

THE RATIONAL COHOMOLOGY OF THE MAPPING CLASS GROUP VANISHES IN ITS VIRTUAL COHOMOLOGICAL DIMENSION

THOMAS CHURCH, BENSON FARB AND ANDREW PUTMAN

ABSTRACT. Let Mod_g be the mapping class group of a genus $g \geq 2$ surface. The group Mod_g has virtual cohomological dimension $4g - 5$. In this note we use a theorem of Broaddus and the combinatorics of chord diagrams to prove that $H^{4g-5}(\text{Mod}_g; \mathbb{Q}) = 0$.

1. INTRODUCTION

Let Mod_g be the mapping class group of a closed, oriented, genus $g \geq 2$ surface, and let \mathcal{M}_g be the moduli space of genus g Riemann surfaces. It is well-known that for each $i \geq 0$,

$$H^i(\text{Mod}_g; \mathbb{Q}) \cong H^i(\mathcal{M}_g; \mathbb{Q}).$$

It is a fundamental open problem to determine the maximal i for which these vector spaces are nonzero. Harer [Ha] proved that the *virtual cohomological dimension* $\text{vcd}(\text{Mod}_g)$ equals $4g - 5$. More precisely, he proved that $H^{4g-5}(\text{Mod}_g; \text{St}_g \otimes \mathbb{Q}) \neq 0$ for a certain Mod_g -module St_g (see below for details) and that $H^i(\text{Mod}_g; V \otimes \mathbb{Q}) = 0$ for all $i > 4g - 5$ and all Mod_g -modules V . Thus the first step of the problem above is to determine whether $H^{4g-5}(\text{Mod}_g; \mathbb{Q}) \neq 0$. The purpose of this note is to answer this question.

Let $\text{Mod}_{g,*}$ (resp. $\text{Mod}_{g,1}$) denote the mapping class group of the genus g surface with one marked point (resp. one boundary component).

Theorem 1. *For any $g \geq 2$,*

$$H^{4g-5}(\text{Mod}_g; \mathbb{Q}) = H^{4g-5}(\mathcal{M}_g; \mathbb{Q}) = 0.$$

Further, the rational cohomology of $\text{Mod}_{g,}$ (resp. the integral cohomology of $\text{Mod}_{g,1}$) vanishes in its virtual cohomological dimension.*

This theorem was announced some years ago by Harer, but he has informed us that his proof will not appear. We recently learned that Morita–Sakasai–Suzuki [MSS] have independently found a proof of Theorem 1 using a completely different method. They apply a theorem of Kontsevich on graph homology to their computation of a generating set for a certain symplectic Lie algebra. Our proof combines some results about the combinatorics of chord diagrams with the work of Broaddus [Br] on the Steinberg module of Mod_g . We thank Allen Hatcher and Takuya Sakasai for their comments on an earlier version of this paper, and John Harer for informing us about the paper [MSS] and his own work.

Theorem 1 is consistent with the well-studied analogy between mapping class groups and arithmetic groups. For example, Theorem 1.3 of Lee–Szczarba [LS] states that the rational cohomology of $\text{SL}(n, \mathbb{Z})$ vanishes in its cohomological dimension.

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2. BACKGROUND

We begin by briefly summarizing previous results that make our computation possible; for details see Broaddus [Br].

Teichmüller space and its boundary. Let S_g be a connected, closed orientable surface of genus $g \geq 2$. Let \mathcal{C}_g be the *curve complex* of S_g defined by Harvey [Harv], i.e. the flag complex whose k -simplices are the $(k + 1)$ -tuples of distinct free homotopy classes of simple closed curves in S_g that can be realized disjointly. Harer [Ha] proved that \mathcal{C}_g is homotopy equivalent to a wedge of spheres $\bigvee_{i=1}^{\infty} S^{2g-2}$.

