, so $X_1^2(s') \equiv 0$, as desired. □

# Part 5. Appendices

## Appendix A. A modified presentation

The goal of this appendix is to transform the presentation for $\mathfrak{K}_g$ from Definition 10.5 to the one needed for our work on the Torelli group in [4, 5]. Recall from §1.13 that $H = \mathbb{Q}^{2g}$ and $H_{\mathbb{Z}} = \mathbb{Z}^{2g}$, and that $\omega\colon H \times H \to \mathbb{Q}$ is the standard symplectic form on $H$.

A.1. **Symmetric kernel and contraction.** Recall from the introduction that the symmetric contraction is the alternating bilinear map

$$\mathfrak{c}\colon ((\wedge^2 H)/\mathbb{Q}) \times ((\wedge^2 H)/\mathbb{Q}) \longrightarrow \mathrm{Sym}^2(H)$$

defined by the formula

$$\mathfrak{c}(x \wedge y, z \wedge w) = \omega(x,z) y{\cdot}w - \omega(x,w) y{\cdot}z - \omega(y,z) x{\cdot}w + \omega(y,w) x{\cdot}z \text{ for } x,y,z,w \in H.$$

This induces a map $((\wedge^2 H)/\mathbb{Q})^{\otimes 2} \longrightarrow \mathrm{Sym}^2(H)$ whose kernel $\mathcal{K}_g$ is the symmetric kernel. Recall that $\kappa_1, \kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ are sym-orthogonal if

$$\mathfrak{c}(\kappa_1, \kappa_2) = -\mathfrak{c}(\kappa_2, \kappa_1) = 0,$$

or equivalently if $\kappa_1 \otimes \kappa_2$ and $\kappa_2 \otimes \kappa_1$ lie in $\mathcal{K}_g$.

A.2. **Presentation.** We recall the definition of $\mathfrak{K}_g$:

**Definition A.1.** Define $\mathfrak{K}_g$ to be the $\mathbb{Q}$-vector space with the following presentation:

- **Generators**. A generator $[\![\kappa_1, \kappa_2]\!]$ for all sym-orthogonal $\kappa_1, \kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ such that either $\kappa_1$ or $\kappa_2$ (or both) is a symplectic pair in $(\wedge^2 H_{\mathbb{Z}})/\mathbb{Z}$.
- **Relations**. For all symplectic pairs $a \wedge b \in (\wedge^2 H_{\mathbb{Z}})/\mathbb{Z}$ and all $\kappa_1, \kappa_2 \in (\wedge^2 H)/\mathbb{Q}$ that are sym-orthogonal to $a \wedge b$ and all $\lambda_1, \lambda_2 \in \mathbb{Q}$, the relations

$$[\![a \wedge b, \lambda_1 \kappa_1 + \lambda_2 \kappa_2]\!] = \lambda_1 [\![a \wedge b, \kappa_1]\!] + \lambda_2 [\![a \wedge b, \kappa_2]\!] \quad \text{and}$$
$$[\![\lambda_1 \kappa_1 + \lambda_2 \kappa_2, a \wedge b]\!] = \lambda_1 [\![\kappa_1, a \wedge b]\!] + \lambda_2 [\![\kappa_2, a \wedge b]\!]. \qquad \square$$

There is a linearization map $\Phi\colon \mathfrak{K}_g \to ((\wedge^2 H)/\mathbb{Q})^{\otimes 2}$ defined by $\Phi([\![\kappa_1, \kappa_2]\!]) = \kappa_1 \otimes \kappa_2$. This takes relations to relations, and thus gives a well-defined map. Since $\kappa_1$ and $\kappa_2$ are sym-orthogonal, the image of $\Phi$ is contained in $\mathcal{K}_g$. Theorem 10.7 says that $\Phi$ is an isomorphism for $g \geq 4$.

