A NEW APPROACH TO TWISTED HOMOLOGICAL STABILITY, WITH APPLICATIONS TO CONGRUENCE SUBGROUPS

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Abstract. We introduce a new method for proving twisted homological stability, and use it to prove such results for symmetric groups and general linear groups. In addition to sometimes slightly improving the stable range given by the traditional method (due to Dwyer), it is easier to adapt to nonstandard situations. As an illustration of this, we generalize to GL_n of many rings R a theorem of Borel which says that passing from GL_n of a number ring to a finite-index subgroup does not change the rational cohomology. Charney proved this generalization for trivial coefficients, and we extend it to twisted coefficients.

1. Introduction

A sequence of groups $\{G_n\}_{n=1}^{\infty}$ exhibits *homological stability* if for each $k \geq 0$, the value $\mathrm{H}_k(G_n)$ is independent of n for $n \gg k$. There is a vast literature on this, starting with early work of Nakaoka on symmetric groups [22] and unpublished work of Quillen dealing with $\mathrm{GL}_n(\mathbb{F}_q)$. Many sequences of groups exhibit homological stability: general linear groups over many rings [29], mapping class groups [16], automorphism groups of free groups [18], etc. See [28] for a general framework that encompasses many of these results, as well as a survey of the literature.

1.1. **Twisted coefficients.** Dwyer [12] showed how to extend this to homology with certain kinds of twisted coefficients. For instance, it follows from his work that if $\Bbbk$ is a field, then for all $m \geq 1$ and $k \geq 0$ we have

$$\mathrm{H}_k(\mathrm{GL}_n(\Bbbk); (\Bbbk^n)^{\otimes m}) \cong \mathrm{H}_k(\mathrm{GL}_{n+1}(\Bbbk); (\Bbbk^{n+1})^{\otimes m}) \quad \text{for } n \gg 0.$$

This has been generalized to many different groups and coefficient systems. The most general result we are aware of is in [28], which shows how to prove such a result for "polynomial coefficient systems" on many classes of groups.

1.2. **New approach.** In this paper, we give an alternate approach to twisted homological stability for polynomial coefficient systems. A small advantage of our approach is that it sometimes gives better stability ranges. More importantly, our approach is more flexible than Dwyer's, which only works well in settings where the traditional proof of homological stability applies in its simplest form.

For constant coefficients, the standard proof of homological stability can give useful information even in settings where it does not apply directly. Examples of this from the author's work include the proofs of [1, Theorem 1.1] and [27, Theorem A]. Our approach to twisted coefficients is similarly flexible. We will illustrate this by proving Theorem C below, which shows how the homology of the general linear group changes when you pass to a finite-index congruence subgroup. With twisted coefficients, this seems hard to prove using the traditional approach.

There is a tension between on the one hand developing abstract frameworks like the one in [28] that systematize the homological stability machine, and on the other hand giving

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flexible tools that can be adapted to nonstandard situations, including ones that are not necessarily about homological stability per se. Both of these goals are important. In writing this paper, we had the latter goal in mind, so our focus will be on basic examples rather than a general abstract result.

Remark 1.1. In forthcoming work [24], we will use our tools to study the stable cohomology of the moduli space of curves with level structures. □

Remark 1.2. Miller–Patzt–Petersen [20] have recently developed an approach to twisted homological stability that shares some ideas with our paper, though the technical details and intended applications are different. Unlike us, they prove a general theorem in the spirit of [28]. Our work was done independently of theirs.[1] □

1.3. **FI-modules.** The easiest example to which our results apply is the symmetric group. For these groups we will encode our twisted coefficient systems using Church–Ellenberg–Farb's theory of `FI`-modules [10]. Let `FI` be the category whose objects are finite sets and whose morphisms are injections. For a commutative ring $\Bbbk$, an `FI`-*module* over $\Bbbk$ is a functor M from `FI` to the category of $\Bbbk$-modules. For $n \geq 0$, let[2] $\overline{n} = \{1, \ldots, n\}$. Every object of `FI` is isomorphic to $\overline{n}$ for some $n \geq 0$, so the data of an `FI`-module M consists of:

- a $\Bbbk$-module $M(\overline{n})$ for each $n \geq 0$, and
- for each injection $f \colon \overline{n} \to \overline{m}$, an induced $\Bbbk$-module homomorphism $f_* \colon M(\overline{n}) \to M(\overline{m})$.

In particular, the inclusions $\overline{n} \to \overline{n+1}$ induce a sequence of morphisms

$$M(\overline{0}) \to M(\overline{1}) \to M(\overline{2}) \to \cdots . \tag{1.1}$$

The group of `FI`-automorphisms of $\overline{n}$ is the symmetric group $\mathfrak{S}_n$. This acts on $M(\overline{n})$, making $M(\overline{n})$ into a $\Bbbk[\mathfrak{S}_n]$-module. More generally, for a finite set S the group $\mathfrak{S}_S$ of bijections of S acts on $M(S)$, making $M(S)$ into a $\Bbbk[\mathfrak{S}_S]$-module.

Example 1.3. The group $\mathfrak{S}_n$ acts on $\Bbbk^n$ for each $n \geq 0$. We can fit the increasing sequence

$$\Bbbk^0 \to \Bbbk^1 \to \Bbbk^2 \to \cdots$$

of $\mathfrak{S}_n$-representations into an `FI`-module M over $\Bbbk$ by defining

$$M(S) = \Bbbk^S \quad \text{for a finite set } S.$$

Here the notation $\Bbbk^S$ means the free $\Bbbk$-module with basis S, and an injective map $f \colon S \to T$ between finite sets induces a map $f_* \colon M(S) \to M(T)$ taking basis elements to basis elements. As a $\Bbbk[\mathfrak{S}_n]$-module, we have $M(\overline{n}) = \Bbbk^{\overline{n}} \cong \Bbbk^n$. □

1.4. **Polynomial FI-modules.** For an `FI`-module M over $\Bbbk$, the inclusions (1.1) induce maps between homology groups for each k:

$$\mathrm{H}_k(\mathfrak{S}_0; M(\overline{0})) \to \mathrm{H}_k(\mathfrak{S}_1; M(\overline{1})) \to \mathrm{H}_k(\mathfrak{S}_2; M(\overline{2})) \to \cdots .$$

We would like to give conditions under which these stabilize. We start with the following. Fix a functorial coproduct $\sqcup$ on `FI`, which we think of as "disjoint union". Letting $*$ be a formal symbol, for a finite set S we then have the finite set $S \sqcup \{*\}$ of cardinality $|S| + 1$.

Definition 1.4. Let $\Bbbk$ be a commutative ring and let M be an `FI`-module over $\Bbbk$.

- The *shifted* `FI`-module of M, denoted ΣM is the `FI`-module over $\Bbbk$ defined via the formula $\Sigma M(S) = M(S \sqcup \{*\})$ for a finite set S.

[1] See [20, Remark 1.2]. We apologize for taking so long to write our approach up.

[2] It is more common to write $[n]$ for $\{1, \ldots, n\}$, but later on when discussing semisimplicial sets we will need to use the notation $[n]$ for $\{0, \ldots, n\}$.

- The *derived* $\mathtt{FI}$-module of M, denoted DM, is the $\mathtt{FI}$-module over $\Bbbk$ defined via the formula
$$DM(S) = \frac{M(S \sqcup \{*\})}{\mathrm{Im}(M(S) \to M(S \sqcup \{*\}))} \quad \text{for a finite set } S. \qquad \square$$

Remark 1.5. Morphisms between $\mathtt{FI}$-modules over $\Bbbk$ are natural transformations between functors. This makes the collection of $\mathtt{FI}$-modules over $\Bbbk$ into an abelian category, where kernels and cokernel are computed pointwise. With these conventions, there is a morphism $M \to \Sigma M$, and $DM = \mathrm{coker}(M \to \Sigma M)$. $\square$

The idea behind the following definition goes back to Dwyer's work [12], and has been elaborated upon by many people; see [28] for a more complete history.

Definition 1.6. Let $\Bbbk$ be a commutative ring and let M be an $\mathtt{FI}$-module over $\Bbbk$. We say that M is *polynomial* of degree $d \geq -1$ starting at $m \in \mathbb{Z}$ if it satisfies the following inductive condition:

- If $d = -1$, then for finite sets S with $|S| \geq m$ we require $M(S) = 0$.
- If $d \geq 0$, then we require the following two conditions:
  - For all injective maps $f\colon S \to T$ between finite sets with $|S| \geq m$, the induced map $f_*\colon M(S) \to M(T)$ must be an injection.
  - The derived $\mathtt{FI}$-module DM must be polynomial of degree $(d-1)$ starting at $(m-1)$. $\square$

Remark 1.7. If the $\mathtt{FI}$-module M over $\Bbbk$ is polynomial of degree d starting at $m \in \mathbb{Z}$, then for all finite sets S with $|S| \geq m$ we have a short exact sequence
$$0 \longrightarrow M(S) \longrightarrow \Sigma M(S) \longrightarrow DM(S) \longrightarrow 0 \tag{1.2}$$
of $\Bbbk[\mathfrak{S}_S]$-modules. $\square$

Remark 1.8. There is also a notion of a polynomial $\mathtt{FI}$-module being *split*, which can slightly improve the bounds on where stability begins. See [28] for the definition. To avoid complicating our arguments, we will not incorporate this into our results. $\square$

Example 1.9. An $\mathtt{FI}$-module M over $\Bbbk$ is polynomial of degree 0 starting at $m \in \mathbb{Z}$ if it satisfies the following two conditions:

- For all injective maps $f\colon S \to T$ between finite sets with $|S| \geq m$, the induced map $f_*\colon M(S) \to M(T)$ must be an injection.
- For all finite sets S of with $|S| \geq m-1$, we must have
$$DM(S) = \frac{M(S \sqcup \{*\})}{\mathrm{Im}(M(S) \to M(S \sqcup \{*\}))} = 0.$$
In other words, the map $M(S) \to M(S \sqcup \{*\})$ must be surjective. This implies more generally that for all injective maps $f\colon S \to T$ between finite sets with $|S| \geq m-1$, the induced map $f_*\colon M(S) \to M(T)$ must be a surjection.

Combining these two facts, we see that M is polynomial of degree 0 starting at $m \in \mathbb{Z}$ if for all injective maps $f\colon S \to T$ between finite sets, the map $f_*\colon M(S) \to M(T)$ is an isomorphism if $|S| \geq m$ and a surjection if $|S| = m-1$. $\square$

Example 1.10. The $\mathtt{FI}$-module M in Example 1.3 is polynomial of degree 1 starting at 0. More generally, for $d \geq 1$ we can define an $\mathtt{FI}$-module $M^{\otimes d}$ via the formula
$$M^{\otimes d}(S) = \left(\Bbbk^S\right)^{\otimes d} \quad \text{for a finite set } S.$$
The $\mathtt{FI}$-module $M^{\otimes d}$ is polynomial of degree d starting at 0. $\square$

Example 1.11. There is a natural notion of an FI-module being generated and related in finite degree (see [10] for the definition). As was observed in [28, Example 4.18], it follows from work of Church–Ellenberg [9] that if M is generated in degree d and related in degree r, then M is polynomial of degree d starting at $r + \min(d, r)$. We remark that though $r + \min(d, r)$ is sharp in general, in practice many such $\mathtt{FI}$-modules are polynomial of degree d starting far earlier than $r + \min(d, r)$, often at 0. □

1.5. **Symmetric groups.** Our main theorem about the symmetric group is as follows:

Theorem A. *Let $\Bbbk$ be a commutative ring and let M be an $\mathtt{FI}$-module over $\Bbbk$ that is polynomial of degree d starting at $m \geq 0$. For each $k \geq 0$, the map*

$$\mathrm{H}_k(\mathfrak{S}_n; M(\overline{n})) \to \mathrm{H}_k(\mathfrak{S}_{n+1}; M(\overline{n+1}))$$

is an isomorphism for $n \geq 2k + \max(d, m-1) + 2$ and a surjection for $n = 2k + \max(d, m-1) + 1$.

In the key case $m = 0$, the stabilization map is thus an isomorphism for $n \geq 2k + d + 2$ and a surjection for $n = 2k + d + 1$. Theorem A is the natural output of the machine we develop, but for $m \gg 0$ the following result sometimes gives better bounds (though often worse ones):

Theorem A′. *Let $\Bbbk$ be a commutative ring and let M be an $\mathtt{FI}$-module over $\Bbbk$ that is polynomial of degree d starting at $m \geq 0$. For each $k \geq 0$, the map*

$$\mathrm{H}_k(\mathfrak{S}_n; M(\overline{n})) \to \mathrm{H}_k(\mathfrak{S}_{n+1}; M(\overline{n+1})) \tag{1.3}$$

is an isomorphism for $n \geq \max(m, 2k + 2d + 2)$ and a surjection for $n \geq \max(m, 2k + 2d)$.

Remark 1.12. Theorem A′ will be derived from Theorem A using (1.2).[3] □

The first twisted homological stability result for the symmetric group was proved by Betley [3], who only considered split coefficient systems starting at $m = 0$ (c.f. Remark 1.8) but proved that (1.3) is an isomorphism for $n \geq 2k + d$. Randal-Williams–Wahl [28, Theorem 5.1] showed how to deal with general polynomial coefficients, but in this level of generality could only prove that (1.3) is an isomorphism for $n \geq \max(2m + 1, 2k + 2d + 2)$, though in the split case they could reduce this to $n \geq \max(m + 1, 2k + d + 2)$.

1.6. **VIC-modules.** We now turn to general linear groups. Let R be a ring, possibly noncommutative. To encode our coefficient systems on $\mathrm{GL}_n(R)$, we will use $\mathtt{VIC}(R)$-modules, which were introduced by the author and Sam [25, 26]. Define $\mathtt{VIC}(R)$ to be the following category:

- The objects of $\mathtt{VIC}(R)$ are finite-rank free right R-modules.[4] For a finite-rank free right R-module A, we will sometimes write $[A]$ for the associated object of $\mathtt{VIC}(R)$ to clarify our statements.
- For finite-rank free right R-modules A_1 and A_2, a $\mathtt{VIC}(R)$-morphism $[A_1] \to [A_2]$ is a pair (f, C), where $f\colon A_1 \to A_2$ is an injection and $C \subset A_2$ is a submodule[5] such that $A_2 = f(A_1) \oplus C$. The composition of $\mathtt{VIC}(R)$-morphisms $(f_1, C_1)\colon [A_1] \to [A_2]$ and $(f_2, C_2)\colon [A_2] \to [A_3]$ is the $\mathtt{VIC}(R)$-morphism $(f_2 \circ f_1, C_2 \oplus f_1(C_1))\colon [A_1] \to [A_3]$.

[3] Plus the slightly better stability range for constant coefficients originally proved by Nakaoka [22].

[4] We use right R-modules since this ensures that $\mathrm{GL}_n(R)$ acts on R^n on the left.

[5] It would also be reasonable to require C to be free with $\mathrm{rk}(C) = \mathrm{rk}(A_2) - \mathrm{rk}(A_1)$. We will not make this restriction. For the rings we consider, this frequently holds automatically, at least when $\mathrm{rk}(A_2) - \mathrm{rk}(A_1)$ is not too large (see Lemma 8.9).

For a commutative ring $\Bbbk$, a $\mathtt{VIC}(R)$-module over $\Bbbk$ is a functor M from $\mathtt{VIC}(R)$ to the category of $\Bbbk$-modules. Every object of $\mathtt{VIC}(R)$ is isomorphic to R^n for some $n \geq 0$, so the data of a $\mathtt{VIC}(R)$-module M consists of:

- a $\Bbbk$-module $M(R^n)$ for each $n \geq 0$, and
- for each $\mathtt{VIC}(R)$-morphism $(f, C)\colon [R^n] \to [R^m]$, an induced $\Bbbk$-module homomorphism $(f, C)_*\colon M(R^n) \to M(R^m)$.

For each n, let $\iota_n\colon R^n \to R^{n+1}$ be the inclusion into the first n coordinates. We then have a $\mathtt{VIC}(R)$-morphism $(\iota_n, 0 \oplus R)\colon [R^n] \to [R^{n+1}]$. These induce a sequence of morphisms

$$M(R^0) \to M(R^1) \to M(R^2) \to \cdots. \tag{1.4}$$

The group of $\mathtt{VIC}(R)$-automorphisms of $[R^n]$ is $\mathrm{GL}_n(R)$. This acts on $M(R^n)$, making $M(R^n)$ into a $\Bbbk[\mathrm{GL}_n(R)]$-module. More generally, for a finite-rank free right R-module A the group $\mathrm{GL}(A)$ acts on $M(A)$.