There exists a constant $\delta > 0$ such that any two closed geodesics on a hyperbolic surface of length $\leq \delta$ are disjoint (the *Margulis constant* for hyperbolic surfaces). Let $\mathcal{T}_g^{\text{thick}}$ be the Teichmüller space of marked hyperbolic surfaces diffeomorphic to S_g having no closed geodesic of length $< \delta$. It is known that $\mathcal{T}_g^{\text{thick}}$ is a $(6g - 6)$ -dimensional manifold with corners. Ivanov [Iv] proved that $\mathcal{T}_g^{\text{thick}}$ is contractible and that its boundary $\partial\mathcal{T}_g^{\text{thick}}$ is homotopy equivalent to \mathcal{C}_g . Briefly, for each simplex σ of \mathcal{C}_g , let \mathcal{T}_σ be the subset of $\partial\mathcal{T}_g^{\text{thick}}$ consisting of surfaces where each curve in σ has length δ . Each \mathcal{T}_σ is contractible, and $\mathcal{T}_\sigma \cap \mathcal{T}_{\sigma'} = \emptyset$ unless $\sigma \cup \sigma'$ is a simplex of \mathcal{C}_g , in which case $\mathcal{T}_\sigma \cap \mathcal{T}_{\sigma'} = \mathcal{T}_{\sigma \cup \sigma'}$.

Duality in the mapping class group. The mapping class group Mod_g acts properly discontinuously on $\mathcal{T}_g^{\text{thick}}$ with finite stabilizers. Defining $\mathcal{M}_g^{\text{thick}} = \mathcal{T}_g^{\text{thick}}/\text{Mod}_g$, it follows that $H^*(\text{Mod}_g; \mathbb{Q}) \cong H^*(\mathcal{M}_g^{\text{thick}}; \mathbb{Q})$. Mumford's compactness criterion states that $\mathcal{M}_g^{\text{thick}}$ is compact. Combining this with the previous two paragraphs, the work of Bieri–Eckmann [BE, Theorem 6.2] shows that $\text{vcd}(\text{Mod}_g) = 4g - 5$ and that

$$(1) \quad H^{4g-5}(\text{Mod}_g; \mathbb{Q}) \cong H_0(\text{Mod}_g; H_{2g-2}(\mathcal{C}_g; \mathbb{Q})).$$

In fact, we can say more. Let St_g denote the *Steinberg module*, i.e. the Mod_g -module $H_{2g-2}(\mathcal{C}_g; \mathbb{Z})$. Then $\text{St}_g \otimes \mathbb{Q}$ is the rational *dualizing module* for Mod_g , meaning that

$$H^{4g-5-k}(\text{Mod}_g; M \otimes \mathbb{Q}) \cong H_k(\text{Mod}_g; M \otimes \text{St}_g \otimes \mathbb{Q})$$

for any k and any M . Moreover St_g is also the dualizing module for $\text{Mod}_{g,*}$ and $\text{Mod}_{g,1}$, which act on St_g via the natural surjections $\text{Mod}_{g,*} \rightarrow \text{Mod}_g$ and $\text{Mod}_{g,1} \rightarrow \text{Mod}_g$ [Ha]. This implies that for $\nu = \text{vcd}(\text{Mod}_{g,*}) = 4g - 3$ we have $H^{\nu-k}(\text{Mod}_{g,*}; M \otimes \mathbb{Q}) \cong H_k(\text{Mod}_{g,*}; M \otimes \text{St}_g \otimes \mathbb{Q})$. For $\text{Mod}_{g,1}$ we obtain a similar result with $\nu = \text{cd}(\text{Mod}_{g,1}) = 4g - 2$, except that since $\text{Mod}_{g,1}$ is torsion-free the result holds integrally: $H^{\nu-k}(\text{Mod}_{g,1}; M) \cong H_k(\text{Mod}_{g,1}; M \otimes \text{St}_g)$.