A.3. **Symplectic summands.** As we said above, our goal is to modify the presentation of $\mathfrak{K}_g$ to the one needed for our papers [4, 5]. This requires some preliminaries. A *symplectic summand* of $H_{\mathbb{Z}}$ is a subgroup $V < H_{\mathbb{Z}}$ such that $H_{\mathbb{Z}} = V \oplus V^{\perp}$. A symplectic summand $V$ of $H_{\mathbb{Z}}$ is isomorphic to $\mathbb{Z}^{2h}$ for some $h$ called its *genus*. If $V$ is a symplectic summand of $H_{\mathbb{Z}}$, then $V^{\perp}$ is too.

A.4. **Symplectic form.** Let $W$ be a symplectic summand of $H_{\mathbb{Z}}$. The symplectic form on $W$ identifies $W$ with its dual. This allows us to identify alternating bilinear forms on $W$ with elements of $\wedge^2 W \subset \wedge^2 H_{\mathbb{Z}}$. In particular, the symplectic form on $W$ is an element $\omega_W$ of $\wedge^2 H_{\mathbb{Z}}$. If $\{a_1, b_1, \ldots, a_h, b_h\}$ is a symplectic basis for $W$, then $\omega_W = a_1 \wedge b_1 + \cdots + a_h \wedge b_h$. With this notation, the symplectic form $\omega$ on $H_{\mathbb{Z}}$ is $\omega = \omega_{H_{\mathbb{Z}}}$.

A.5. **Symplectic pairs.** Recall that $(\wedge^2 H)/\mathbb{Q}$ is the quotient of $\wedge^2 H$ by the $\mathbb{Q}$-span of $\omega \in \wedge^2 H_{\mathbb{Z}}$. For a symplectic summand $W$ of $H_{\mathbb{Z}}$, let $\overline{\omega}_W$ be the image of $\omega_W \in \wedge^2 H_{\mathbb{Z}}$ in $(\wedge^2 H)/\mathbb{Q}$. Since $\omega = \omega_W + \omega_{W^{\perp}}$, we have $\overline{\omega}_{W^{\perp}} = -\overline{\omega}_W$. The elements $\overline{\omega}_W$ with $W$ a genus-1 symplectic summand of $H_{\mathbb{Z}}$ are exactly the symplectic pairs.

A.6. **Generators and summands.** Recall that the generators for $\mathfrak{K}_g$ are as follows:

- Let $V$ be a genus-1 symplectic summand of $H_{\mathbb{Z}}$ and let $\kappa \in (\wedge^2 H)/\mathbb{Q}$ be sym-orthogonal to $\overline{\omega}_V$. We then have generators $[\![\overline{\omega}_V, \kappa]\!]$ and $[\![\kappa, \overline{\omega}_V]\!]$.

The following lemma says that the element $\overline{\omega}_V$ is determined by $V$:

**Lemma A.2.** *Assume that $g \geq 3$. Let $V$ and $W$ be genus-1 symplectic summands of $H_{\mathbb{Z}}$ such that $\overline{\omega}_V = \overline{\omega}_W$. Then $V = W$.*

*Proof.* Let $\omega \in \wedge^2 H$ be the element corresponding to the symplectic form. Since $\overline{\omega}_V = \overline{\omega}_W$, there exists some $\lambda \in \mathbb{Q}$ such that $\omega_V - \omega_W = \lambda\omega$. The orthogonal complements $V^\perp$ and $W^\perp$ in $H$ both have codimension 2. Since $g \geq 3$, it follows that $H$ has dimension at least 6 and hence we can find some nonzero $x \in V^\perp \cap W^\perp$. View elements of $\wedge^2 H$ as alternating bilinear forms on $H$. Since $\omega$ is a nondegenerate pairing on $H$, we can find $y \in H$ such that $\omega(x, y) = 1$. We then have

$$0 = \omega_V(x, y) - \omega_W(x, y) = \lambda\omega(x, y) = \lambda,$$

so $\omega_V = \omega_W$. The alternating bilinear forms $\omega_V$ and $\omega_W$ determine $V$ and $W$; for instance, the kernel of the form $\omega_V$ is $V^\perp$ and $V = (V^\perp)^\perp$. We conclude that $V = W$. □