Example 1.13. We can fit the increasing sequence

$$\mathbb{Z}^0 \to \mathbb{Z}^1 \to \mathbb{Z}^2 \to \cdots$$

of $\mathrm{GL}_n(\mathbb{Z})$-representations into a $\mathtt{VIC}(\mathbb{Z})$-module M over $\mathbb{Z}$ by defining

$$M(A) = A \quad \text{for a finite-rank free } \mathbb{Z}\text{-module } A.$$

For a $\mathtt{VIC}(\mathbb{Z})$-morphism $(f, C)\colon [A_1] \to [A_2]$, the associated map $(f, C)_*\colon M(A_1) \to M(A_2)$ is simply f. □

Example 1.14. Example 1.13 can be generalized as follows. Let R be a ring, let $\Bbbk$ be a commutative ring, let V be a $\Bbbk$-module, and let $\lambda\colon R \to \mathrm{End}_{\Bbbk}(V)$ be a ring homomorphism. Example 1.13 will correspond to $R = \Bbbk = V = \mathbb{Z}$ and

$$\lambda(r)(v) = rv \quad \text{for } r \in R = \mathbb{Z} \text{ and } v \in V = \mathbb{Z}.$$

Another example would be $R = \Bbbk[G]$ for a group G and V a representation of G over $\Bbbk$. For each $n \geq 0$, the ring homomorphism λ induces a group homomorphism

$$\mathrm{GL}_n(R) \to \mathrm{GL}_n(\mathrm{End}_{\Bbbk}(V)),$$

endowing the $\Bbbk$-module $V^{\oplus n}$ with the structure of a $\Bbbk[\mathrm{GL}_n(R)]$-module. We can fit the increasing sequence

$$V^{\oplus 0} \to V^{\oplus 1} \to V^{\oplus 2} \to \cdots$$

of $\mathrm{GL}_n(R)$-representations into a $\mathtt{VIC}(R)$-module M by defining

$$M(A) = A \otimes_R V \quad \text{for a finite-rank free } R\text{-module } A.$$

Here we use λ to regard V as a left R-module. For a $\mathtt{VIC}(R)$-morphism $(f, C)\colon [A_1] \to [A_2]$, the induced map $(f, C)_*\colon M(A_1) \to M(A_2)$ is $f \otimes \mathrm{id}$. As a $\Bbbk[\mathrm{GL}_n(R)]$-module, we have

$$M(R^n) = R^n \otimes_R V = V^{\oplus n}.$$

□

Example 1.15. In Examples 1.13 and 1.14, the C in a $\mathtt{VIC}(R)$-morphism played no role. For an easy example of a $\mathtt{VIC}(R)$-module where it is important, consider the dual M^* of the $\mathtt{VIC}(\mathbb{Z})$-module M over $\mathbb{Z}$ from Example 1.13:

$$M^*(A) = \mathrm{Hom}(A, \mathbb{Z}) \quad \text{for a finite-rank free } \mathbb{Z}\text{-module } A.$$

For a $\mathtt{VIC}(\mathbb{Z})$-morphism $(f, C)\colon [A_1] \to [A_2]$, the associated map $(f, C)_*\colon M^*(A_1) \to M^*(A_2)$ takes $\phi \in \mathrm{Hom}(A_1, \mathbb{Z})$ to the composition

$$A_2 \xrightarrow{/C} f(A_1) \xrightarrow{f^{-1}} A_1 \xrightarrow{\phi} \mathbb{Z},$$

where the first map is the projection $A_2 = f(A_1) \oplus C \to f(A_1)$. □

1.7. **Polynomial VIC-modules.** For a $\mathtt{VIC}(R)$-module M over $\Bbbk$, the inclusions (1.4) induce maps between homology groups for each k:

$$\mathrm{H}_k(\mathrm{GL}_0(R); M(R^0)) \to \mathrm{H}_k(\mathrm{GL}_1(R); M(R^1)) \to \mathrm{H}_k(\mathrm{GL}_2(R); M(R^2)) \to \cdots .$$

Just like for for $\mathtt{FI}$-modules, to make this stabilize we will need to impose a polynomiality condition.[6] The definitions are similar to those for $\mathtt{FI}$-modules:

Definition 1.16. Let R be a ring, $\Bbbk$ be a commutative ring, and M be a $\mathtt{VIC}(R)$-module over $\Bbbk$.

- The *shifted* $\mathtt{VIC}(R)$-module of M, denoted ΣM is the $\mathtt{VIC}(R)$-module over $\Bbbk$ defined via the formula $\Sigma M(A) = M(A \oplus R^1)$ for a finite-rank free right R-module A.
- The *derived* $\mathtt{VIC}(R)$-module of M, denoted DM, is the $\mathtt{VIC}(R)$-module over $\Bbbk$ defined via the formula

$$DM(S) = \frac{M(A \oplus R^1)}{\mathrm{Im}(M(A) \to M(A \oplus R^1))} \quad \text{for a finite-rank free right } R\text{-module } A. \qquad \square$$

Definition 1.17. Let R be a ring, $\Bbbk$ be a commutative ring, and M be a $\mathtt{VIC}(R)$-module over $\Bbbk$. We say that M is *polynomial* of degree $d \geq -1$ starting at $m \in \mathbb{Z}$ if it satisfies the following inductive condition:

- If $d = -1$, then for all finite-rank free right R-modules A with $\mathrm{rk}(A) \geq m$ we require $M(A) = 0$.
- If $d \geq 0$, then we require the following two conditions:
  - For all $\mathtt{VIC}(R)$-morphisms $(f, C) \colon : [A_1] \to [A_2]$ with $\mathrm{rk}(A_1) \geq m$, the induced map $(f, C)_* \colon M(A_1) \to M(A_2)$ must be an injection.
  - The derived $\mathtt{VIC}(R)$-module DM must be polynomial of degree $(d-1)$ starting at $(m-1)$. $\square$

Example 1.18. The $\mathtt{VIC}(R)$-modules in Examples 1.13, 1.14, and 1.15 are all polynomial of degree 1 starting at 0. Letting M be one of these, for $d \geq 0$ we can define another $\mathtt{VIC}(R)$-module $M^{\otimes d}$ via the formula

$$M^{\otimes d}(A) = (M(A))^{\otimes d} \quad \text{for a finite-rank free } R\text{-module } A.$$

This is easily seen to be polynomial of degree d starting at 0. $\square$

1.8. **General linear groups.** We now turn to our stability theorem, which will concern the homology of $\mathrm{GL}_n(R)$. We will need to impose a "stable rank" condition on R called (SR_r) that was introduced by Bass [2]. See §8 below for the definition and a survey. Here we will simply say this condition is satisfied by many rings; in particular, fields satisfy (SR_2) and PIDs satisfy (SR_3). More generally a ring R that is finitely generated as a module over a Noetherian commutative ring of Krull dimension r satisfies (SR_{r+2}). Thus for instance if $\Bbbk$ is a field and G is a finite group, then the group ring $\Bbbk[G]$ satisfies (SR_2).

Theorem B. *Let R be a ring satisfying (SR_r), let $\Bbbk$ be a commutative ring, and let M be a $\mathtt{VIC}(R)$-module over $\Bbbk$ that is polynomial of degree d starting at $m \geq 0$. For each $k \geq 0$, the map*

$$\mathrm{H}_k(\mathrm{GL}_n(R); M(R^n)) \to \mathrm{H}_k(\mathrm{GL}_{n+1}(R); M(R^{n+1}))$$

is an isomorphism for $n \geq 2k + \max(2d + r, m+1)$ and a surjection for $n = 2k + \max(2d + r - 1, m)$.

[6]For finite rings R, the author and Sam proved in [25, 26] a homological stability result which replaces the polynomiality condition by a much weaker "finite generation" condition. That proof is very different from the one we will give, and cannot possibly work for infinite rings like $R = \mathbb{Z}$.

In the key case $m = 0$, the stabilization map is thus an isomorphism for $n \geq 2k + 2d + r$ and a surjection for $n = 2k + 2d + r - 1$. Just like for the symmetric group, we will derive from Theorem B the following variant result which for $m \gg 0$ sometimes gives better bounds:

Theorem B′. *Let R be a ring satisfying (SR_r), let $\Bbbk$ be a commutative ring, and let M be a $\mathtt{VIC}(R)$-module over $\Bbbk$ that is polynomial of degree d starting at $m \geq 0$. For each $k \geq 0$, the map*

$$\mathrm{H}_k(\mathrm{GL}_n(R); M(R^n)) \to \mathrm{H}_k(\mathrm{GL}_{n+1}(R); M(R^{n+1})) \tag{1.5}$$

is an isomorphism for $n \geq \max(m, 2k + 2d + r + 1)$ and a surjection for $n \geq \max(m, 2k + 2d + r - 1)$.

Dwyer [12] proved a version of this for R a PID, though he only worked with split coefficient systems starting at $m = 0$ (c.f. Remark 1.8). He did not identify a stable range. Later van der Kallen [29, Theorem 5.6] extended this to rings satisfying (SR_r), though again he only worked with split coefficient systems starting at $m = 0$. He proved that (1.5) is an isomorphism for $n \geq 2k + d + r$. Randal-Williams–Wahl [28, Theorem 5.11] then showed how to deal with general polynomial modules, though they only stated a result for ones that started at $m = 0$. They proved that (1.5) is an isomorphism for $n \geq 2k + 2d + r + 1$.

Remark 1.19. The result we will prove is more general than Theorems B and B′ and applies to certain subgroups of $\mathrm{GL}_n(R)$ as well.[7] For instance, if R is commutative then it applies to $\mathrm{SL}_n(R)$. See §8.10 for the definition of the groups we will consider and Theorems 11.1 and 11.2 for the statement of our theorem. □

1.9. **Congruence subgroups.** Our final theorem illustrates how our machinery can be applied to prove a theorem that is not (directly) about homological stability. Borel [5, 6] proved that if Γ is a lattice in $\mathrm{SL}_n(\mathbb{R})$ and V is a rational representation of the algebraic group SL_n, then for $n \gg k$ the homology group $\mathrm{H}_k(\Gamma; V)$ depends only on n and V, not on the lattice Γ. In particular, it is unchanged if you pass from Γ to a finite-index subgroup.

We will prove a version of this for $\mathrm{GL}_n(R)$ for rings R satisfying (SR_r). The basic flavor of the result will be that passing from $\mathrm{GL}_n(R)$ to appropriate finite-index subgroups does not change rational homology, at least in some stable range. The finite-index subgroups we will consider are the finite-index congruence subgroups, which are defined as follows:

Definition 1.20. Let R be a ring and let α be a 2-sided ideal of R. The *level-α congruence subgroup*[8] of $\mathrm{GL}_n(R)$, denoted $\mathrm{GL}_n(R, \alpha)$, is the kernel of the natural group homomorphism $\mathrm{GL}_n(R) \to \mathrm{GL}_n(R/\alpha)$. □

Our theorem is as follows:

Theorem C. *Let R be a ring satisfying (SR_r), let $\Bbbk$ be a field of characteristic 0, and let M be a $\mathtt{VIC}(R)$-module over $\Bbbk$ that is polynomial of degree d starting at $m \geq 0$. Assume furthermore that $M(R^n)$ is a finite-dimensional vector space over $\Bbbk$ for all $n \geq 0$. Then for all 2-sided ideals α of R such that R/α is finite, the map*

$$\mathrm{H}_k(\mathrm{GL}_n(R, \alpha); M(R^n)) \to \mathrm{H}_k(\mathrm{GL}_n(R); M(R^n))$$

is an isomorphism for $n \geq \max(m, 2k + 2d + 2r)$.

[7] The work of van der Kallen also applies to these subgroups.

[8] In many contexts it is common to call these *principal congruence subgroups*, and to define a congruence subgroup as a subgroup that contains a principal congruence subgroup. For general rings R, this more general notion of "congruence subgroups" can have some pathological properties. For number rings R, all nontrivial ideals α of R satisfy $|R/\alpha| < \infty$, so for number rings all principal congruence subgroups are finite-index. This is false for general rings. Even worse, as Tom Church pointed out to me there might exist finite-index subgroups $\Gamma < \mathrm{GL}_n(R)$ that contain principal congruence subgroups, but do not contain finite-index principal congruence subgroups.

We emphasize that Theorem C is *not* a homological stability theorem:[9] rather than increasing $\mathrm{GL}_n(R)$ to $\mathrm{GL}_{n+1}(R)$, we are decreasing $\mathrm{GL}_n(R)$ by passing to the finite-index subgroup $\mathrm{GL}_n(R,\alpha)$. For untwisted rational coefficients, something like Theorem C is implicit in work of Charney [8]. Though she did not state this, it can easily be derived from her work that for R and $\Bbbk$ and α as in Theorem C, the map

$$\mathrm{H}_k(\mathrm{GL}_n(R,\alpha);\Bbbk) \to \mathrm{H}_k(\mathrm{GL}_n(R);\Bbbk)$$

is an isomorphism for $n \geq 2k+2r+4$. Later Cohen [11] proved an analogous result for the symplectic group, and more importantly for us showed how to simplify Charney's argument. Our proof of Theorem C follows the outline of Cohen's proof, but using our new approach to twisted homological stability. It seems very hard to do this using the traditional proof of twisted homological stability.

Remark 1.21. The fact that the field $\Bbbk$ in Theorem C has characteristic 0 is essential. This theorem is false over fields of finite characteristic or over more general rings like $\mathbb{Z}$. □

Remark 1.22. Just like for Theorems B and B′, we will actually prove something more general than Theorem C that will apply to congruence subgroups of certain subgroups of $\mathrm{GL}_n(R)$, e.g., to $\mathrm{SL}_n(R)$ if R is commutative. □

Remark 1.23. The proof of Theorem C also requires some some recent work of Harman [17] classifying certain kinds of finitely generated $\mathtt{VIC}(\mathbb{Z})$-modules. □

Remark 1.24. If the ring R in Theorem C is a *finite* ring, then we can take $\alpha = R$, so $\mathrm{GL}_n(R,\alpha)$ is trivial. The case $k=0$ of the theorem thus implies that under its hypotheses, for R a finite ring the action of $\mathrm{GL}_n(R)$ on $M(R^n)$ is *trivial* for $n \geq \max(m, 2d+2r)$. It is enlightening to go through our proof and see how it proves this special case. This will also clarify to the reader how the work of Harman discussed in the previous remark is used. □

1.10. **Outline.** We start with three sections of preliminary results in §2 – §4. We then discuss our twisted homological stability machine in §5. To make this useful we also need an accompanying "vanishing theorem", which is in §6. We then have §7 on symmetric groups, which proves Theorems A and A′. After that, we have three sections of background on rings and general linear groups: §8 discusses the stable rank condition (SR_r), and §9 and §10 introduce some important simplicial complexes associated to $\mathrm{GL}_n(R)$. We then have §11 on general linear groups, which proves Theorems B and B′. We finally turn our attention to congruence subgroups. This requires some preliminary results on unipotent representations that are discussed in §12. We close with §13 on congruence subgroups, which proves Theorem C.

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2. Background I: simplicial complexes

This section contains background material on simplicial complexes. Its main purpose is to establish notation. See [15] for more details.

[9]Though in light of Theorem B it implies that $\mathrm{H}_k(\mathrm{GL}_n(R,\alpha); M(R^n))$ stabilizes.

2.1. **Basic definitions.** A *simplicial complex* X consists of the following data:

- A set $X^{(0)}$ called the 0-*simplices* or *vertices*.
- For each $k \geq 1$, a set $X^{(k)}$ of $(k+1)$-element subsets of the vertices called the k-*simplices*. These are required to satisfy the following condition:
  - Consider $\sigma \in X^{(k)}$. Then for $\sigma' \subseteq \sigma$ with $|\sigma'| > 0$, we must have $\sigma' \in X^{(|\sigma'|-1)}$. In this case, we call σ' a *face* of σ.

A simplicial complex X has a geometric realization $|X|$ obtained by gluing together geometric k-simplices (one for each k-simplex in $X^{(k)}$) according to the face relation. Whenever we talk about topological properties of X (e.g. being connected), we are referring to its geometric realization.

2.2. **Links and Cohen–Macaulay.** Let X be a simplicial complex. The *link* of a simplex σ of X, denoted $\mathrm{Link}_X(\sigma)$, is the subcomplex of X consisting of all simplices τ satisfying the following two conditions.

- The simplices τ and σ are disjoint, i.e., have no vertices in common.
- The union $\tau \cup \sigma$ is a simplex.

We say that X is *weakly Cohen–Macaulay* of dimension $n \in \mathbb{Z}$ if it satisfies the following:

- The complex X must be $(n-1)$-connected. Here our convention is that a space is (-1)-connected if it is nonempty, and all spaces are k-connected for $k \leq -2$.
- For all k-simplices σ of X, the complex $\mathrm{Link}_X(\sigma)$ must be $(n-k-2)$-connected.

Example 2.1. Let X be a simplicial triangulation of an n-sphere. We claim that X is weakly Cohen–Macaulay of dimension n. There are two things to check:

- The simplicial complex X is $(n-1)$-connected, which is clear.
- For a k-simplex σ of X, the complex $\mathrm{Link}_X(\sigma)$ is $(n-k-2)$-connected. In fact, since our triangulation is simplicial the link of σ is a simplicial triangulation of an $(n-k-1)$-sphere. □

Remark 2.2. The adjective "weak" is here since we do not require X to be n-dimensional, nor do we require links of k-simplices to be $(n-k-1)$-dimensional. □

One basic property of weakly Cohen–Macaulay complexes is as follows:

Lemma 2.3. *Let X be a simplicial complex that is weakly Cohen–Macaulay of dimension n and let σ be a k-simplex of X. Then $\mathrm{Link}_X(\sigma)$ is weakly Cohen–Macaulay of dimension $(n-k-1)$.*

Proof. By definition, $\mathrm{Link}_X(\sigma)$ is $(n-k-2)$-connected. Also, if τ is an ℓ-simplex of $\mathrm{Link}_X(\sigma)$, then $\tau \cup \sigma$ is a $(k+\ell+1)$-simplex of X and $\mathrm{Link}_{\mathrm{Link}_X(\sigma)}(\tau) = \mathrm{Link}_X(\tau \cup \sigma)$. This is $n-(k+\ell+1)-2 = n-k-\ell-3$ connected by assumption. □

3. Background II: semisimplicial sets

The category of simplicial complexes has some undesirable features that make it awkward for homological stability proofs. For instance, if X is a simplicial complex and G is a group acting on X, then one might expect the quotient X/G to be a simplicial complex whose k-simplices are the G-orbits of simplices of X. Unfortunately, this need not hold. In this section, we discuss the category of semisimplicial sets, which does not have such pathologies. See [15][10] and [13] for more details.