An alternate model for St_g . Fix a finite-volume hyperbolic metric on $S_g - \{*\}$. Another model for St_g comes from the *arc complex* \mathcal{A}_g , the flag complex whose k -simplices are the disjoint $(k + 1)$ -tuples of simple geodesics on $S_g - \{*\}$ beginning and ending at the cusp $*$. Let \mathcal{A}_g^∞ be the subcomplex consisting of collections of geodesics $\gamma_1, \dots, \gamma_{k+1}$ for which $S - \bigcup \gamma_i$ has some non-contractible component. Harer proved that \mathcal{A}_g^∞ is homotopy equivalent to \mathcal{C}_g [Ha], and that \mathcal{A}_g is contractible [Ha2] (see also [Hat]). Thus

$$\text{St}_g = H_{2g-2}(\mathcal{C}_g) \simeq H_{2g-2}(\mathcal{A}_g^\infty) \simeq H_{2g-1}(\mathcal{A}_g/\mathcal{A}_g^\infty).$$

Chord diagrams. By examining how the geodesics are arranged in a neighborhood of $*$, an $(n - 1)$ -simplex of \mathcal{A}_g can be encoded by a n -chord diagram; see [Br, §4.1]. An *ordered n -chord diagram* is an ordered sequence $U = (u_1, \dots, u_n)$, where u_i is an unordered pair of distinct points on S^1 (a *chord*) and $u_i \cap u_j = \emptyset$ if $i \neq j$. We will visually depict U by drawing arcs connecting the

points in each u_i (see Figure 1 for examples). Two ordered chord diagrams are identified if they differ by an orientation-preserving homeomorphism of the circle.

Filling systems. An unlabeled k -filling system of genus g is a $(2g+k)$ -chord diagram satisfying the conditions described in [Br, §4.1]: no chord should be parallel to another chord or to the boundary circle, and the chords should determine exactly $k+1$ boundary cycles. These conditions, which guarantee that these chords define a simplex of $\mathcal{A}_g - \mathcal{A}_g^\infty$, have the following simple combinatorial formulation. Given $U = (u_1, \dots, u_n)$, consider two permutations of the $2n$ points $u_1 \cup \dots \cup u_n$: let ω be the $2n$ -cycle which takes each point to the point immediately adjacent in the clockwise direction, while τ exchanges the two points of each chord u_i and thus is a product of n transpositions. Then a $(2g+k)$ -chord diagram is a k -filling system of genus g if $\tau \circ \omega$ has $k+1$ orbits, none of which have length 1 or 2. Finally, let t_i be the straight line in D^2 connecting the two points of u_i . Then we say that U is *disconnected* if the set $t_1 \cup \dots \cup t_n \subset D^2$ is not connected.

The chord diagram chain complex. Fix a genus g , and set $n = 2g+k$. Let \mathcal{U}_k be the free abelian group spanned by ordered k -filling systems of genus g modulo the following relation. For $\sigma \in S_n$ and $U = (u_1, \dots, u_n)$, define $\sigma \cdot U = (u_{\sigma(1)}, \dots, u_{\sigma(n)})$. We impose the relation $\sigma \cdot U = (-1)^\sigma U$. The differential $\partial : \mathcal{U}_k \rightarrow \mathcal{U}_{k-1}$ is defined as follows. Consider an ordered k -filling system $U = (u_1, \dots, u_n)$ of genus g . For $1 \leq i \leq n$, let $\partial_i U$ equal $(u_1, \dots, \hat{u}_i, \dots, u_n)$ if this is an ordered $(k-1)$ -filling system of genus g ; otherwise, let $\partial_i U = 0$. Then

$$\partial(U) = \sum_{i=1}^n (-1)^{i-1} \partial_i U.$$

Broaddus's results. We will need the following theorem of Broaddus [Br]. Recall that if Γ is a group and M is a Γ -module, then the *module of coinvariants*, denoted M_Γ , is the quotient $M/\langle g \cdot m - m \mid g \in \Gamma, m \in M \rangle$. Let X be the 0-filling system of genus g depicted in Figure 1a.

Theorem 2 (Broaddus [Br]). *For each $g \geq 0$, the following hold.*

- (i) $(\text{St}_g)_{\text{Mod}_g} \cong \mathcal{U}_0/\partial(\mathcal{U}_1)$.
- (ii) The abelian group $\mathcal{U}_0/\partial(\mathcal{U}_1)$ is spanned by the image $[X] \in \mathcal{U}_0/\partial(\mathcal{U}_1)$ of $X \in \mathcal{U}_0$.
- (iii) If v is a disconnected 0-filling system of genus g , then the image of v in $\mathcal{U}_0/\partial(\mathcal{U}_1)$ is 0.