A.7. **Lifting sym-orthogonal subspace.** Recall that for a subspace $U$ of $\wedge^2 H$, we denote by $\overline{U}$ the image of $U$ in $(\wedge^2 H)/\mathbb{Q}$. Also, for a subgroup $V$ of $H_{\mathbb{Z}}$ we write $V_{\mathbb{Q}}$ for the subspace $V \otimes \mathbb{Q}$ of $H$. For a genus-1 symplectic summand $V$, Lemma 10.1 says that the elements of $(\wedge^2 H)/\mathbb{Q}$ that are sym-orthogonal to $\overline{\omega}_V$ are those lying in $\overline{\wedge^2 V_{\mathbb{Q}}^\perp}$. The following lets us lift these to elements of $\wedge^2 V_{\mathbb{Q}}^\perp$:

**Lemma A.3.** *Let $V$ be a genus-1 symplectic summand of $H_{\mathbb{Z}}$. Then the map $\wedge^2 V_{\mathbb{Q}}^\perp \to \overline{\wedge^2 V_{\mathbb{Q}}^\perp}$ obtained by restricting the projection $\wedge^2 H \to (\wedge^2 H)/\mathbb{Q}$ is an isomorphism.*

*Proof.* Let $\omega \in \wedge^2 H$ be the element corresponding to the symplectic form. We must prove that $\omega \notin \wedge^2 V_{\mathbb{Q}}^\perp$. Let $\mathcal{B} = \{a_1, b_1, \ldots, a_g, b_g\}$ be a symplectic basis for $H_{\mathbb{Z}}$ such that $V = \langle a_g, b_g \rangle$. Let $\prec$ be the total order on $\mathcal{B}$ indicated in the above list. Then $\wedge^2 H$ has the basis $\{x \wedge y \mid x, y \in \mathcal{B},\ x \prec y\}$, the subspace $\wedge^2 V_{\mathbb{Q}}^\perp$ has for a basis the subset $\{x \wedge y \mid x, y \in \mathcal{B} \setminus \{a_g, b_g\},\ x \prec y\}$, and

$$\omega = a_1 \wedge b_1 + \cdots + a_g \wedge b_g.$$

Since $a_g \wedge b_g$ is a basis element *not* included in the basis for $\wedge^2 V_{\mathbb{Q}}^\perp$, the lemma follows. □

A.8. **Symplectic pairs in sym-orthogonal complement.** For a genus-1 symplectic summand $V$ of $H_{\mathbb{Z}}$, we have $\overline{\omega}_V = -\overline{\omega}_{V^\perp} \in \overline{\wedge^2 V_{\mathbb{Q}}^\perp}$. In particular, by Lemma 10.1 the element $\overline{\omega}_V$ is sym-orthogonal to itself. The following lemma says that this is the only non-obvious $\overline{\omega}_W$ contained in the sym-orthogonal complement of $\overline{\omega}_V$:

**Lemma A.4.** *Let $V$ and $W$ be genus-1 symplectic summand of $H_{\mathbb{Z}}$ such $\overline{\omega}_V$ and $\overline{\omega}_W$ are sym-orthogonal. Then either $W \subset V^\perp$ or $W = V$.*

*Proof.* Assume for the sake of contradiction that $W \neq V$ and $W \not\subset V^\perp$. Since $V$ and $W$ are both genus-1 symplectic summands of $H_{\mathbb{Z}}$, the assumption $W \neq V$ implies that $W \not\subset V$. Since $H_{\mathbb{Z}} = V \oplus V^\perp$, the assumptions that $W \not\subset V^\perp$ and $W \not\subset V$ imply that there exist nonzero $x_1 \in V$ and $x_2 \in V^\perp$ such that $x_1 + x_2 \in W$.