[10]This reference calls semisimplicial sets Δ-sets

3.1. **Semisimplicial sets.** Let Δ be the category whose objects are the finite sets $[k] = \{0, \ldots, k\}$ with $k \geq 0$ and whose morphisms $[\ell] \to [k]$ are order-preserving injections. A *semisimplicial set* is a contravariant functor $\mathbb{X}\colon \Delta \to \mathtt{Set}$. Unwinding this, $\mathbb{X}$ consists of the following two pieces of data:

- For each $k \geq 0$, a set $\mathbb{X}^k$ called the *k-simplices*.
- For each order-preserving injection $\iota\colon [\ell] \to [k]$, a map $\iota^*\colon \mathbb{X}^k \to \mathbb{X}^\ell$ called a *face map*. For $\sigma \in \mathbb{X}^k$, the image $\iota^*(\sigma) \in \mathbb{X}^\ell$ is called a *face* of σ.

A semisimplicial set $\mathbb{X}$ has a geometric realization $|\mathbb{X}|$ obtained by gluing geometric n-simplices together (one for each n-simplex) according to the face maps. See [15] for more details. Whenever we talk about topological properties of $\mathbb{X}$ (e.g. being connected), we are referring to its geometric realization.

3.2. **Morphisms.** Below we will compare semisimplicial sets with simplicial complexes, but first we describe their morphisms. A morphism $f\colon \mathbb{X} \to \mathbb{Y}$ between semisimplicial sets $\mathbb{X}$ and $\mathbb{Y}$ is a natural transformation between the functors $\mathbb{X}$ and $\mathbb{Y}$. In other words, f consists of set maps $f_k\colon \mathbb{X}^k \to \mathbb{Y}^k$ for each $k \geq 0$ that commute with the face maps. Such a morphism induces a continuous map $|f|\colon |\mathbb{X}| \to |\mathbb{Y}|$. Using this definition, an action of a group G on a semisimplicial set $\mathbb{X}$ consists of actions of G on each set $\mathbb{X}^k$ that commute with the face maps, and $\mathbb{X}/G$ is a semisimplicial set with $(\mathbb{X}/G)^k = \mathbb{X}^k/G$ for each $k \geq 0$.

3.3. **Semisimplicial sets vs simplicial complexes.** Let $\mathbb{X}$ be a semisimplicial set. The *vertices* of $\mathbb{X}$ are the elements of the set $\mathbb{X}^0$ of 0-simplices. For each k-simplex $\sigma \in \mathbb{X}^k$, we can define an ordered $(k+1)$-tuple $(v_0, \ldots, v_k)$ of vertices via the formula

$$v_i = \iota_i^*(\sigma) \text{ with } \iota_i\colon [0] \to [k] \text{ the map } \iota_i(0) = i.$$

We will call $v_0, \ldots, v_k$ the vertices of σ. This is similar to a simplicial complex, whose vertices are $(k+1)$-element sets of vertices. However, there are three essential differences:

- The vertices of a simplex in a semisimplicial set have a natural ordering.
- The vertices of a simplex in a semisimplicial set need not be distinct.
- Simplices in a semisimplicial set are *not* determined by their vertices.

3.4. **Small ordering.** Consider a simplicial complex X. A *small ordering*[11] of X is a semisimplicial set $\mathbb{X}$ with the following properties:

(a) The vertices of $\mathbb{X}$ are the same as the vertices of X.
(b) The (unordered) set of vertices of a k-simplex of $\mathbb{X}$ is a k-simplex of X.
(c) The map $\mathbb{X}^k \to X^{(k)}$ taking a simplex to its set of vertices is a bijection.

Condition (b) implies that all the vertices of a simplex of $\mathbb{X}$ are distinct, and condition (c) implies that a simplex is determined by its set of vertices. The only difference between $\mathbb{X}$ and X is therefore that the vertices of a simplex of $\mathbb{X}$ have a natural ordering, while the vertices of a simplex of X are unordered. It is clear that the geometric realizations $|\mathbb{X}|$ and $|X|$ are homeomorphic.

Small orderings always exist; for instance, if you choose a total ordering on the the vertices of X, then you can construct a small ordering $\mathbb{X}$ of X as follows:

- For $k \geq 0$, let $\mathbb{X}^k = \{(v_0, \ldots, v_k) \in V^{k+1} \mid \{v_0, \ldots, v_k\} \in X^{(k)} \text{ and } v_0 < \cdots < v_k\}$.
- For an order-preserving injection $\iota\colon [\ell] \to [k]$, define the face map $\iota^*\colon \mathbb{X}^k \to \mathbb{X}^\ell$ via the formula

$$\iota^*(v_0, \ldots, v_k) = (v_{\iota(1)}, \ldots, v_{\iota(\ell)}) \quad \text{for } (v_0, \ldots, v_k) \in \mathbb{X}^k.$$

[11] Our applications in this paper will only use large orderings (defined below), but we will state our technical results to also apply to appropriate small orderings so they can be used in followup work, e.g., in [24].

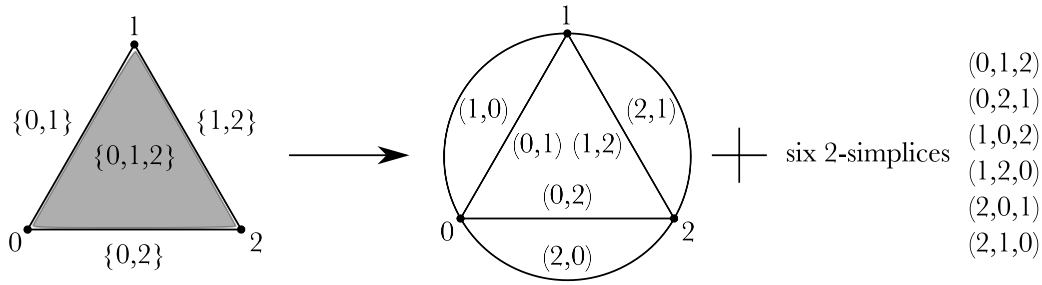

Figure 1. *Left: the* 2*-simplex* Sim_2*. Right: its large ordering* $\mathbb{S}\mathrm{im}_2$*. The drawn portion is the* 1*-skeleton.*

However, since this depends on the total ordering on $X^{(0)}$ it might have fewer symmetries than X. The example we will give of this will be used repeatedly, so we make a formal definition.

Definition 3.1. For $n \geq 0$, let Sim_n be the *n-simplex*,[12] i.e., the simplicial complex whose vertex set is $[n] = \{0, \ldots, n\}$ and whose k-simplices are all $(k+1)$-element subsets of $[n]$. The geometric realization $|\,\mathrm{Sim}_n\,|$ is the usual geometric n-simplex. □

Example 3.2. The symmetric group $\mathfrak{S}_{n+1}$ acts on Sim_n via its natural action on $[n] = \{0, \ldots, n\}$. However, if we choose a total ordering on $[n]$ and use this to define a small ordering $\mathbb{X}$ of Sim_n as above, then $\mathfrak{S}_{n+1}$ does not act on $\mathbb{X}$ except in the degenerate case $n = 0$. Indeed, it is easy to see that the automorphism group of $\mathbb{X}$ is trivial. □

3.5. **Large ordering.** If a sufficiently symmetric small ordering of a simplicial complex does not exist, there is another construction that is often useful. Let X be a simplicial complex. The *large ordering* of X, denoted X_{ord}, is the following semisimplicial set:

- For each $k \geq 0$, the k-simplices X^k_{ord} are ordered $(k+1)$-tuples $(v_0, \ldots, v_k)$ of distinct vertices of X such that $\{v_0, \ldots, v_k\} \in X^{(k)}$.
- For an order-preserving injection $\iota\colon [\ell] \to [k]$, the face map $\iota^*\colon X^k_{\mathrm{ord}} \to X^\ell_{\mathrm{ord}}$ is
$$\iota^*(v_0, \ldots, v_k) = (v_{\iota(1)}, \ldots, v_{\iota(\ell)}) \quad \text{for } (v_0, \ldots, v_k) \in \mathbb{X}^k.$$

The following example of this will be used several times, so we introduce notation for it.

Definition 3.3. For $n \geq 0$, let $\mathbb{S}\mathrm{im}_n$ be the large ordering $(\mathrm{Sim}_n)_{\mathrm{ord}}$ of the n-simplex Sim_n. The k-simplices of $\mathbb{S}\mathrm{im}_n$ are thus ordered sequences $(i_0, \ldots, i_k)$ of distinct elements of $[n]$. □

Example 3.4. Each Sim_n is contractible. However, none of the $\mathbb{S}\mathrm{im}_n$ are contractible except for $\mathbb{S}\mathrm{im}_0$. For instance, $\mathbb{S}\mathrm{im}_1$ has two 0-cells 0 and 1 and two 1-cells $(0,1)$ and $(1,0)$, and its geometric realization is homeomorphic to S^1. For an even more complicated example, see the picture of $\mathbb{S}\mathrm{im}_2$ in Figure 1. □

The semisimplicial set X_{ord} is much larger than X; indeed, each k-simplex of X corresponds to $(k+1)!$ simplices of X_{ord}, one for each total ordering of its vertices. It is clear from its construction that if a group G acts on X, then G also acts on X_{ord}. However, examples like Example 3.4 might lead one to think that there is no simple relationship between the topologies of X and X_{ord}. This makes the following theorem of Randal-Williams–Wahl somewhat surprising:

Theorem 3.5 ([28, Theorem 2.14])**.** *Let X be a simplicial complex that is weakly Cohen–Macaulay of dimension n. Then X_{ord} is $(n-1)$-connected.*

[12]It would be natural to instead denote this by Δ^n or Δ_n, but we are already using Δ for the category of finite sets used to define semisimplicial sets.

Example 3.6. Since Sim_n is clearly weakly Cohen–Macaulay of dimension n, Theorem 3.5 implies that $\mathbb{S}\mathrm{im}_n$ is $(n-1)$-connected. The semisimplicial set $\mathbb{S}\mathrm{im}_n$ is also called the *complex of injective words* on $(n+1)$ letters, and the fact that it is $(n-1)$-connected was originally proved by Farmer [14]. □

3.6. **Forward link and forward Cohen–Macaulay.** Let X be a simplicial complex and let $\mathbb{X}$ be either a small or large ordering of X. Simplices of $\mathbb{X}$ are thus certain ordered sequences of distinct vertices of X, and we will write them as $(v_0, \ldots, v_k)$ where the v_i are vertices. As notation, if $\sigma = (v_0, \ldots, v_k)$ and $\tau = (w_0, \ldots, w_\ell)$ are ordered sequences of vertices, we will write $\sigma \cdot \tau$ for $(v_0, \ldots, v_k, w_0, \ldots, w_\ell)$. Of course, $\sigma \cdot \tau$ need not be a simplex; for instance, its vertices need not be distinct.

Given a simplex σ of $\mathbb{X}$, the *forward link* of σ, denoted $\overrightarrow{Link}_{\mathbb{X}}(\sigma)$, is the semisimplicial set whose ℓ-simplices are ℓ-simplices τ of $\mathbb{X}$ such that $\sigma \cdot \tau$ is a $(k+\ell+1)$-simplex of $\mathbb{X}$. We will say that $\mathbb{X}$ is *weakly forward Cohen–Macaulay* of dimension n if $\mathbb{X}$ is $(n-1)$-connected and for all k-simplices σ of $\mathbb{X}$, the forward link $\overrightarrow{Link}_{\mathbb{X}}(\sigma)$ is $(n-k-2)$-connected.

If X is a simplicial complex that is weakly Cohen–Macaulay of dimension n, then we will say that a small ordering $\mathbb{X}$ of X is a *CM-small ordering* if $\mathbb{X}$ is weakly forward Cohen–Macaulay of dimension n. These need not exist. However, Lemma 2.3 and Theorem 3.5 together imply the following:

Lemma 3.7. *Let X be a simplicial complex that is weakly Cohen–Macaulay of dimension n. Then its large ordering X_{ord} is weakly forward Cohen–Macaulay of dimension n.*

4. Background III: coefficient systems

In this section, we define coefficient systems on semisimplicial sets. Informally, these are natural assignments of abelian groups to each simplex.

4.1. **Simplex category.** To formalize this, we introduce the *simplex category* of a semisimplicial set $\mathbb{X}$, which is the following category[13] $\mathtt{Simp}(\mathbb{X})$:

- The objects of $\mathtt{Simp}(\mathbb{X})$ are the simplices of $\mathbb{X}$.
- For $\sigma, \sigma' \in \mathtt{Simp}(\mathbb{X})$ with $\sigma \in \mathbb{X}^k$ and $\sigma' \in \mathbb{X}^\ell$, the morphisms from σ to σ' are
$$\mathrm{Mor}(\sigma, \sigma') = \left\{\iota\colon [\ell] \to [k] \mid \iota \text{ order-preserving injection w/ } \iota^*(\sigma) = \sigma'\right\}.$$
This is nonempty precisely when σ' is a face of σ.

The *augmented simplex category* of $\mathbb{X}$, denoted $\widetilde{\mathtt{Simp}}(\mathbb{X})$, is obtained by adjoining a terminal object $*$ to $\mathtt{Simp}(\mathbb{X})$ that we will call the (-1)-*simplex*.

4.2. **Coefficient systems.** Let $\Bbbk$ be a commutative ring. A *coefficient system* over $\Bbbk$ on a semisimplicial set $\mathbb{X}$ is a covariant functor $\mathcal{F}$ from $\mathtt{Simp}(\mathbb{X})$ to the category of $\Bbbk$-modules. Unpacking this, $\mathcal{F}$ consists of the following data:

- For each simplex σ of $\mathbb{X}$, a $\Bbbk$-module $\mathcal{F}(\sigma)$.
- For $\sigma \in \mathbb{X}^k$ and $\sigma' \in \mathbb{X}^\ell$ and $\iota\colon [\ell] \to [k]$ an order-preserving injection with $\iota^*(\sigma) = \sigma'$, a $\Bbbk$-module morphism $\iota^*\colon \mathcal{F}(\sigma) \to \mathcal{F}(\sigma')$.

These must satisfy the evident compatibility conditions. Similarly, an *augmented coefficient system* on $\mathbb{X}$ is a covariant functor $\mathcal{F}$ from $\widetilde{\mathtt{Simp}}(\mathbb{X})$ to the category of $\Bbbk$-modules.

Example 4.1. If $\mathbb{X}$ is a semisimplicial set and $\Bbbk$ is a commutative ring, then we have the constant coefficient system $\underline{\Bbbk}$ on $\mathbb{X}$ with $\underline{\Bbbk}(\sigma) = \Bbbk$ for all simplices σ. This can be extended to an augmented coefficient system by setting $\underline{\Bbbk}(*) = \Bbbk$ for the (-1)-simplex $*$. □

[13] This is related to the poset of simplices of $\mathbb{X}$, but is *not* always a poset since there can be multiple morphisms between simplices in it.

Example 4.2. Recall that $\mathbb{S}\mathrm{im}_n$ is the large ordering of the n-simplex Sim_n. For an $\mathtt{FI}$-module M, we can define a coefficient system $\mathcal{F}_{M,n}$ on $\mathbb{S}\mathrm{im}_n$ via the formula

$$\mathcal{F}_{M,n}(i_0, \ldots, i_k) = M([n] \setminus \{i_0, \ldots, i_k\}) \quad \text{for a simplex } (i_0, \ldots, i_k) \text{ of } \mathbb{S}\mathrm{im}_n.$$

For an order-preserving injective map $\iota\colon [\ell] \to [k]$, the induced map

$$\iota^*\colon \mathcal{F}_{M,n}(i_0, \ldots, i_k) \to \mathcal{F}_{M,n}(i_{\iota(0)}, \ldots, i_{\iota(\ell)})$$

is the one induced by the inclusion

$$[n] \setminus \{i_0, \ldots, i_k\} \hookrightarrow [n] \setminus \{i_{\iota(0)}, \ldots, i_{\iota(\ell)}\}.$$

This can be extended to an augmented coefficient system by setting $\mathcal{F}_{M,n}(*) = M([n])$ for the (-1)-simplex $*$. □

The collection of coefficient systems (resp. augmented coefficient systems) over $\mathbb{k}$ on $\mathbb{X}$ forms an abelian category.

4.3. **Equivariant coefficient systems.** Let G be a group, let $\mathbb{X}$ be a semisimplicial set on which G acts, and let $\mathbb{k}$ be a commutative ring. Consider a (possibly augmented) coefficient system $\mathcal{F}$ on $\mathbb{X}$. We want to equip $\mathcal{F}$ with an "action" of G that is compatible with the G-action on $\mathbb{X}$. For $g \in G$ and a simplex σ of $\mathbb{X}$, the data of such an action should include isomorphisms $\mathcal{F}(\sigma) \to \mathcal{F}(g \cdot \sigma)$, and more generally for $h \in G$ should include isomorphisms $\mathcal{F}(h \cdot \sigma) \to \mathcal{F}(gh \cdot \sigma)$. Moreover, these isomorphisms should be be compatible with $\mathcal{F}$ in an appropriate sense.

We formalize this as follows. For $h \in G$, let $\mathcal{F}_h$ be the coefficient system over $\mathbb{k}$ on $\mathbb{X}$ defined via the formula

$$\mathcal{F}_h(\sigma) = \mathcal{F}(h \cdot \sigma) \quad \text{for a simplex } \sigma \text{ of } \mathbb{X}.$$

We say that $\mathcal{F}$ is a *G-equivariant coefficient system* if for all $g, h \in G$ we are given a natural transformation $\Phi_{g,h}\colon \mathcal{F}_h \Rightarrow \mathcal{F}_{gh}$. These natural transformations should satisfy the following two properties:

- For all $h \in G$, the natural transformation $\Phi_{1,h}\colon \mathcal{F}_h \Rightarrow \mathcal{F}_h$ should be the identity natural transformation.
- For all $g_1, g_2 \in G$ and all $h \in H$, we require the two natural transformations
$$\mathcal{F}_h \xRightarrow{\Phi_{g_1 g_2,h}} \mathcal{F}_{g_1 g_2 h} \quad \text{and} \quad \mathcal{F}_h \xRightarrow{\Phi_{g_2,h}} \mathcal{F}_{g_2 h} \xRightarrow{\Phi_{g_1,g_2 h}} \mathcal{F}_{g_1 g_2 h}$$
to be equal.