For part (i) of Theorem 2, see [Br, Proposition 3.3] together with the remark preceding [Br, Example 4.1]; for part (ii), see [Br, Theorem 4.2]; and for part (iii), see [Br, Proposition 4.5].

3. PROOF OF THEOREM 1

For any group Γ and any Γ -module M , recall that $H_0(\Gamma; M) = M_\Gamma$. Since the actions of $\text{Mod}_{g,*}$ and $\text{Mod}_{g,1}$ on St_g factor through Mod_g , to prove Theorem 1 it suffices by (1) to show that $(\text{St}_g)_{\text{Mod}_g} = 0$. By Theorem 2(i), this is equivalent to showing that $\mathcal{U}_0/\partial(\mathcal{U}_1) = 0$.

For $v \in \mathcal{U}_0$, let $[v]$ denote the associated element of $\mathcal{U}_0/\partial(\mathcal{U}_1)$. Let $X = (x_1, \dots, x_{2g})$ be the 0-filling system depicted in Figure 1(a). By Theorem 2(ii), it is enough to show that $[X] = 0$. Let $Y = (x_1, \dots, x_{2g}, y)$ be the 1-filling system depicted in Figure 1(b). Observe that

$$\partial_1 Y = (x_2, \dots, x_{2g}, y) = (x_1, \dots, x_{2g}) = X,$$

where the second equality holds since the indicated chord diagrams differ by an orientation preserving homeomorphism of S^1 . Similarly, $\partial_{2g+1} Y = X$. Also, $\partial_2 Y = 0$ (resp. $\partial_{2g} Y = 0$) by definition,

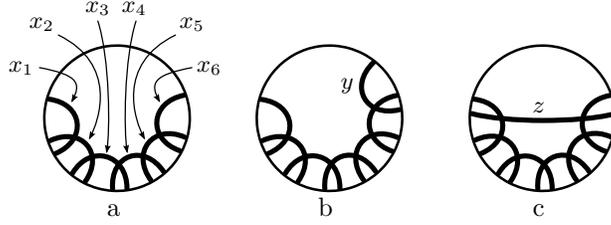


Figure 1. (a) The oriented 0-filling system $X = (x_1, \dots, x_{2g})$. For concreteness, we depict it for $g = 3$. In general, X has $2g$ chords arranged in the same pattern as the chords shown. (b) The 1-filling system $Y = (x_1, \dots, x_{2g}, y)$. The chord y intersects the chord x_{2g} . (c) The 1-filling system $Z = (z, x_1, \dots, x_{2g})$. The chord z intersects both x_1 and x_{2g} .

since the chord x_1 (resp. x_{2g+1}) becomes parallel to the boundary. We thus have

$$\partial(Y) = 2X + \sum_{i=3}^{2g-1} (-1)^{i-1} \partial_i Y.$$

For $3 \leq i \leq 2g-1$, the 0-filling system $\partial_i Y$ is disconnected, so Theorem 2(iii) implies that $[\partial_i Y] = 0$. We conclude that $2[X] = 0$.

Now consider the 1-filling system $Z = (z, x_1, \dots, x_{2g})$ depicted in Figure 1(c). Removing any chord from Figure 1(c) yields Figure 1(a) up to rotation, so $\partial_i Z = \pm X$ for each i . In fact, it is clear that $\partial_1 Z = X$, that $\partial_2 Z = -X$, that $\partial_3 Z = X$, and so on, with $\partial_i Z = (-1)^{i-1} X$. This shows that

$$\partial(Z) = X + X + \cdots + X = (2g + 1)X,$$

so $(2g + 1)[X] = 0$.

Summing up, we have shown that $2[X] = (2g + 1)[X] = 0$. This implies that $[X] = 0$, as desired.

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E-mail: tchurch@math.uchicago.edu, farb@math.uchicago.edu, andyp@rice.edu