Since $H = V_{\mathbb{Q}} \oplus V_{\mathbb{Q}}^\perp$, we have

$$\wedge^2 H = \left(\wedge^2 V_{\mathbb{Q}}\right) \oplus \left(\wedge^2 V_{\mathbb{Q}}^\perp\right) \oplus \left(V_{\mathbb{Q}} \wedge V_{\mathbb{Q}}^\perp\right). \tag{A.1}$$

Let $\omega \in \wedge^2 H$ be the symplectic form. Both $\omega_V$ and $\omega = \omega_V + \omega_{V^\perp}$ lie in the subspace

$$\left(\wedge^2 V_{\mathbb{Q}}\right) \oplus \left(\wedge^2 V_{\mathbb{Q}}^\perp\right) \tag{A.2}$$

of (A.1). Since $\overline{\omega}_W$ is sym-orthogonal to $\overline{\omega}_V$, Lemma 10.1 says that $\overline{\omega}_W \in \overline{\wedge^2 V_{\mathbb{Q}}^\perp}$. Equivalently, modulo $\mathbb{Q}\omega$ the element $\omega_W$ lies in $\wedge^2 V_{\mathbb{Q}}^\perp$, so $\omega_W$ lies in (A.2) as well.

Recall that we have nonzero $x_1 \in V$ and $x_2 \in V^\perp$ with $x_1 + x_2 \in W$. Regard $x_1 + x_2$ as an element of $W_{\mathbb{Q}}$. Pick $y_1 \in V$ and $y_2 \in V^\perp$ such that $W_{\mathbb{Q}} = \langle x_1 + x_2, y_1 + y_2 \rangle$. We have

$$\omega_W = (x_1 + x_2) \wedge (y_1 + y_2) = x_1 \wedge y_1 + x_2 \wedge y_2 + x_1 \wedge y_2 - y_1 \wedge x_2.$$

Since $\omega_W$ lies in (A.2), we must have $x_1 \wedge y_2 = y_1 \wedge x_2$ in $V_{\mathbb{Q}} \wedge V_{\mathbb{Q}}^\perp$. Since $x_1$ and $x_2$ are nonzero, this implies that there exists some $\lambda_1, \lambda_2 \in \mathbb{Q}$ such that $y_1 = \lambda_1 x_1$ and $y_2 = \lambda_2 x_2$. Since $\omega(x_1, x_2) = 0$, we conclude that

$$\omega(x_1 + x_2, y_1 + y_2) = \omega(x_1 + x_2, \lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 \omega(x_1, x_1) + \lambda_2 \omega(x_2, x_2) = 0.$$

This implies that $\omega$ vanishes identically on $W_{\mathbb{Q}} = \langle x_1 + x_2, y_1 + y_2 \rangle$, contradicting the fact that it is a symplectic summand. □

A.9. **Modified presentation.** Define the following:

**Definition A.5.** Define $\mathfrak{K}'_g$ to be the vector space with the following presentation:

- **Generators**. For all genus-1 symplectic summands $V$ of $H_{\mathbb{Z}}$ and all $\kappa \in \wedge^2 V_{\mathbb{Q}}^\perp$, generators $[\![V, \kappa]\!]$ and $[\![\kappa, V]\!]$.
- **Relations**. The following families of relations:
  - For all genus-1 symplectic summands $V$ of $H_{\mathbb{Z}}$ and all $\kappa_1, \kappa_2 \in \wedge^2 V_{\mathbb{Q}}^\perp$ and all $\lambda_1, \lambda_2 \in \mathbb{Q}$, the relations
  $$[\![V, \lambda_1\kappa_1 + \lambda_2\kappa_2]\!] = \lambda_1 [\![V, \kappa_1]\!] + \lambda_2 [\![V, \kappa_2]\!] \quad \text{and}$$
  $$[\![\lambda_1\kappa_1 + \lambda_2\kappa_2, V]\!] = \lambda_1 [\![\kappa_1, V]\!] + \lambda_2 [\![\kappa_2, V]\!].$$
  - For all orthogonal genus-1 symplectic summands $V$ and $W$ of $H_{\mathbb{Z}}$, the relation
  $$[\![V, \omega_W]\!] = [\![\omega_V, W]\!]$$
  - For all genus-1 symplectic summands $V$ of $H_{\mathbb{Z}}$, the relation
  $$[\![V, \omega_{V^\perp}]\!] = [\![\omega_{V^\perp}, V]\!].$$ □