Let us unpack this a bit. For all simplices σ of $\mathbb{X}$ and all $g, h \in G$, the natural transformation $\Phi_{g,h}$ gives a homomorphism $\mathcal{F}(h \cdot \sigma) \to \mathcal{F}(gh \cdot \sigma)$. This homomorphism is an isomorphism whose inverse is given by the map $\mathcal{F}(gh \cdot \sigma) \to \mathcal{F}(h \cdot \sigma)$ induced by $\Phi_{g^{-1},gh}$. Moreover, it must be natural in the sense that if σ' is a face of σ via some face map, then the diagram

$$\begin{array}{ccc} \mathcal{F}(h \cdot \sigma) & \longrightarrow & \mathcal{F}(gh \cdot \sigma) \\ \downarrow & & \downarrow \\ \mathcal{F}(h \cdot \sigma') & \longrightarrow & \mathcal{F}(gh \cdot \sigma') \end{array}$$

must commute. The fact that the natural transformations respect the group law implies that the stabilizer subgroup G_σ acts on $\mathcal{F}(\sigma)$, making it into a $\mathbb{k}[G_\sigma]$-module. If σ' is a face of σ via some face map, the induced map $\mathcal{F}(\sigma) \to \mathcal{F}(\sigma')$ is a map of $\mathbb{k}[G_\sigma]$-modules, where

G_σ acts on $\mathcal{F}(\sigma')$ via the inclusion $G_\sigma \hookrightarrow G_{\sigma'}$. Another consequence of the fact that the natural transformations respect the group law is that for all $k \geq 0$, the direct sum

$$\bigoplus_{\sigma \in \mathbb{X}^k} \mathcal{F}(\sigma)$$

is a $\Bbbk[G]$-module in a natural way, where the G-action restricts to the G_σ-action on $\mathcal{F}(\sigma)$ for each $\sigma \in \mathbb{X}^k$.

Example 4.3. Let M be an $\mathtt{FI}$-module over $\Bbbk$ and let $\mathcal{F}_{M,n}$ be the augmented coefficient system on $\mathbb{S}\mathrm{im}_n$ from Example 4.2 defined via the formula

$$\mathcal{F}_{M,n}(i_0, \ldots, i_k) = M([n] \setminus \{i_0, \ldots, i_k\}) \quad \text{for a simplex } (i_0, \ldots, i_k) \text{ of } \mathbb{S}\mathrm{im}_n.$$

Recalling that $[n] = \{0, \ldots, n\}$, the symmetric group $\mathfrak{S}_{n+1}$ acts on $\mathbb{S}\mathrm{im}_n$. The augmented coefficient system $\mathcal{F}_{M,n}$ can be endowed with the structure of an $\mathfrak{S}_{n+1}$-equivariant augmented coefficient system in the following way. Consider $g, h \in \mathfrak{S}_{n+1}$. We then define $\Phi_{g,h}$ to be the following natural transformation:

- For a simplex $(i_0, \ldots, i_k)$ of $\mathbb{S}\mathrm{im}_n$, let the induced map

  $$\mathcal{F}_{M,n}(h(i_0), \ldots, h(i_k)) \to \mathcal{F}_{M,n}(gh(i_0), \ldots, gh(i_k))$$

  be the map

  $$M([n] \setminus \{h(i_0), \ldots, h(i_k)\}) \to M([n] \setminus \{gh(i_0), \ldots, gh(i_k)\}$$

  induced by the bijection

  $$[n] \setminus \{h(i_0), \ldots, h(i_k)\} \to [n] \setminus \{gh(i_0), \ldots, gh(i_k)\}$$

  obtained by restricting $g \in \mathfrak{S}_{n+1}$ to $[n] \setminus \{h(i_0), \ldots, h(i_k)\}$. $\square$

4.4. **Chain complex and homology.** Let $\mathbb{X}$ be a semisimplicial set and let $\mathcal{F}$ be a coefficient system on $\mathbb{X}$. Define the *simplicial chain complex* of $\mathbb{X}$ with coefficients in $\mathcal{F}$ to be the chain complex $\mathrm{C}_\bullet(\mathbb{X}; \mathcal{F})$ defined as follows:

- For $k \geq 0$, we have

  $$\mathrm{C}_k(\mathbb{X}; \mathcal{F}) = \bigoplus_{\sigma \in \mathbb{X}^k} \mathcal{F}(\sigma).$$

- The boundary map $d\colon \mathrm{C}_k(\mathbb{X}; \mathcal{F}) \to \mathrm{C}_{k-1}(\mathbb{X}; \mathcal{F})$ is $d = \sum_{i=0}^k (-1)^i d_i$, where the map $d_i\colon \mathrm{C}_k(\mathbb{X}; \mathcal{F}) \to \mathrm{C}_{k-1}(\mathbb{X}; \mathcal{F})$ is as follows. Consider $\sigma \in \mathbb{X}^k$. Let $\iota\colon [k-1] \to [k]$ be the order-preserving map whose image omits i. Then on the $\mathcal{F}(\sigma)$ factor of $\mathrm{C}_n(\mathbb{X}; \mathcal{F})$, the map d_i is

  $$\mathcal{F}(\sigma) \xrightarrow{\iota^*} \mathcal{F}(\iota^*(\sigma)) \hookrightarrow \bigoplus_{\sigma' \in \mathbb{X}^{k-1}} \mathcal{F}(\sigma') = \mathrm{C}_{k-1}(\mathbb{X}; \mathcal{F}).$$

Define

$$\mathrm{H}_k(\mathbb{X}; \mathcal{F}) = \mathrm{H}_k(\mathrm{C}_\bullet(\mathbb{X}; \mathcal{F})).$$

For an augmented coefficient system $\mathcal{F}$ on $\mathbb{X}$, define $\widetilde{\mathrm{C}}_\bullet(\mathbb{X}; \mathcal{F})$ to be the augmented chain complex defined just like we did above but with $\widetilde{\mathrm{C}}_{-1}(\mathbb{X}; \mathcal{F}) = \mathcal{F}(*)$ and define

$$\widetilde{\mathrm{H}}_k(\mathbb{X}; \mathcal{F}) = \mathrm{H}_k(\widetilde{\mathrm{C}}_\bullet(\mathbb{X}; \mathcal{F})).$$

Example 4.4. For a semisimplicial set $\mathbb{X}$ and a commutative ring $\Bbbk$, we have

$$\mathrm{H}_k(\mathbb{X}; \underline{\Bbbk}) = \mathrm{H}_k(|\mathbb{X}|; \Bbbk) \quad \text{and} \quad \widetilde{\mathrm{H}}_k(\mathbb{X}; \underline{\Bbbk}) = \widetilde{\mathrm{H}}_k(|\mathbb{X}|; \Bbbk). \qquad \square.$$

Remark 4.5. With our definition, $\widetilde{\mathrm{H}}_{-1}(\mathbb{X};\mathcal{F})$ is a quotient of $\mathcal{F}(*)$. This quotient can sometimes be nonzero. It vanishes precisely when the map

$$\bigoplus_{v\in\mathbb{X}^0}\mathcal{F}(v)\to\mathcal{F}(*)$$

is surjective. □

Note that if a group G acts on $\mathbb{X}$ and $\mathcal{F}$ is G-equivariant coefficient system, then

$$\cdots\to\widetilde{\mathrm{C}}_2(\mathbb{X};\mathcal{F})\to\widetilde{\mathrm{C}}_1(\mathbb{X};\mathcal{F})\to\widetilde{\mathrm{C}}_0(\mathbb{X};\mathcal{F})\to\widetilde{\mathrm{C}}_{-1}(\mathbb{X};\mathcal{F})=\mathcal{F}(*)\to 0$$

is a chain complex of $\Bbbk[G]$-modules, and each $\widetilde{\mathrm{H}}_k(\mathbb{X};\mathcal{F})$ is a $\Bbbk[G]$-module.

4.5. **Long exact sequences.** Consider a short exact sequence

$$0\longrightarrow\mathcal{F}_1\longrightarrow\mathcal{F}_2\longrightarrow\mathcal{F}_3\longrightarrow 0$$

of coefficient systems over $\Bbbk$ on $\mathbb{X}$. For each simplex σ of $\mathbb{X}$, we thus have a short exact sequence

$$0\longrightarrow\mathcal{F}_1(\sigma)\longrightarrow\mathcal{F}_2(\sigma)\longrightarrow\mathcal{F}_3(\sigma)\longrightarrow 0$$

of $\Bbbk$-modules. These fit together into a short exact sequence

$$0\longrightarrow\mathrm{C}_\bullet(\mathbb{X};\mathcal{F}_1)\longrightarrow\mathrm{C}_\bullet(\mathbb{X};\mathcal{F}_2)\longrightarrow\mathrm{C}_\bullet(\mathbb{X};\mathcal{F}_3)\longrightarrow 0$$

of chain complexes, and thus induce a long exact sequence in homology of the form

$$\cdots\longrightarrow\mathrm{H}_k(\mathbb{X};\mathcal{F}_1)\longrightarrow\mathrm{H}_k(\mathbb{X};\mathcal{F}_2)\longrightarrow\mathrm{H}_k(\mathbb{X};\mathcal{F}_3)\longrightarrow\mathrm{H}_{k-1}(\mathbb{X};\mathcal{F}_1)\longrightarrow\cdots.$$

A similar result holds for augmented coefficient systems and reduced homology.

5. Stability I: stability machine

In this section, we describe our machine for proving twisted homological stability.

5.1. **Classical homological stability.** An *increasing sequence of groups* is an indexed sequence of groups $\{G_n\}_{n=0}^{\infty}$ such that

$$G_0\subseteq G_1\subseteq G_2\subseteq\cdots.$$

For each $k\geq 0$, we get maps

$$\mathrm{H}_k(G_0)\to\mathrm{H}_k(G_1)\to\mathrm{H}_k(G_2)\to\cdots.$$

The classical homological stability machine gives conditions under which these stabilize, i.e., such that the maps $\mathrm{H}_k(G_{n-1})\to\mathrm{H}_k(G_n)$ are isomorphisms for $n\gg k$. One version of it is as follows.

Theorem 5.1 (Classical homological stability)**.** *Let $\{G_n\}_{n=0}^{\infty}$ be an increasing sequence of groups. For each $n\geq 1$, let $\mathbb{X}_n$ be a semisimplicial set upon which G_n acts. Assume for some $c\geq 2$ that the following hold:*

(a) For all $-1\leq k\leq\frac{n-2}{c}$, we have $\widetilde{\mathrm{H}}_k(\mathbb{X}_n)=0$.
(b) For all $0\leq k<n$, the group G_{n-k-1} is the G_n-stabilizer of a k-simplex of $\mathbb{X}_n$.
(c) For all $0\leq k<n$, the group G_n acts transitively on the k-simplices of $\mathbb{X}_n$.
(d) For all $n\geq 2$ and all 1-simplices e of $\mathbb{X}_n$ whose proper faces consist of 0-simplices v and v', there exists some $\lambda\in G_n$ with $\lambda(v)=v'$ such that λ commutes with all elements of $(G_n)_e$.

Then for all k the map $\mathrm{H}_k(G_{n-1})\to\mathrm{H}_k(G_n)$ is an isomorphism for $n\geq ck+2$ and a surjection for $n=ck+1$.

Proof. This can be proved exactly like [19, Theorem 1.1] – the only major difference between our theorem and [19, Theorem 1.1] is that we assume a weaker connectivity range on the $\mathbb{X}_n$, which causes stability to happen at a slower rate. We thus omit the details of the proof, but to clarify our indexing conventions we make a few remarks.

The proof is by induction on k. The base case is $k \leq -1$, where the result is trivial since $\mathrm{H}_k(G) = 0$ for all groups G when k is negative. We could also start with $k = 0$ since $\mathrm{H}_0(G) = \mathbb{Z}$, but later when we work with twisted coefficients even the H_0 statement will be nontrivial. In any case, to go from H_{k-1} to H_k two steps are needed (which is why require $c \geq 2$).

The first step is to prove that the map $\mathrm{H}_k(G_{n-1}) \to \mathrm{H}_k(G_n)$ is surjective for all n sufficiently large, which requires that $\widetilde{\mathrm{H}}_i(\mathbb{X}_n) = 0$ for $-1 \leq i \leq k-1$. Once this has been done, we then prove that $\mathrm{H}_k(G_{n-1}) \to \mathrm{H}_k(G_n)$ is also injective for n one step larger than needed for surjective stability. This requires $\widetilde{\mathrm{H}}_i(\mathbb{X}_n) = 0$ for $-1 \leq i \leq k$. We remark that condition (d) is used for injective stability but not surjective stability, which is why we only assume it for $n \geq 2$.

This explains our indexing conventions:

- Surjective stability for H_0 starts with $\mathrm{H}_0(G_0) \to \mathrm{H}_0(G_1)$, so we need $\widetilde{\mathrm{H}}_{-1}(\mathbb{X}_1) = 0$.
- Injective stability for H_0 starts with $\mathrm{H}_0(G_1) \to \mathrm{H}_0(G_2)$, so we need $\widetilde{\mathrm{H}}_{-1}(\mathbb{X}_2) = \widetilde{\mathrm{H}}_0(\mathbb{X}_2) = 0$. □

5.2. **Setup for twisted coefficients.** We want to give a version of this with twisted coefficients. Fix a commutative ring $\Bbbk$. An *increasing sequence of groups and modules* is an indexed sequence of pairs $\{(G_n, M_n)\}_{n=0}^{\infty}$, where the G_n and the M_n are as follows:

- The $\{G_n\}_{n=0}^{\infty}$ are an increasing sequence of groups.
- Each M_n is a $\Bbbk[G_n]$-module.
- As abelian groups, the M_n satisfy
$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots .$$
- For each $n \geq 0$, the inclusion $M_n \hookrightarrow M_{n+1}$ is G_n-equivariant, where G_n acts on M_{n+1} via the inclusion $G_n \hookrightarrow G_{n+1}$.

Given an increasing sequence of groups and modules $\{(G_n, M_n)\}_{n=0}^{\infty}$, for each k we get maps
$$\mathrm{H}_k(G_0; M_0) \to \mathrm{H}_k(G_1; M_1) \to \mathrm{H}_k(G_2; M_2) \to \cdots$$
between the associated twisted homology groups. We want to show that this stabilizes via a machine similar to Theorem 5.1.

We will incorporate the M_n into our machine via a G_n-equivariant augmented coefficient $\mathcal{M}_n$ on the semisimplicial set $\mathbb{X}_n$ with $\mathcal{M}_n(*) = M_n$. Having done this, we will be forced to replace the requirement that $\widetilde{\mathrm{H}}_k(\mathbb{X}_n) = 0$ in condition (a) by $\widetilde{\mathrm{H}}_k(\mathbb{X}_n; \mathcal{M}_n) = 0$. This is not easy to check, but we will give a useful criterion in §6 below. The twisted analogue of Theorem 5.1 is as follows.

Theorem 5.2 (Twisted homological stability)**.** *Let $\{(G_n, M_n)\}_{n=0}^{\infty}$ be an increasing sequence of groups and modules. For each $n \geq 1$, let $\mathbb{X}_n$ be a semisimplicial set upon which G_n acts and let $\mathcal{M}_n$ be a G_n-equivariant augmented coefficient system on $\mathbb{X}_n$. Assume for some $c \geq 2$ that the following hold:*

(a) For all $-1 \leq k \leq \frac{n-2}{c}$, we have $\widetilde{\mathrm{H}}_k(\mathbb{X}_n; \mathcal{M}_n) = 0$.
(b) For all $-1 \leq k < n$, the group G_{n-k-1} is the G_n-stabilizer of a k-simplex σ_k of $\mathbb{X}_n$ with $\mathcal{M}_n(\sigma_k) = M_{n-k-1}$. In particular, $\mathcal{M}_n(*) = M_n$.
(c) For all $0 \leq k < n$, the group G_n acts transitively on the k-simplices of $\mathbb{X}_n$.

(d) *For all $n \geq 2$ and all 1-simplices e of $\mathbb{X}_n$ whose proper faces consist of 0-simplices v and v', there exists some $\lambda \in G_n$ with $\lambda(v) = v'$ such that λ commutes with all elements of $(G_n)_e$ and fixes all elements of $\mathcal{M}_n(e)$.*

Then for $k \geq 0$ the map $\mathrm{H}_k(G_{n-1}; M_{n-1}) \to \mathrm{H}_k(G_n; M_n)$ is an isomorphism for $n \geq ck+2$ and a surjection for $n = ck+1$.

The proof of Theorem 5.2 is almost identical to that of Theorem 5.1 (for which we referred to [19, Theorem 1.1]). We will therefore not give full details, but only describe how to construct the key spectral sequence (see Example 5.6 below), which will explain the role played by condition (a).