The actions of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $H_{\mathbb{Z}}$ and $H$ induce an action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathfrak{K}'_g$. The main result of this appendix is:

**Theorem A.6.** *For $g \geq 4$, there is an $\mathrm{Sp}_{2g}(\mathbb{Z})$-equivariant isomorphism between $\mathfrak{K}'_g$ and the symmetric kernel $\mathcal{K}_g$. In particular, $\mathfrak{K}'_g$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$.*

*Proof.* By Theorem 10.7, it is enough to construct an $\mathrm{Sp}_{2g}(\mathbb{Z})$-equivariant isomorphism from $\mathfrak{K}'_g$ to $\mathfrak{K}_g$. For $\kappa \in \wedge^2 H$, let $\overline{\kappa}$ be its image in $(\wedge^2 H)/\mathbb{Q}$. Define a map $f\colon \mathfrak{K}'_g \to \mathfrak{K}_g$ on generators via the formulas

$$f([\![V, \kappa]\!]) = [\![\overline{\omega}_V, \overline{\kappa}]\!] \quad \text{and} \quad f([\![\kappa, V]\!]) = [\![\overline{\kappa}, \overline{\omega}_V]\!].$$

This takes relations to relations; indeed, the linearity relations are obvious, and the other relations can be checked as follows:

- Consider orthogonal genus-1 symplectic summands $V$ and $W$ of $H_{\mathbb{Z}}$. We must prove that
$$f([\![V, \omega_W]\!]) = [\![\overline{\omega}_V, \overline{\omega}_W]\!] \quad \text{and} \quad f([\![\omega_V, W]\!]) = [\![\overline{\omega}_V, \overline{\omega}_W]\!]$$
are equal, which is clear.

- Consider a genus-1 symplectic summand $V$ of $H_{\mathbb{Z}}$. We must prove that
$$f([\![V, \omega_{V^\perp}]\!]) = [\![\overline{\omega}_V, \overline{\omega}_{V^\perp}]\!] \quad \text{and} \quad f([\![\omega_{V^\perp}, V]\!]) = [\![\overline{\omega}_{V^\perp}, \overline{\omega}_V]\!]$$
are equal, which follows from the calculation
$$[\![\overline{\omega}_V, \overline{\omega}_{V^\perp}]\!] = [\![\overline{\omega}_V, -\overline{\omega}_V]\!] = -[\![\overline{\omega}_V, \overline{\omega}_V]\!] = [\![-\overline{\omega}_V, \overline{\omega}_V]\!] = [\![\overline{\omega}_{V^\perp}, \overline{\omega}_V]\!].$$

This implies that $f$ is a well-defined map.