Remark 5.3. Theorem 5.2 is related to [23, Theorem D]. □

5.3. **Homology of stabilizers.** This requires the following construction. Fix a commutative ring $\Bbbk$. Let G be a group acting on a semisimplicial set $\mathbb{X}$ and let $\mathcal{M}$ be a G-equivariant augmented coefficient system on $\mathbb{X}$ over $\Bbbk$. For a simplex $\widetilde{\sigma}$ of $\mathbb{X}$, the value $\mathcal{M}(\widetilde{\sigma})$ is a $\Bbbk[G_{\widetilde{\sigma}}]$-module. For $q \geq 0$, define an augmented coefficient system $\mathcal{H}_q(\mathcal{M})$ on $\mathbb{X}/G$ as follows. Consider a simplex σ of $\mathbb{X}/G$, and let $\widetilde{\sigma}$ be a lift of σ to $\mathbb{X}$. We then define

$$\mathcal{H}_q(\mathcal{M})(\sigma) = \mathrm{H}_q(G_{\widetilde{\sigma}}; \mathcal{M}(\widetilde{\sigma})).$$

To see that this is well-defined, let $\widetilde{\sigma}'$ be another lift of σ to $\mathbb{X}$. There exists some $g \in G$ with $g\widetilde{\sigma} = \widetilde{\sigma}'$, so $gG_{\widetilde{\sigma}}g^{-1} = G_{\widetilde{\sigma}'}$ and $g \cdot \mathcal{M}(\widetilde{\sigma}) = \mathcal{M}(\widetilde{\sigma}')$. Conjugation/multiplication by g thus induces an isomorphism

$$\mathrm{H}_q(G_{\widetilde{\sigma}}; \mathcal{M}(\widetilde{\sigma})) \cong \mathrm{H}_q(G_{\widetilde{\sigma}'}; \mathcal{M}(\widetilde{\sigma}')).$$

What is more, since inner automorphisms induce the identify on homology (even with twisted coefficients; see [7, Proposition III.8.1]), this isomorphism is independent of the choice of g, and thus is completely canonical. That it is a coefficient system follows immediately.

Remark 5.4. If $*$ is the (-1)-simplex of $\mathbb{X}/G$, then the only possible lift $\widetilde{*}$ is the (-1)-simplex of $\mathbb{X}$, and by definition its stabilizer is the entire group G. It follows that $\mathcal{H}_q(\mathcal{M})(*) = \mathrm{H}_q(G; \mathcal{M}(\widetilde{*}))$. □

5.4. **Spectral sequence.** The spectral sequence that underlies Theorem 5.2 is as follows. In it, our convention is that $\mathcal{H}_q(\mathcal{M}) = \underline{0}$ for $q < 0$.

Theorem 5.5 (Spectral sequence)**.** *Let G be a group acting on a semisimplicial set $\mathbb{X}$, let $\Bbbk$ be a commutative ring, let $\mathcal{M}$ be a G-equivariant augmented coefficient system over $\Bbbk$ on $\mathbb{X}$. Assume that $\widetilde{\mathrm{H}}_k(\mathbb{X}; \mathcal{M}) = 0$ for $0 \leq k \leq r$. Then there exists a spectral sequence $E^{\bullet}_{pq}$ with the following properties:*

(i) *We have $E^1_{pq} = \widetilde{\mathrm{C}}_p(\mathbb{X}/G; \mathcal{H}_q(\mathcal{M}))$, and the differential $E^1_{pq} \to E^1_{p-1,q}$ is the differential on $\widetilde{\mathrm{C}}_{\bullet}(\mathbb{X}/G; \mathcal{H}_p(\mathcal{M}))$. In particular, $E^1_{pq} = 0$ if $p < -1$ or if $q < 0$.*
(ii) *For $p + q \leq r$, we have $E^{\infty}_{pq} = 0$.*

Example 5.6. Let the notation and assumptions be as in Theorem 5.2, and apply Theorem 5.5 to G_n and $\mathbb{X}_n$ and M_n and $\mathcal{M}_n$. Since G_n acts transitively on the k-simplices of $\mathbb{X}_n$ for $0 \leq k < n$, there is a single k-simplex in $\mathbb{X}_n/G_n$. Since G_{n-k-1} is the G_n stabilizer of a k-simplex σ_k of $\mathbb{X}_n$ with $\mathcal{M}_n(\sigma_k) = M_{n-k-1}$, we conclude that our spectral sequence has

$$E^1_{pq} \cong \widetilde{\mathrm{C}}_p(\mathbb{X}_n/G_n; \mathcal{H}_q(\mathcal{M}_n)) = \mathrm{H}_q(G_{n-p-1}; M_{n-p-1}) \quad \text{for } -1 \leq p < n.$$ □

Proof of Theorem 5.5. Let

$$\cdots \to F_2(G) \to F_1(G) \to F_0(G) \to \Bbbk$$

be a resolution of the trivial $\Bbbk[G]$-module $\Bbbk$ by free $\Bbbk[G]$-modules. The action of G on $\mathbb{X}$ makes $\widetilde{\mathrm{C}}_\bullet(\mathbb{X};\mathcal{M})$ into a chain complex of $\Bbbk[G]$-modules, so we can consider the double complex $C_{\bullet,\bullet}$ defined by

$$C_{pq} = \widetilde{\mathrm{C}}_p(\mathbb{X};\mathcal{M}) \otimes_G F_q(G). \tag{5.1}$$

The spectral sequence we are looking for converges to the homology of this double complex. In fact, there are two spectral sequences converging to the homology of a double complex, one arising by filtering it "horizontally" and the other by filtering it "vertically" (see, e.g., [7, §VII.3]). We will use the horizontal filtration to show that the homology of (5.1) vanishes up to degree r (conclusion (ii)), and then prove that the vertical filtration gives the spectral sequence described in the theorem.

The spectral sequence arising from the horizontal filtration has

$$E^1_{pq} \cong \mathrm{H}_p\left(\widetilde{\mathrm{C}}_\bullet\left(\mathbb{X};\mathcal{M}\right) \otimes_G F_q\left(G\right)\right).$$

Our assumptions imply that $\widetilde{\mathrm{C}}_\bullet(\mathbb{X};\mathcal{M})$ is exact up to degree r, and since $F_q(G)$ is a free $\mathbb{Z}[G]$-module it follows that $\widetilde{\mathrm{C}}_\bullet(\mathbb{X};\mathcal{M}) \otimes_G F_q(G)$ is also exact up to degree r. It follows that $E^1_{pq} = 0$ for $p \leq r$. We deduce that the homology of the double complex (5.1) vanishes up to degree r.

The spectral sequence arising from the vertical filtration has

$$E^1_{pq} \cong \mathrm{H}_q\left(\widetilde{\mathrm{C}}_p\left(\mathbb{X};\mathcal{M}\right) \otimes_G F_\bullet\left(G\right)\right).$$

This is the spectral sequence that is referred to in the theorem, and we must prove that it satisfies (i). Since $F_\bullet(G)$ is a free resolution of the trivial $\Bbbk[G]$-module $\mathbb{Z}$, the above expression simplifies to

$$E^1_{pq} \cong \mathrm{H}_q\left(G;\widetilde{\mathrm{C}}_p\left(\mathbb{X};\mathcal{M}\right)\right).$$

For each p-simplex σ of $\mathbb{X}/G$, fix a lift $\widetilde{\sigma}$ to $\mathbb{X}$. As a $\Bbbk[G]$-module, we have

$$\widetilde{\mathrm{C}}_p\left(\mathbb{X};\mathcal{M}\right) \cong \bigoplus_{\sigma\in(\mathbb{X}/G)^p} \mathrm{Ind}_{G_{\widetilde{\sigma}}}^{G} \mathcal{M}\left(\widetilde{\sigma}\right).$$

Plugging this in, we get

$$E^1_{pq} \cong \bigoplus_{\sigma\in(\mathbb{X}/G)^p} \mathrm{H}_q\left(G;\mathrm{Ind}_{G_{\widetilde{\sigma}}}^{G} \mathcal{M}\left(\widetilde{\sigma}\right)\right).$$

Applying Shapiro's Lemma, the right hand side simplifies to

$$\bigoplus_{\sigma\in(\mathbb{X}/G)^p} \mathrm{H}_q\left(G_{\widetilde{\sigma}};\mathcal{M}\left(\widetilde{\sigma}\right)\right) = \widetilde{\mathrm{C}}_p(\mathbb{X}/G;\mathcal{H}_q(\mathcal{M})).$$

That the differential is as described in (i) is clear. The theorem follows. □

5.5. **Stability machine and finite-index subgroups.** We close this section by proving a variant of Theorem 5.2 that we will use to analyze congruence subgroups of $\mathrm{GL}_n(R)$. To motivate its statement, consider a finite-index normal subgroup G' of a group G and a $\Bbbk[G]$-module M over a field $\Bbbk$ of characteristic 0. The conjugation action of G on G' induces an action of G on each $\mathrm{H}_k(G';M)$, and using the transfer map (see [7, Chapter III.9]) one can show that $\mathrm{H}_k(G;M)$ is the G-coinvariants of the action of G on $\mathrm{H}_k(G';M)$. From this, we see that the map

$$\mathrm{H}_k(G';M) \to \mathrm{H}_k(G;M)$$

induced by the inclusion $G' \hookrightarrow G$ is an isomorphism if and only if the action of G on $\mathrm{H}_k(G'; M)$ is trivial. The point of the following is that under the conditions of our stability machine, it is enough to check a weaker weaker version of this triviality (condition (h)).

Theorem 5.7. *Let $\{(G_n, M_n)\}_{n=0}^{\infty}$ be an increasing sequence of groups and modules. For each $n \geq 1$, let $\mathbb{X}_n$ be a semisimplicial set upon which G_n acts and let $\mathcal{M}_n$ be a G_n-equivariant augmented coefficient system on $\mathbb{X}_n$. Assume for some $c \geq 2$ that conditions (a)-(d) of Theorem 5.2 hold, so by that theorem $\mathrm{H}_k(G_n; M_n)$ stabilizes. Furthermore, assume that $\{G'_n\}_{n=0}^{\infty}$ is an increasing sequence of groups such that each G'_n is a finite-index normal subgroup of G_n, and that the following hold:*

- *(e) Each M_n is a vector space over a field $\Bbbk$ of characteristic 0.*
- *(f) For the k-simplex σ_k of $\mathbb{X}_n$ from condition (b) whose G_n-stabilizer is G_{n-k-1}, the G'_n-stabilizer of σ_k is G'_{n-k-1}.*
- *(g) The quotient $\mathbb{X}_n/G'_n$ is $\frac{n-2}{c}$-connected.*
- *(h) For $k \geq 0$ and $n \geq ck+2$, the action of G_n on $\mathrm{H}_k(G'_n; M_n)$ induced by the conjugation action of G_n on G'_n fixes the image of the stabilization map*

$$\mathrm{H}_k(G'_{n-1}; M(R^{n-1})) \to \mathrm{H}_k(G'_n; M(R^n)). \tag{5.2}$$

Then for $n \geq ck+2$ the map $\mathrm{H}_k(G'_n; M_n) \to \mathrm{H}_k(G_n; M_n)$ induced by the inclusion $G'_n \hookrightarrow G_n$ is an isomorphism.

Proof. The proof will be by induction on k. The base case $k \leq -1$ is trivial, so assume that $k \geq 0$ and that the result is true for all smaller k. Consider some $n \geq ck+2$. As we discussed before the theorem, because of (e) to prove that $\mathrm{H}_k(G'_n; M_n) \cong \mathrm{H}_k(G_n; M_n)$ it is enough to prove that G_n acts trivially on $\mathrm{H}_k(G'_n; M_n)$. Condition (h) implies that to do this, it is enough to prove that $\mathrm{H}_k(G'_n; M_n)$ is spanned by the G_n-orbit of the image of the stabilization map (5.2).

Condition (a) says that $\mathrm{H}_i(\mathbb{X}_n; \mathcal{M}_n) = 0$ for $-1 \leq i \leq \frac{n-2}{c}$. Since $n \geq ck+2$, this vanishing holds for $-1 \leq i \leq k$. Applying Theorem 5.5, we obtain a spectral sequence $E^{\bullet}_{pq}$ with the following properties:

- (i) We have $E^1_{pq} = \widetilde{\mathrm{C}}_p(\mathbb{X}_n/G'_n; \mathcal{H}_q(\mathcal{M}_n))$, and the differential $E^1_{pq} \to E^1_{p-1,q}$ is the differential on $\widetilde{\mathrm{C}}_\bullet(\mathbb{X}_n/G'_n; \mathcal{H}_p(\mathcal{M}_n))$. In particular, $E^1_{pq} = 0$ if $p < -1$ or if $q < 0$.
- (ii) For $p+q \leq k$, we have $E^{\infty}_{pq} = 0$.

Unwinding the definition of $\mathcal{H}_q(\mathcal{M}_n)$, we see that for $*$ the (-1)-simplex of $\mathbb{X}_n$ we have

$$E^1_{-1,k} = \widetilde{\mathrm{C}}_{-1}(\mathbb{X}_n/G'_n; \mathcal{H}_k(\mathcal{M}_n)) = \mathrm{H}_k((G'_n)_*; \mathcal{M}_n(*)) = \mathrm{H}_k(G'_n; M_n).$$

For each simplex τ of $\mathbb{X}_n/G'_n$, fix a lift $\widetilde{\tau}$ to $\mathbb{X}_n$. We then have

$$E^1_{0,k} = \widetilde{\mathrm{C}}_0(\mathbb{X}_n/G'_n; \mathcal{H}_k(\mathcal{M}_n)) = \bigoplus_{\tau \in (\mathbb{X}_n/G'_n)^0} \mathrm{H}_k((G'_n)_{\widetilde{\tau}}; \mathcal{M}_n(\widetilde{\tau})). \tag{5.3}$$

By conditions (b) and (f), we have

$$\mathrm{H}_k((G'_n)_{\sigma_0}; \mathcal{M}_n(\sigma_0)) = \mathrm{H}_k(G'_{n-1}; M_{n-1}).$$

Condition (c) says that G_n acts transitively on the 0-simplices of $\mathbb{X}_n$, so the different $(G'_n)_{\widetilde{\tau}}$ appearing in (5.3) are in the G_n-orbit of G'_{n-1} and the $\mathcal{M}_n(\widetilde{\tau})$ are the corresponding G_n-orbits under the G_n-equivariant structure on $\mathcal{M}_n$.

From these observations, we see that the image of the differential

$$E^1_{0,k} = \bigoplus_{\tau \in (\mathbb{X}_n/G'_n)^0} \mathrm{H}_k((G'_n)_{\widetilde{\tau}}; \mathcal{M}_n(\widetilde{\tau})) \to \mathrm{H}_k(G'_n; M_n) = E^1_{-1,k}$$

is spanned by G_n-orbits of the image of the stabilization map

$$\mathrm{H}_k(G'_{n-1}; M_{n-1}) \to \mathrm{H}_k(G'_n; M_n).$$

To see that $\mathrm{H}_k(G'_n; M_n)$ is spanned by the G_n-orbits of the image of this stabilization map, it is therefore enough to prove that the differential $E^1_{0,k} \to E^1_{-1,k}$ is surjective.

The only nonzero differentials that can possibly hit $E^{\bullet}_{-1,k}$ are

$$\begin{gathered} E^1_{0,k} \to E^1_{-1,k} \\ E^2_{1,k-1} \to E^2_{-1,k} \\ \vdots \\ E^{k+1}_{k,0} \to E^{k+1}_{-1,k}. \end{gathered}$$

By (ii) above we have $E^{\infty}_{-1,k} = 0$, so to prove that the differential $E^1_{0,k} \to E^1_{-1,k}$ is surjective it is enough to prove that $E^{i+1}_{i,k-i} = 0$ for $1 \leq i \leq k$. We will actually prove the following more general result:

Claim. *Fix some $0 \leq q \leq k-1$. Then $E^2_{pq} = 0$ for $-1 \leq p \leq k-q$.*

The terms E^1_{pq} with $-1 \leq p \leq k-q+1$ along with the relevant E^1-differentials are

$$\widetilde{\mathrm{C}}_{-1}(\mathbb{X}_n/G'_n; \mathcal{H}_q(\mathcal{M}_n)) \leftarrow \widetilde{\mathrm{C}}_0(\mathbb{X}_n/G'_n; \mathcal{H}_q(\mathcal{M}_n)) \leftarrow \cdots \leftarrow \widetilde{\mathrm{C}}_{k-q+1}(\mathbb{X}_n/G'_n; \mathcal{H}_q(\mathcal{M}_n)).$$

To prove that the E^2_{pq} with $-1 \leq p \leq k-q$ are all zero, we must prove that this chain complex is exact except possibly at its rightmost term $\widetilde{\mathrm{C}}_{k-q+1}(\mathbb{X}_n/G'_n; \mathcal{H}_q(\mathcal{M}_n))$.

For a p-simplex τ of $\mathbb{X}_n/G'_n$, we have by definition

$$\mathcal{H}_q(\mathcal{M}_n)(\tau) = \mathrm{H}_q((G'_n)_{\widetilde{\tau}}; \mathcal{M}_n(\widetilde{\tau})).$$

Here $\widetilde{\tau}$ is our chosen lift of τ to $\mathbb{X}_n$. Setting $V = \mathrm{H}_q(G_n; M_n)$, the inclusion $(G'_n)_{\widetilde{\tau}} \hookrightarrow G_n$ along with the map

$$\mathcal{M}_n(\widetilde{\tau}) \to \mathcal{M}_n(*) = M_n \quad \text{with } * \text{ the } (-1)\text{-simplex of } \mathbb{X}_n$$

coming from our coefficient system induce a map

$$\mathcal{H}_q(\mathcal{M}_n)(\tau) = \mathrm{H}_q((G'_n)_{\widetilde{\tau}}; \mathcal{M}_n(\widetilde{\tau})) \to \mathrm{H}_q(G_n; M_n) = V. \tag{5.4}$$

Letting $\underline{V}$ be the constant coefficient system on $\mathbb{X}_n/G'_n$ with value V, the maps (5.4) assemble to a map of coefficient systems $\mathcal{H}_q(\mathcal{M}_n) \to \underline{V}$.

Letting $f_p\colon \widetilde{\mathrm{C}}_p(\mathbb{X}_n/G'_n; \mathcal{H}_q(\mathcal{M}_n)) \to \widetilde{\mathrm{C}}_p(\mathbb{X}_n/G'_n; \underline{V})$ be the induced map on reduced chain complexes, we have a commutative diagram

$$\begin{tikzcd} \widetilde{\mathrm{C}}_{-1}(\mathbb{X}_n/G'_n; \mathcal{H}_q(\mathcal{M}_n)) \arrow[d, "f_{-1}"] & \widetilde{\mathrm{C}}_0(\mathbb{X}_n/G'_n; \mathcal{H}_q(\mathcal{M}_n)) \arrow[l] \arrow[d, "f_0"] & \cdots \arrow[l] & \widetilde{\mathrm{C}}_{k-q+1}(\mathbb{X}_n/G'_n; \mathcal{H}_q(\mathcal{M}_n)) \arrow[l] \arrow[d, "f_{k-q+1}"] \\ \widetilde{\mathrm{C}}_{-1}(\mathbb{X}_n/G'_n; \underline{V}) & \widetilde{\mathrm{C}}_0(\mathbb{X}_n/G'_n; \underline{V}) \arrow[l] & \cdots \arrow[l] & \widetilde{\mathrm{C}}_{k-q+1}(\mathbb{X}_n/G'_n; \underline{V}). \arrow[l] \end{tikzcd}$$

Condition (g) says that $\mathbb{X}_n/G'_n$ is $\frac{n-2}{c}$. Since $n \geq ck+2$, this means that $\mathbb{X}_n/G'_n$ is k-connected. Since $0 \leq q \leq k-1$, we deduce that the bottom chain complex of this diagram is exact except possibly at its rightmost term. To prove that the top chain complex is exact except possibly at its rightmost term, it is thus enough to prove that f_p is an isomorphism for $-1 \leq p \leq k-q$ and a surjection for $p = k-q+1$.

Letting τ be a p-simplex of $\mathbb{X}_n/G'_n$, the map $\mathcal{H}_q(\mathcal{M}_n)(\tau) \to \underline{V}(\tau)$ is the map

$$\mathrm{H}_q((G'_n)_{\widetilde{\tau}}; \mathcal{M}_n(\widetilde{\tau})) \to \mathrm{H}_q(G_n; M_n) \tag{5.5}$$

We must prove that this is an isomorphism for $-1 \leq p \leq k-q$ and a surjection for $p = k-q+1$. Condition (c) says that G_n acts transitively on the p-simplices of $\mathbb{X}_n$, so $\widetilde{\tau}$ is in the same G_n-orbit as the p-simplex σ_p from conditions (b) and (f), where

$$\mathrm{H}_q((G'_n)_{\sigma_p}; \mathcal{M}_n(\sigma_p)) = \mathrm{H}_q(G'_{n-p-1}; M_{n-p-1}).$$

Whether or not (5.5) is an isomorphism/surjection is invariant under the G_n-action, so it is enough to prove that

$$\mathrm{H}_q(G'_{n-p-1}; M_{n-p-1}) \to \mathrm{H}_q(G_n; M_n)$$

is an isomorphism for $-1 \leq p \leq k-q$ and a surjection for $p = k-q+1$. Factor this as

$$\mathrm{H}_q(G'_{n-p-1}; M_{n-p-1}) \to \mathrm{H}_q(G_{n-p-1}; M_{n-p-1}) \to \mathrm{H}_q(G_n; M_n).$$

Since G'_{n-p-1} is a finite-index subgroup of G_{n-p-1}, the transfer map (see [7, Chapter III.9]) implies that the first map is always a surjection, and our inductive hypothesis says that it is an isomorphism if $n-p-1 \geq cq+2$. Also, Theorem 5.2 says that the second map is an isomorphism if $n-p-1 \geq cq+2$ and a surjection if $n-p-1 = cq+1$. To prove the theorem, it is thus enough to prove that $n-p-1 \geq cq+2$ if $-1 \leq p \leq k-q$ and that $n-p-1 \geq cg+1$ if $p = k-q+1$.

We check this as follows. If $-1 \leq p \leq k-q$, then since $n \geq ck+2$ and $q \leq k-1$ and $c \geq 2$ we have

$$\begin{aligned} n-p-1 &\geq (ck+2)-(k-q)-1 = (c-1)k+q+1 \\ &\geq (c-1)(q+1)+q+1 = cq+c \\ &\geq cq+2. \end{aligned}$$

If instead $p = k-q+1$, then we have

$$\begin{aligned} n-p-1 &\geq (ck+2)-(k-q+1)-1 = (c-1)+q \\ &\geq (c-1)(q+1)+q = cq+c-1 \\ &\geq cq+1. \end{aligned}$$

□

6. Stability II: the vanishing theorem

Let X be a simplicial complex and let $\mathbb{X}$ be either a small or large ordering of X. To use Theorem 5.2, we need a way to prove that $\widetilde{\mathrm{H}}_k(\mathbb{X}; \mathcal{F}) = 0$ in a range for an augmented coefficient system $\mathcal{F}$. This section contains a useful criterion for this that applies in many situations.

Notation 6.1. The following notation for $\mathbb{X}$ and $\mathcal{F}$ will be used throughout this section:

- Simplices of $\mathbb{X}$ are ordered sequences of distinct vertices of X, and we will write them as $(v_0, \ldots, v_k)$ with the v_i vertices. If $\tau = (v_0, \ldots, v_k)$ and $\sigma = (w_0, \ldots, w_\ell)$ are simplices such that σ is in the forward link $\overrightarrow{Link}_{\mathbb{X}}(\tau)$, then we will write $\tau \cdot \sigma$ for the simplex $(v_0, \ldots, v_k, w_0, \ldots, w_\ell)$. Finally, we will write $\emptyset$ or $()$ for the unique (-1)-simplex used to define the augmentation.
- For a coefficient system $\mathcal{F}$, we will write $\mathcal{F}(v_0, \ldots, v_k)$ for its value on $(v_0, \ldots, v_k)$ rather than the more awkward but technically correct $\mathcal{F}((v_0, \ldots, v_k))$. Also, if σ' is a face of σ, then there is only one face map taking σ to σ', so we will just talk about the induced map $\mathcal{F}(\sigma) \to \mathcal{F}(\sigma')$. □

6.1. **Polynomiality.** Our criterion applies to augmented coefficient systems $\mathcal{F}$ that are *polynomial* of degree $d \geq -1$ up to dimension $e \geq 0$. This condition is inspired by the notion of polynomial $\texttt{FI}$-modules (see Definition 1.6). It is defined inductively in the degree d as follows:

- A coefficient system $\mathcal{F}$ is polynomial of degree -1 up to dimension e if for all simplices σ of dimension at most e, we have $\mathcal{F}(\sigma) = 0$.
- A coefficient system $\mathcal{F}$ is polynomial of degree $d \geq 0$ up to dimension e if it satisfies the following two conditions:
  - If σ is a simplex of dimension at most e, then the map $\mathcal{F}(\sigma) \to \mathcal{F}(\emptyset)$ is injective.
  - Let $\tau = (w_0, \ldots, w_\ell)$ be a simplex with $\ell \leq e$ and let $\mathbb{Y}$ be the forward link $\overrightarrow{Link}_{\mathbb{X}}(\tau)$. Let $\mathcal{G}$ be the coefficient system on $\mathbb{Y}$ defined by the formula
  $$\mathcal{G}(\sigma) = \frac{\mathcal{F}(\sigma)}{\operatorname{Im}\left(\mathcal{F}\left(w_\ell \cdot \sigma\right) \to \mathcal{F}\left(\sigma\right)\right)} \quad \text{for a simplex } \sigma \text{ of } \mathbb{Y}.$$
  Then $\mathcal{G}$ must be polynomial of degree $d-1$ up to dimension $e-\ell$.

6.2. **Key example.** The following lemma provides motivation for this definition.

Lemma 6.2. *Let M be an $\texttt{FI}$-module that is polynomial of degree d starting at $m \geq 0$ (see Definition 1.6). Fix some $n \geq m$, and let $\mathcal{F}_{M,n}$ be the coefficient system on $\mathbb{S}\mathrm{im}_n$ discussed in Example 4.2, so*

$$\mathcal{F}_{M,n}(i_0, \ldots, i_k) = M([n] \setminus \{i_0, \ldots, i_k\}) \quad \textit{for all simplices } (i_0, \ldots, i_k) \textit{ of } \mathbb{S}\mathrm{im}_n.$$

Then $\mathcal{F}_{M,n}$ is polynomial of degree d up to dimension $n-m$.

Proof. The proof will be by induction on d. If $d = -1$, then consider a simplex $\sigma = (i_0, \ldots, i_k)$ with $k \leq n-m$. We then have

$$\mathcal{F}_{M,n}(\sigma) = M([n] \setminus \{i_0, \ldots, i_k\}).$$

Since

$$|[n] \setminus \{i_0, \ldots, i_k\}| = (n+1) - (k+1) = n-k \geq m, \tag{6.1}$$

it follows from the fact that M is polynomial of degree -1 starting at m that $\mathcal{F}_{M,n}(\sigma) = 0$, as desired.

Now assume that $d \geq 0$. There are two things to check. For the first, let $\sigma = (i_0, \ldots, i_k)$ be a simplex with $k \leq n-m$. We must prove that the map $\mathcal{F}_{M,n}(\sigma) \to \mathcal{F}_{M,n}(\emptyset)$ is injective, i.e., that the map

$$M([n] \setminus \{i_0, \ldots, i_k\}) \to M([n])$$

is injective. The calculation (6.1) shows that this injectivity follows from the fact that M is polynomial of degree d starting at m.

For the second, let $\tau = (w_0, \ldots, w_\ell)$ be any simplex with $\ell \leq n-m$ and let $\mathbb{Y}$ be the forward link $\overrightarrow{Link}_{\mathbb{S}\mathrm{im}_n}(\tau)$. Let $\mathcal{G}$ be the coefficient system on $\mathbb{Y}$ defined by the formula

$$\mathcal{G}(\sigma) = \frac{\mathcal{F}_{M,n}(\sigma)}{\operatorname{Im}\left(\mathcal{F}_{M,n}\left(w_\ell \cdot \sigma\right) \to \mathcal{F}_{M,n}\left(\sigma\right)\right)} \quad \text{for a simplex } \sigma \text{ of } \mathbb{Y}.$$

We must prove that $\mathcal{G}$ is polynomial of degree $d-1$ up to dimension $n-m-\ell$. Without loss of generality, $\tau = (n-\ell, \ldots, n)$, so $\mathbb{Y} = \overrightarrow{Link}_{\mathbb{S}\mathrm{im}_n}(\tau) = \mathbb{S}\mathrm{im}_{n-\ell-1}$. Recall that we defined the derived $\texttt{FI}$-module DM in Definition 1.4. By construction, there is an isomorphism between the coefficient systems $\mathcal{G}$ and $\mathcal{F}_{DM,n-\ell-1}$ on $\mathbb{S}\mathrm{im}_{n-\ell-1}$. Since M is polynomial of degree d starting at m, the $\texttt{FI}$-module DM is polynomial of degree $d-1$ starting at $m-1$. By induction, $\mathcal{G}$ is polynomial of degree $d-1$ starting at $(n-\ell-1)-(m-1) = n-m-\ell$, as desired. $\square$

6.3. **Statement of vanishing theorem.** Having made these definitions, our vanishing theorem is as follows:

Theorem 6.3. *For some $N \geq -1$ and $d \geq 0$, let X be a simplicial complex that is weakly Cohen–Macaulay of dimension $N+d+1$ and let $\mathbb{X}$ be either a CM-small or large ordering of X. Let $\mathcal{F}$ be an augmented coefficient system on $\mathbb{X}$ that is polynomial of degree d up to dimension N. Then $\widetilde{\mathrm{H}}_k(\mathbb{X};\mathcal{F}) = 0$ for $-1 \leq k \leq N$.*

Proof. By definition when $\mathbb{X}$ is a CM-small ordering of X and by Lemma 3.7 when $\mathbb{X}$ is the large ordering of X, we have that

$$\mathbb{X} \text{ is weakly forward Cohen–Macaulay of dimension } N+d+1. \tag{6.2}$$

The proof will be by induction on d. For the base case $d = -1$, by definition the coefficient system $\mathcal{F}$ equals the constant coefficient system $\underline{0}$ on the N-skeleton of $\mathbb{X}$. This trivially implies that for $-1 \leq k \leq N$ we have $\widetilde{\mathrm{H}}_k(\mathbb{X};\mathcal{F}) = 0$.

Assume now that $d \geq 0$ and that the theorem is true for smaller d. We divide the rest of the proof into four steps. The descriptions of the steps describe the objects that are introduced during that step and what the step reduces the theorem to, with the fifth step proving the result. The inductive hypothesis is invoked in the fourth step.

Step 1. *We introduce the shifted coefficient systems $\mathcal{F}_n$ and reduce the theorem to proving that $\widetilde{\mathrm{H}}_k(\mathbb{X};\mathcal{F}) \cong \widetilde{\mathrm{H}}_k(\mathbb{X};\mathcal{F}_{N+1})$ for $-1 \leq k \leq N$.*

For $n \geq -1$, define an augmented coefficient system $\mathcal{F}_n$ on $\mathbb{X}$ via the formula

$$\mathcal{F}_n(v_0,\ldots,v_k) = \begin{cases} \mathcal{F}(\emptyset) & \text{if } n \geq k \\ \mathcal{F}(v_{n+1},\ldots,v_k) & \text{if } n < k \end{cases} \quad \text{for a simplex } (v_0,\ldots,v_k) \text{ of } \mathbb{X}.$$

The maps $\mathcal{F}_n(\sigma) \to \mathcal{F}_n(\sigma')$ when σ' is a face of σ are the ones induced by $\mathcal{F}$. The augmented coefficient system $\mathcal{F}_{N+1}$ equals the constant coefficient system $\underline{\mathcal{F}(\emptyset)}$ on all simplices of dimension at most $(N+1)$, so the fact that $\mathbb{X}$ is weakly forward Cohen–Macaulay of dimension $(N+d+1)$ (see (6.2)) implies that $\mathbb{X}$ is $N+d \geq N$ connected and hence

$$\widetilde{\mathrm{H}}_k(\mathbb{X};\mathcal{F}_{N+1}) = \widetilde{\mathrm{H}}_k(\mathbb{X};\underline{\mathcal{F}(\emptyset)}) = 0 \quad \text{for } -1 \leq k \leq N.$$

There is a map $\mathcal{F} \to \mathcal{F}_{N+1}$, and to prove the theorem it is enough to prove that this induces an isomorphism $\widetilde{\mathrm{H}}_k(\mathbb{X};\mathcal{F}) \cong \widetilde{\mathrm{H}}_k(\mathbb{X};\mathcal{F}_{N+1})$ for $-1 \leq k \leq N$.

Step 2. *We introduce the quotiented shifted coefficient systems $\overline{\mathcal{F}}_n$ and reduce the theorem to proving that $\widetilde{\mathrm{H}}_k(\mathbb{X};\overline{\mathcal{F}}_n) = 0$ for $0 \leq n \leq k \leq N+1$.*

We can factor the map $\mathcal{F} \to \mathcal{F}_{N+1}$ as

$$\mathcal{F} = \mathcal{F}_{-1} \to \mathcal{F}_0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_{N+1},$$

so it is enough to prove that the map $\mathcal{F}_{n-1} \to \mathcal{F}_n$ induces an isomorphism $\widetilde{\mathrm{H}}_k(\mathbb{X};\mathcal{F}_{n-1}) \cong \widetilde{\mathrm{H}}_k(\mathbb{X};\mathcal{F}_n)$ for $0 \leq n \leq N+1$ and $-1 \leq k \leq N$.