To prove that $f$ is an isomorphism, we must construct an inverse $h\colon \mathfrak{K}_g \to \mathfrak{K}'_g$. Let $V$ be a genus-1 symplectic summand of $H_{\mathbb{Z}}$ and let $\kappa \in (\wedge^2 H)/\mathbb{Q}$ be sym-orthogonal to $\overline{\omega}_V$. We must define $h$ on $[\![\overline{\omega}_V, \kappa]\!]$ and $[\![\kappa, \overline{\omega}_V]\!]$. Lemma 10.1 implies that $\kappa \in \overline{\wedge^2 V_{\mathbb{Q}}^\perp}$, and Lemma A.3 says that $\kappa$ can be uniquely lifted to $\widetilde{\kappa} \in \wedge^2 V_{\mathbb{Q}}^\perp$. Lemma A.2 says that $\overline{\omega}_V$ determines $V$, so we can define
$$h_1([\![\overline{\omega}_V, \kappa]\!]) = [\![V, \widetilde{\kappa}]\!] \quad \text{and} \quad h_2([\![\kappa, \overline{\omega}_V]\!]) = [\![\widetilde{\kappa}, V]\!].$$
The reason for distinguishing between $h_1$ and $h_2$ is that it is possible for a generator of $\mathfrak{K}_g$ to be of both of these forms. To define $h$, we must check that:

**Claim.** *Let $V$ and $W$ be genus-1 symplectic summands of $H_{\mathbb{Z}}$ such that $\overline{\omega}_V$ and $\overline{\omega}_W$ are sym-orthogonal. Then $h_1([\![\overline{\omega}_V, \overline{\omega}_W]\!]) = h_2([\![\overline{\omega}_V, \overline{\omega}_W]\!])$.*

*Proof of claim.* Since $\overline{\omega}_V$ and $\overline{\omega}_W$ are sym-orthogonal, Lemma A.4 implies that one of the following holds:

- $W \subset V^\perp$. The unique lift of $\overline{\omega}_W \in \overline{\wedge^2 V_{\mathbb{Q}}^\perp}$ to $\wedge^2 V_{\mathbb{Q}}^\perp$ is $\omega_W$, and the unique lift of $\overline{\omega}_V \in \overline{\wedge^2 W_{\mathbb{Q}}^\perp}$ is $\omega_V$. We now calculate as follows, where the orange $=$ are applications of relations in $\mathfrak{K}'_g$:
$$h_1([\![\overline{\omega}_V, \overline{\omega}_W]\!]) = [\![V, \omega_W]\!] = [\![\omega_V, W]\!] = h_2([\![\overline{\omega}_V, \overline{\omega}_W]\!]).$$
- $W = V$. The unique lift of
$$\overline{\omega}_V = -\overline{\omega}_{V^\perp} \in \overline{\wedge^2 V_{\mathbb{Q}}^\perp}$$
to $\wedge^2 V_{\mathbb{Q}}^\perp$ is $-\omega_{V^\perp}$. We now calculate as follows, where the orange $=$ are applications of relations in $\mathfrak{K}'_g$:
$$\begin{aligned} h_1([\![\overline{\omega}_V, \overline{\omega}_V]\!]) &= [\![V, -\omega_{V^\perp}]\!] = -[\![V, \omega_{V^\perp}]\!] \\ &= -[\![\omega_{V^\perp}, V]\!] = [\![-\omega_{V^\perp}, V]\!] = h_2([\![\overline{\omega}_V, \overline{\omega}_V]\!]). \end{aligned}$$ □

In light of this claim, we can unambiguously define a map $h\colon \mathfrak{K}_g \to \mathfrak{K}'_g$ on generators $[\![\kappa_1, \kappa_2]\!]$ by letting $h([\![\kappa_1, \kappa_2]\!])$ equal whichever one of $h_1([\![\kappa_1, \kappa_2]\!])$ or $h_2([\![\kappa_1, \kappa_2]\!])$ is defined. The map $h$ takes relations to relations, and thus gives a well-defined map. By construction, $f$ and $h$ are inverses to each other. The theorem follows. □

*Remark* A.7. The isomorphism $\mathfrak{K}'_g \to \mathcal{K}_g$ takes $[\![V, \kappa]\!]$ to $\overline{\omega}_V \otimes \overline{\kappa}$ and $[\![\kappa, V]\!]$ to $\overline{\kappa} \otimes \overline{\omega}_V$. □

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