Define

$$\begin{aligned} \overline{\mathcal{F}}_n &= \mathrm{coker}(\mathcal{F}_{n-1} \to \mathcal{F}_n), \\ \mathcal{F}'_{n-1} &= \mathrm{Im}(\mathcal{F}_{n-1} \to \mathcal{F}_n), \\ \mathcal{F}''_{n-1} &= \ker(\mathcal{F}_{n-1} \to \mathcal{F}_n). \end{aligned}$$

We thus have short exact sequences of coefficient systems

$$0 \longrightarrow \mathcal{F}'_{n-1} \longrightarrow \mathcal{F}_n \longrightarrow \overline{\mathcal{F}}_n \longrightarrow 0 \tag{6.3}$$

and

$$0 \longrightarrow \mathcal{F}''_{n-1} \longrightarrow \mathcal{F}_{n-1} \longrightarrow \mathcal{F}'_{n-1} \longrightarrow 0. \tag{6.4}$$

Both of these induce long exact sequences in homology. Let us focus first on the one associated to (6.4), which contains segments of the form

$$\widetilde{\mathrm{H}}_k(\mathbb{X};\mathcal{F}''_{n-1}) \longrightarrow \widetilde{\mathrm{H}}_k(\mathbb{X};\mathcal{F}_{n-1}) \longrightarrow \widetilde{\mathrm{H}}_k(\mathbb{X};\mathcal{F}'_{n-1}) \longrightarrow \widetilde{\mathrm{H}}_{k-1}(\mathbb{X};\mathcal{F}''_{n-1}). \tag{6.5}$$

Since $\mathcal{F}$ is polynomial of degree d up to dimension N, for all simplices σ of dimension at most N and all faces σ' of σ the map $\mathcal{F}(\sigma) \to \mathcal{F}(\sigma')$ is injective. This implies that the map $\mathcal{F}_{n-1}(\sigma) \to \mathcal{F}_n(\sigma)$ is injective as long as σ has dimension at most N, and thus that $\mathcal{F}''_{n-1}(\sigma) = 0$. It follows that $\widetilde{\mathrm{H}}_k(\mathbb{X};\mathcal{F}''_{n-1}) = 0$ for $k \leq N$. Combining this with (6.5), we see that

$$\widetilde{\mathrm{H}}_k(\mathbb{X};\mathcal{F}_{n-1}) \cong \widetilde{\mathrm{H}}_k(\mathbb{X};\mathcal{F}'_{n-1}) \quad \text{for } k \leq N. \tag{6.6}$$

We now turn to the long exact sequence associated to (6.3), which contains the segment

$$\widetilde{\mathrm{H}}_{k+1}(\mathbb{X};\overline{\mathcal{F}}_n) \longrightarrow \widetilde{\mathrm{H}}_k(\mathbb{X};\mathcal{F}'_{n-1}) \longrightarrow \widetilde{\mathrm{H}}_k(\mathbb{X};\mathcal{F}_n) \longrightarrow \widetilde{\mathrm{H}}_k(\mathbb{X};\overline{\mathcal{F}}_n).$$

In light of (6.6), this implies that to prove that the map $\mathcal{F}_{n-1} \to \mathcal{F}_n$ induces an isomorphism $\widetilde{\mathrm{H}}_k(\mathbb{X};\mathcal{F}_{n-1}) \cong \widetilde{\mathrm{H}}_k(\mathbb{X};\mathcal{F}_n)$ for $0 \leq n \leq N+1$ and $-1 \leq k \leq N$, it is enough to prove that

$$\widetilde{\mathrm{H}}_k(\mathbb{X};\overline{\mathcal{F}}_n) = 0 \quad \text{for } 0 \leq n \leq N+1 \text{ and } -1 \leq k \leq N+1.$$

This can be simplified a little further: if σ is a k-simplex with $k \leq n-1$, then $\mathcal{F}_{n-1}(\sigma) = \mathcal{F}_n(\sigma) = \mathcal{F}(\emptyset)$, so $\overline{\mathcal{F}}_n(\sigma) = 0$. The group $\widetilde{\mathrm{H}}_k(\mathbb{X};\overline{\mathcal{F}}_n)$ is thus automatically 0 for $k \leq n-1$, so it is actually enough to prove that

$$\widetilde{\mathrm{H}}_k(\mathbb{X};\overline{\mathcal{F}}_n) = 0 \quad \text{for } 0 \leq n \leq k \leq N+1.$$

Step 3. *For each n-simplex τ with $0 \leq n \leq N+1$, we construct coefficient systems $\overline{\mathcal{F}}_{n,\tau}$ and prove that $\overline{\mathcal{F}}_n$ is the direct sum of the $\overline{\mathcal{F}}_{n,\tau}$ as τ ranges over the n-simplices, reducing the theorem to proving that $\widetilde{\mathrm{H}}_k(\mathbb{X};\overline{\mathcal{F}}_{n,\tau}) = 0$ for $0 \leq n \leq k \leq N+1$.*

Fix some $0 \leq n \leq N+1$, and consider a k-simplex σ of $\mathbb{X}$ with $k \geq n$. We claim that the following holds:

- Let σ' be a face of σ. Then the map $\overline{\mathcal{F}}_n(\sigma) \to \overline{\mathcal{F}}_n(\sigma')$ is the zero map unless the first $(n+1)$ vertices of σ and σ' are equal.

To prove this, it is enough to prove the following special case:

- Let $(w_0,\ldots,w_n,v_0,\ldots,v_m)$ be a simplex of $\mathbb{X}$ and let $0 \leq i \leq n$. Then the map $\overline{\mathcal{F}}_n(w_0,\ldots,w_n,v_0,\ldots,v_m) \to \overline{\mathcal{F}}_n(w_0,\ldots,\widehat{w_i},\ldots,w_n,v_0,\ldots,v_m)$ is the zero map.

For this, observe that

$$\overline{\mathcal{F}}_n(w_0,\ldots,w_n,v_0,\ldots,v_m) = \frac{\mathcal{F}(v_0,\ldots,v_m)}{\mathcal{F}(w_n,v_0,\ldots,v_m)}$$

and

$$\overline{\mathcal{F}}_n(w_0,\ldots,\widehat{w_i},\ldots,w_n,v_0,\ldots,v_m) = \frac{\mathcal{F}(v_1,\ldots,v_m)}{\mathcal{F}(v_0,\ldots,v_m)}.$$

The map between these is visibly the zero map.

For each n-simplex τ of $\mathbb{X}$, this suggests defining a coefficient system $\overline{\mathcal{F}}_{n,\tau}$ on $\mathbb{X}$ via the formula

$$\overline{\mathcal{F}}_{n,\tau}(\sigma) = \begin{cases} \overline{\mathcal{F}}_n(\sigma) & \text{if } \sigma \text{ starts with } \tau \\ 0 & \text{otherwise} \end{cases} \qquad \text{for a simplex } \sigma \text{ of } \mathbb{X}.$$

By the above, we have a decomposition

$$\overline{\mathcal{F}}_n = \bigoplus_{\tau \in \mathbb{X}^n} \overline{\mathcal{F}}_{n,\tau}$$

of coefficient systems. To prove that

$$\widetilde{\mathrm{H}}_k(\mathbb{X}; \overline{\mathcal{F}}_n) = 0 \quad \text{for } 0 \leq n \leq k \leq N+1,$$

it is thus enough to prove that for all $\tau \in \mathbb{X}^n$ we have

$$\widetilde{\mathrm{H}}_k(\mathbb{X}; \overline{\mathcal{F}}_{n,\tau}) = 0 \quad \text{for } 0 \leq n \leq k \leq N+1.$$

Step 4. *Fix an n-simplex τ with $0 \leq n \leq N+1$. We prove that $\widetilde{\mathrm{H}}_k(\mathbb{X}; \overline{\mathcal{F}}_{n,\tau}) = 0$ for $0 \leq n \leq k \leq N+1$.*

Define $\mathbb{Y} = \overrightarrow{Link}_{\mathbb{X}}(\tau)$. Recalling our notation for simplices of $\mathbb{X}$ in Notation 6.1, define a coefficient system $\mathcal{G}$ on $\mathbb{Y}$ via the formula

$$\mathcal{G}(\sigma) = \overline{\mathcal{F}}_{n,\tau}(\tau \cdot \sigma) \quad \text{for a simplex } \sigma \text{ of } \mathbb{Y}.$$

On the level of reduced chain complexes, up to multiplying the differentials by $(-1)^{n+1}$ we have

$$\widetilde{\mathrm{C}}_\bullet(\mathbb{Y}; \mathcal{G}) \cong \widetilde{\mathrm{C}}_{\bullet+n+1}(\mathbb{X}; \overline{\mathcal{F}}_{n,\tau}),$$

so $\widetilde{\mathrm{H}}_k(\mathbb{Y}; \mathcal{G}) \cong \widetilde{\mathrm{H}}_{k+n+1}(\mathbb{X}; \overline{\mathcal{F}}_{n,\tau})$. It is thus enough to prove that $\widetilde{\mathrm{H}}_k(\mathbb{Y}; \mathcal{G}) = 0$ for $-1 \leq k \leq N-n$.

This will be an application of our inductive hypothesis. For this, we make two observations:

- Since $\mathbb{X}$ is weakly forward Cohen–Macaulay of dimension $(N+d+1)$ (see (6.2)), the forward link $\mathbb{Y}$ of the n-simplex τ is weakly forward Cohen–Macaulay of dimension $(N-n+d)$.
- Write $\tau = (w_0, \ldots, w_n)$. For a simplex σ of $\mathbb{Y}$, we have

  $$\mathcal{G}(\sigma) = \overline{\mathcal{F}}_{n,\tau}((w_0, \ldots, w_n) \cdot \sigma) = \frac{\mathcal{F}(\sigma)}{\mathcal{F}((w_n) \cdot \sigma)}.$$

  Since $\mathcal{F}$ is a polynomial coefficient system of degree d up to dimension N, it follows that $\mathcal{G}$ is a polynomial coefficient of degree $(d-1)$ up to dimension $(N-n)$.

Since

$$N - n + d = (N-n) + (d-1) + 1,$$

we can apply our inductive hypothesis to conclude that $\widetilde{\mathrm{H}}_k(\mathbb{Y}; \mathcal{G}) = 0$ for $-1 \leq k \leq N-n$. ☐

7. Stability for symmetric groups

We now turn to applications of our machinery, starting with Theorems A and A′.

Proof of Theorem A. We first recall the statement. Let $\Bbbk$ be a commutative ring and let M be an $\mathtt{FI}$-module over $\Bbbk$ that is polynomial of degree d starting at $m \geq 0$. For each $k \geq 0$, we must prove that the map

$$\mathrm{H}_k(\mathfrak{S}_n; M(\overline{n})) \to \mathrm{H}_k(\mathfrak{S}_{n+1}; M(\overline{n+1}))$$

is an isomorphism for $n \geq 2k + \max(d, m-1) + 2$ and a surjection for $n = 2k + \max(d, m-1) + 1$. Recall that $\overline{n} = \{1, \ldots, n\}$ and $[n-1] = \{0, \ldots, n-1\}$. The group $\mathfrak{S}_n$ acts on both $M(\overline{n})$ and $M([n-1])$, and there is a $\Bbbk[\mathfrak{S}_n]$-module isomorphism $M(\overline{n}) \cong M([n-1])$. In light of this, it is enough to deal with the map

$$\mathrm{H}_k(\mathfrak{S}_n; M([n-1])) \to \mathrm{H}_k(\mathfrak{S}_{n+1}; M([n])),$$

which will fit into our framework a little better.

The group $\mathfrak{S}_n$ acts on $\mathbb{S}\mathrm{im}_{n-1}$. Let $\mathcal{F}_{M,n-1}$ be the $\mathfrak{S}_n$-equivariant augmented system of coefficients on $\mathbb{S}\mathrm{im}_{n-1}$ from Example 4.3, so

$$\mathcal{F}_{M,n-1}(i_0,\ldots,i_k) = M([n-1] \setminus \{i_0,\ldots,i_k\}) \quad \text{for a simplex } (i_0,\ldots,i_k) \text{ of } \mathbb{S}\mathrm{im}_{n-1}.$$

The following claim will be used to show that with an appropriate degree shift, this all satisfies the hypotheses of Theorem 5.2.

Claim. *The following hold:*

(a) For all $-1 \leq k \leq n - \max(d, m-1) - 2$, we have $\widetilde{\mathrm{H}}_k(\mathbb{S}\mathrm{im}_{n-1}; \mathcal{F}_{M,n-1}) = 0$.
(b) For all $-1 \leq k \leq n-1$, the group $\mathfrak{S}_{n-k-1}$ is the $\mathfrak{S}_n$-stabilizer of a k-simplex σ_k of $\mathbb{S}\mathrm{im}_{n-1}$ with $\mathcal{F}_{M,n-1}(\sigma_k) = M([n-k-2])$.
(c) For all $0 \leq k \leq n-1$, the group $\mathfrak{S}_n$ acts transitively on the k-simplices of $\mathbb{S}\mathrm{im}_{n-1}$.
(d) For all $n \geq 2$ and all 1-simplices e of $\mathbb{S}\mathrm{im}_{n-1}$ of the form $e = (i_0, i_1)$, there exists some $\lambda \in \mathfrak{S}_n$ with $\lambda(i_0) = i_1$) such that λ commutes with all elements of $(\mathfrak{S}_n)_e$ and fixes all elements of $\mathcal{F}_{M,n-1}(e)$.

Proof of claim. For (a), Lemma 6.2 says that $\mathcal{F}_{M,n-1}$ is a polynomial coefficient system of degree d up to dimension $n-m-1$. Also, $\mathbb{S}\mathrm{im}_{n-1}$ is the large ordering of the $(n-1)$-simplex Sim_{n-1} and Sim_{n-1} is weakly Cohen–Macaulay of dimension $(n-1)$. Letting

$$N = \min(n-m-1, n-d-2) = n - \max(d, m-1) - 2,$$

the complex Sim_{n-1} is weakly Cohen–Macaulay of dimension $N + d + 1$ and $\mathcal{F}_{M,n-1}$ is a polynomial coefficient system of degree d up to dimension N. Theorem 6.3 thus implies that $\widetilde{\mathrm{H}}_k(\mathbb{S}\mathrm{im}_{n-1}; \mathcal{F}_{M,n-1}) = 0$ for $-1 \leq k \leq N$.

For (b), the group $\mathfrak{S}_{n-k-1}$ is the $\mathfrak{S}_n$-stabilizer of the k-simplex

$$\sigma_k = \begin{cases} (\emptyset) & \text{if } k = -1, \\ (n-k-1, n-k+1, \ldots, n-1) & \text{if } 0 \leq k \leq n-1 \end{cases}$$

of $\mathbb{S}\mathrm{im}_{n-1}$, and by definition

$$\mathcal{F}_{M,n-1}(\sigma_k) = M([n-1] \setminus \{n-k-1, n-k+1, \ldots, n-1\}) = M([n-k-2]).$$

Condition (c) is obvious.

For (d), simply let λ be the transposition (i_0, i_1), which acts trivially on

$$\mathcal{F}_{M,n-1}(e) = M([n-1] \setminus \{i_0, i_1\})$$

by basic properties of $\mathtt{FI}$-modules. □

Letting $e = \max(d, m-1)$, the Claim verifies that the conditions of Theorem 5.2 are satisfied for

$$G_n = \mathfrak{S}_{n+e+1} \quad \text{and} \quad M_n = M([n+e]) \quad \text{and} \quad \mathbb{X}_n = \mathbb{S}\mathrm{im}_{n+e} \quad \text{and} \quad \mathcal{M}_n = \mathcal{F}_{M,n+e}.$$

with $c = 2$. The shift by $e+1$ is needed for condition (a) of Theorem 5.2, which requires that $\widetilde{\mathrm{H}}_k(\mathbb{X}_n; \mathcal{M}_n) = 0$ for all $-1 \leq k \leq \frac{n-2}{2}$. Conclusion (a) of the Claim says that $\widetilde{\mathrm{H}}_k(\mathbb{S}\mathrm{im}_{n+e}; \mathcal{F}_{M,n+e}) = 0$ for $-1 \leq k \leq (n+e+1) - e - 2$, which implies the desired range of vanishing for $\widetilde{\mathrm{H}}_k(\mathbb{X}_n; \mathcal{M}_n)$ since

$$(n+e+1) - e - 2 = n - 1 \geq \frac{n-2}{2} \quad \text{for } n \geq 0.$$

Applying Theorem 5.2, we deduce that the map

$$\mathrm{H}_k(\mathfrak{S}_{n+e}; M([n+e-1])) \to \mathrm{H}_k(\mathfrak{S}_{n+e+1}; M([n+e]))$$

is an isomorphism for $n \geq 2k+2$ and a surjection for $n = 2k+1$, which implies that

$$\mathrm{H}_k(\mathfrak{S}_n; M([n-1])) \to \mathrm{H}_k(\mathfrak{S}_{n+1}; M([n]))$$

is an isomorphism for $n \geq 2k + e + 2$ and a surjection for $n = 2k + e + 1$. $\square$

Proof of Theorem A′. We first recall the statement. Let $\Bbbk$ be a commutative ring and let M be an $\mathtt{FI}$-module over $\Bbbk$ that is polynomial of degree d starting at $m \geq 0$. For each $k \geq 0$, we must prove that the map

$$\mathrm{H}_k(\mathfrak{S}_n; M(\overline{n})) \to \mathrm{H}_k(\mathfrak{S}_{n+1}; M(\overline{n+1})) \tag{7.1}$$

is an isomorphism for $n \geq \max(m, 2k + 2d + 2)$ and a surjection for $n \geq \max(m, 2k + 2d)$.

The proof will be by double induction on d and m. There are three base cases:

- The first is where $m = 0$ and $d \geq 1$. Theorem A says in this case that (7.1) is an isomorphism for
$$n \geq 2k + \max(d, m-1) + 2 = 2k + \max(d, -1) + 2 = 2k + d + 2$$
and a surjection for
$$n = 2k + \max(d, m-1) + 1 = 2k + \max(d, -1) + 1 = 2k + d + 1.$$
Since $d \geq 1$, these bounds are even stronger than our purported bounds of
$$n \geq \max(m, 2k + 2d + 2) = \max(0, 2k + 2d + 3) = 2k + 2d + 2$$
for (7.1) to be an isomorphism and
$$n \geq \max(m, 2k + 2d) = \max(0, 2k + 2d + 1) = 2k + 2d$$
for (7.1) to be a surjection.
- The second is where $m = 0$ and $d = 0$. As we discussed in Example 1.9, this implies that the $M(\overline{n})$ are all the same trivial $\Bbbk[\mathfrak{S}_n]$-representation. We can thus appeal to the classical theorem of Nakaoka [22, Corollary 6.7] saying that for these trivial constant coefficient systems the map (7.1) is an isomorphism for $n \geq 2k$. This is again even stronger than our purported bounds of
$$n \geq \max(m, 2k + 2d + 2) = \max(0, 2k + 2) = 2k + 2$$
for (7.1) to be an isomorphism and
$$n \geq \max(m, 2k + 2d) = \max(0, 2k) = 2k$$
for (7.1) to be a surjection.
- The third is where $m \geq 0$ and $d = -1$. In this case, by the definition of an $\mathtt{FI}$-module being polynomial of degree -1 starting at m we have for $n \geq m$ that $M(\overline{n}) = 0$ and hence $\mathrm{H}_k(\mathfrak{S}_n; M(\overline{n})) = 0$. In other words, for $n \geq m$ the domain and codomain of (7.1) are both 0, so it is trivially an isomorphism.

Assume now that $m \geq 1$ and $d \geq 0$, and that the theorem is true for all smaller m and d. As in Definition 1.6, let ΣM be the shifted $\mathtt{FI}$-module and DM be the derived $\mathtt{FI}$-module. For $n \geq m$, we have a short exact sequence

$$0 \longrightarrow M(\overline{n}) \longrightarrow \Sigma M(\overline{n}) \longrightarrow DM(\overline{n}) \longrightarrow 0 \tag{7.2}$$

of $\Bbbk[\mathfrak{S}_n]$-modules (see Remark 1.7). The $\mathtt{FI}$-module ΣM is polynomial of degree d starting at $(m-1)$, and the $\mathtt{FI}$-module DM is polynomial of degree $(d-1)$ starting at $(m-1)$.

To simplify our notation, for all $r \geq 0$ and all $\Bbbk[\mathfrak{S}_r]$-modules N, we will denote $\mathrm{H}_k(\mathfrak{S}_r; N)$ by $\mathrm{H}_k(N)$. The long exact sequence in $\mathfrak{S}_n$-homology associated to (7.2) maps to the one in $\mathfrak{S}_{n+1}$-homology, so for $n \geq m$ and all k we have a commutative diagram

$$\begin{tikzcd}
\mathrm{H}_{k+1}(\Sigma M(\overline{n})) \arrow[r] \arrow[d, "g_1"] & \mathrm{H}_{k+1}(DM(\overline{n})) \arrow[r] \arrow[d, "g_2"] & \mathrm{H}_k(M(\overline{n})) \arrow[r] \arrow[d, "f_1"] & \mathrm{H}_k(\Sigma M(\overline{n})) \arrow[r] \arrow[d, "f_2"] & \mathrm{H}_k(DM(\overline{n})) \arrow[d, "f_3"] \\
\mathrm{H}_{k+1}(\Sigma M(\overline{n+1})) \arrow[r] & \mathrm{H}_{k+1}(DM(\overline{n+1})) \arrow[r] & \mathrm{H}_k(M(\overline{n+1})) \arrow[r] & \mathrm{H}_k(\Sigma M(\overline{n+1})) \arrow[r] & \mathrm{H}_k(DM(\overline{n+1}))
\end{tikzcd}$$

with exact rows. Our inductive hypothesis says the following about the g_i and f_i:

- Since ΣM is polynomial of degree d starting at $(m-1)$, the map f_2 is an isomorphism for $n \geq \max(m-1, 2k+2d+2)$ and a surjection for $n \geq \max(m-1, 2k+2d)$. Also, the map g_1 is an isomorphism for
$$n \geq \max(m-1, 2(k+1)+2d+2) = \max(m-1, 2k+2d+4)$$
and a surjection for $n \geq \max(m-1, 2k+2d+2)$.
- Since DM is polynomial of degree $(d-1)$ starting at $(m-1)$, the map f_3 is an isomorphism for
$$n \geq \max(m-1, 2k+2(d-1)+2) = \max(m-1, 2k+2d)$$
and a surjection for $n \geq \max(m-1, 2k+2d-2)$. Also, the map g_2 is an isomorphism for
$$n \geq \max(m-1, 2(k+1)+2(d-1)+2) = \max(m-1, 2k+2d+2)$$
and a surjection for $n \geq \max(m-1, 2k+2d)$.

For $n \geq \max(m, 2k+2d+2)$, the maps g_2 and f_2 and f_3 are isomorphisms and the map g_1 is a surjection, so by the five-lemma the map f_1 is an isomorphism. For $n \geq \max(m, 2k+2d)$, the maps g_2 and f_2 are surjections and the map f_3 is an isomorphism, so by the five-lemma[14] the map f_1 is a surjection. The claim follows. □

8. The stable rank of rings

The rest of the paper is devoted to the general linear group and its congruence subgroups. We begin with a discussion of a ring-theoretic condition called the stable rank that was introduced by Bass [2]. To make this paper a bit more self-contained,[15] we will include the proofs of the results we need (most of which are due to Bass) if they are short.

8.1. **Unimodular vectors.** The starting point is the following.

Definition 8.1. Let R be a ring. A vector $v \in R^n$ is *unimodular* if there is a homomorphism $\phi\colon R^n \to R$ of right R-modules such that $\phi(v) = 1$. □

Example 8.2. The columns of matrices in $\mathrm{GL}_n(R)$ are unimodular. □

The following lemma might clarify this definition.

Lemma 8.3. *Let R be a ring and let $v = (c_1, \ldots, c_n) \in R^n$. Then v is unimodular if and only if there exist $a_1, \ldots, a_n \in R$ such that $a_1 c_1 + \cdots + a_n c_n = 1$.*

Proof. Immediate from the fact that all right R-module morphisms $\phi\colon R^n \to R$ are of the form $\phi(x_1, \ldots, x_n) = a_1 x_1 + \cdots + a_n x_n$ for some $a_1, \ldots, a_n \in R$. □

8.2. **Stable rank.** In light of Example 8.2, one might hope that all unimodular vectors in R^n can appear as columns of matrices in $\mathrm{GL}_n(R)$. This holds if R is a field or if $R = \mathbb{Z}$, but does not hold in general. To clarify this, we make the following definition.

Definition 8.4. A ring R satisfies *Bass's stable rank condition* (SR_r) if the following holds for all $n \geq r$. Let $(c_1, \ldots, c_n) \in R^n$ be a unimodular vector. Then there exist $b_1, \ldots, b_{n-1} \in R$ such that $(c_1 + b_1 c_n, \ldots, c_{n-1} + b_{n-1} c_n) \in R^{n-1}$ is unimodular. □

Example 8.5. This condition only make sense for $r \geq 2$. It is easy to see that fields satisfy (SR_2) and that PIDs satisfy (SR_3). More generally, if a ring R is finitely generated as a module over a Noetherian commutative ring S of Krull dimension r, then R satisfies (SR_{r+2}) (see [2, Theorem V.3.5]). □

[14] Or, more precisely, one of the four-lemmas.

[15] And to avoid depending on [2], which is out of print.

8.3. **Elementary matrices and their action on unimodular vectors.** Recall that an *elementary matrix* in $\mathrm{GL}_n(R)$ is a matrix that differs from the identity matrix at exactly one off-diagonal position. Let $\mathrm{EL}_n(R)$ be the subgroup of $\mathrm{GL}_n(R)$ generated by elementary matrices. One of the key properties of rings satisfying Bass's stable rank condition is the following lemma, which implies in particular that under its hypotheses all unimodular vectors appear as columns of matrices in $\mathrm{GL}_n(R)$. In fact, they even appear as columns of matrices in $\mathrm{EL}_n(R)$.

Lemma 8.6. *Let R be a ring satisfying (SR_r) and let $n \geq r$. Then $\mathrm{EL}_n(R)$ acts transitively on the set of unimodular vectors in R^n.*

Proof. Let $v \in R^n$ be a unimodular vector. It is enough to find some $M \in \mathrm{EL}_n(R)$ such that $M \cdot v = (0, \ldots, 0, 1)$. Write $v = (c_1, \ldots, c_n)$. Using elementary matrices, for any distinct $1 \leq i, j \leq n$ and any $\lambda \in R$ we can add λ times the i^{th} entry to the j^{th} one, changing the entry c_j to $c_j + \lambda c_i$. We will use a series of these moves to transform v into $(0, \ldots, 0, 1)$.

Applying (SR_r), we can add multiples of the last entry to the other ones to ensure that the first $(n-1)$ entries of v form a unimodular vector in R^{n-1}. Using Lemma 8.3, we can find $a_1, \ldots, a_{n-1} \in R$ such that $a_1 c_1 + \cdots + a_{n-1} c_{n-1} = 1$. For each $1 \leq i \leq n-1$, add $(1 - c_n) a_i$ times the i^{th} entry to the n^{th} one. This transforms v into $(c_1, \ldots, c_{n-1}, 1)$. Finally, add appropriate multiples of the last entry of v to the other ones to transform it into $(0, \ldots, 0, 1)$. $\square$

8.4. **Generation by elementary matrices.** If $\Bbbk$ is a field then $\mathrm{EL}_n(\Bbbk) = \mathrm{SL}_n(\Bbbk)$, so $\mathrm{GL}_n(\Bbbk)$ is generated by $\mathrm{EL}_n(\Bbbk)$ and $\mathrm{GL}_1(\Bbbk)$, which is embedded in $\mathrm{GL}_n(\Bbbk)$ via the upper left hand corner matrix embedding. The following lemma shows that the stable rank condition implies a similar type of result:

Lemma 8.7. *Let R be a ring satisfying (SL_r) and let $n \geq r-1$. Then $\mathrm{GL}_n(R)$ is generated by $\mathrm{EL}_n(R)$ and $\mathrm{GL}_{r-1}(R) \subset \mathrm{GL}_n(R)$.*

Proof. The proof will be by induction on n. The base case $n = r-1$ is trivial, so assume that $n \geq r$ and that the lemma is true for smaller n. Let $\{v_1, \ldots, v_n\}$ be the standard basis for the right R-module R^n and let $C = \oplus_{i=1}^{n-1} v_i \cdot R$. Consider some $M \in \mathrm{GL}_n(R)$. Applying Lemma 8.8 below, we can find some $N \in \mathrm{EL}_n(R)$ such that $N \cdot (M \cdot v_n) = v_n$ and $N \cdot (M \cdot C) = C$. It follows that $NM \in \mathrm{GL}_{n-1}(R)$, so by induction NM lies in the subgroup generated by elementary matrices and $\mathrm{GL}_{r-1}(R)$. We conclude that M does as well. $\square$

The above proof used the following lemma, which refines Lemma 8.6.

Lemma 8.8. *Let R be a ring satisfying (SR_r) and let $n \geq r$. Let $x, y \in R^n$ be unimodular vectors and $C, D \subset R^n$ be R-submodules such that $R^n = C \oplus x \cdot R$ and $R^n = D \oplus y \cdot R$. Then there exists some $M \in \mathrm{EL}_n(R)$ such that $M \cdot x = y$ and $M \cdot C = D$.*

Proof. Let $\{v_1, \ldots, v_n\}$ be the standard basis for the right R-module R^n. It is enough to deal with the case where $x = v_n$ and $C = \oplus_{i=1}^{n-1} v_i \cdot R$. What is more, Lemma 8.6 says that $\mathrm{EL}_n(R)$ acts transitively on unimodular vectors in R^n, so we can assume without loss of generality that $y = v_n$ as well. Let $\rho\colon R^n \to R$ be the projection with $D = \ker(\rho)$ and $\rho(v_n) = 1$. For $1 \leq i \leq n-1$, let $\lambda_i = \rho(v_i)$. Define $M \in \mathrm{GL}_n(R)$ via the formula

$$M \cdot v_i = v_i - v_n \cdot \lambda_i \quad \text{for } 1 \leq i \leq n-1, \text{ and } M \cdot v_n = v_n.$$

It is easy to see that M can be written as a product of $(n-1)$ elementary matrices, so $M \in \mathrm{EL}_n(R)$. For $1 \leq i \leq n-1$, we have

$$\rho(M \cdot v_i) = \rho(v_i) - \rho(v_n)\lambda_i = \lambda_i - \lambda_i = 0,$$

so $M \cdot v_i \in D$. We conclude that $M \cdot C = D$. $\square$

8.5. **Stable freeness.** For a general ring R, there can exist non-free R-modules C that are stably free in the sense that $C \oplus R^k \cong R^n$ for some $k \geq 1$. The stable rank condition prevents this, at least if $n-k$ not too large:

Lemma 8.9. *Let R be a ring satisfying* (SR_r). *Let C be a right R-module such that $C \oplus R^k \cong R^n$. Assume that $k \leq n-r+1$. Then $C \cong R^{n-k}$.*

Proof. It is enough to deal with the case where $k=1$, so $n \geq r$. Identify C with a submodule of R^n and let $x \in R^n$ be the unimodular vector with $R^n = C \oplus x \cdot R$. Letting $\{v_1, \ldots, v_n\}$ be the standard basis for R^n, Lemma 8.8 implies that there exists some $M \in \mathrm{GL}_n(R)$ with $M \cdot x = v_n$ and $M \cdot C = \oplus_{i=1}^{n-1} v_i \cdot R$. In particular, $M \cdot C \cong R^{n-1}$, so $C \cong R^{n-1}$ as well. $\square$

8.6. **Quotients of rings.** The following lemma shows that the stable rank condition is preserved by quotients:

Lemma 8.10. *Let R be a ring satisfying* (SR_r) *and let $\mathfrak{q}$ be a two-sided ideal in R. Then $R/\mathfrak{q}$ satisfies* (SR_r).

Proof. Let $n \geq r$ and let $\overline{v} \in (R/\mathfrak{q})^n$ be a unimodular vector. It is enough to prove that $\overline{v}$ can be lifted to a unimodular vector in R^n. Let $(c_1, \ldots, c_n) \in R^n$ be any vector projecting to $\overline{v}$. Since $\overline{v}$ is unimodular, Lemma 8.3 implies that there exist $a_1, \ldots, a_n \in R$ and $q \in \mathfrak{q}$ such that $a_1 c_1 + \cdots + a_n c_n = 1 + q$. It follows that $(c_1, \ldots, c_n, q) \in R^{n+1}$ is unimodular, so by (SR_r) we can find $b_1, \ldots, b_n \in R$ such that $v = (c_1 + b_1 q, \ldots, c_n + b_n q) \in R^n$ is unimodular. The vector v projects to $\overline{v}$. $\square$

8.7. **Stable rank modulo an ideal.** The following lemma gives a variant of the stable rank condition that takes into account an ideal.

Lemma 8.11. *Let R be a ring satisfying* (SR_r) *and let $\mathfrak{q}$ be a two-sided ideal of A. For some $n \geq r$, let $(c_1, \ldots, c_n) \in R^n$ be a unimodular vector with $c_n \in \mathfrak{q}$. Then there exists $b_1, \ldots, b_{n-1} \in \mathfrak{q}$ such that $(c_1 + b_1 c_n, \ldots, c_{n-1} + b_{n-1} c_n) \in R^{n-1}$ is unimodular.*

Proof. Using Lemma 8.3, we can find $a_1, \ldots, a_n \in R$ with $a_1 c_1 + \ldots + a_n c_n = 1$. We claim that $(c_1, \ldots, c_{n-1}, c_n a_n c_n) \in R^n$ is unimodular. Indeed, we have

$$\begin{aligned}
&(a_1 + a_n c_n a_1)\, c_1 + \cdots + (a_{n-1} + a_n c_n a_{n-1})\, c_{n-1} + (a_n)\, c_n a_n c_n \\
&\quad = (a_1 c_1 + \cdots + a_{n-1} c_{n-1}) + a_n c_n\, (a_1 c_1 + \cdots + a_n c_n) \\
&\quad = (a_1 c_1 + \cdots + a_{n-1} c_{n-1}) + a_n c_n = 1,
\end{aligned}$$

as desired. Applying (SR_r), we can find $b_1', \ldots, b_{n-1}' \in R$ such that $(c_1 + b_1' c_n a_n c_n, \ldots, c_{n-1} + b_{n-1}' c_n a_n c_n) \in R^{n-1}$ is unimodular. Since $c_n \in \mathfrak{q}$, so is $b_i = b_i' c_n a_n$. The lemma follows. $\square$

8.8. **Elementary congruence subgroups and their action on unimodular vectors.** Recall that if $\mathfrak{q}$ is an ideal in R, then $\mathrm{GL}_n(R, \mathfrak{q})$ is the level-$\mathfrak{q}$ congruence subgroup of $\mathrm{GL}_n(R)$, i.e., the kernel of the map $\mathrm{GL}_n(R) \to \mathrm{GL}_n(R/\mathfrak{q})$. An elementary matrix lies in $\mathrm{GL}_n(R, \mathfrak{q})$ precisely when its single off-diagonal entry lies in $\mathfrak{q}$. Let $\mathrm{EL}_n(R, \mathfrak{q})$ be the subgroup of $\mathrm{EL}_n(R)$ that is **normally** generated by elementary matrices lying in $\mathrm{GL}_n(R, \mathfrak{q})$.

Lemma 8.6 says that the stable rank condition implies that under suitable conditions $\mathrm{EL}_n(R)$ acts transitively on the set of unimodular vectors. The following lemma strengthens this:

Lemma 8.12. *Let R be a ring satisfying* (SR_r) *and let $\mathfrak{q}$ be an ideal in R. For some $n \geq r$, let $v, v' \in R^n$ be unimodular vectors that map to the same vector in $(R/\mathfrak{q})^n$. Then there exists some $M \in \mathrm{EL}_n(R, \mathfrak{q})$ with $M \cdot v = v'$.*

Proof. We will prove this in two steps.

Step 1. *This is true if* $v' = (1, 0, \ldots, 0)$.

Since v and $(1, 0, \ldots, 0)$ map to the same vector in $(R/\mathfrak{q})^n$, we can write $v = (1 + q_1, q_2, \ldots, q_n)$ with $q_1, \ldots, q_n \in \mathfrak{q}$. Recall that an elementary matrix lies in $\mathrm{GL}_n(R, \mathfrak{q})$ if its single off-diagonal entry lies in $\mathfrak{q}$. Just like in the proof of Lemma 8.6, using such elementary matrices, for any distinct $1 \leq i, j \leq n$ and any $\lambda \in \mathfrak{q}$ we can add λ times the i^{th} entry to the j^{th} one. We will use a series of these moves (plus one extra trick) to transform v into $(1, 0, \ldots, 0)$.

Applying the relative version of (SR_r) given by Lemma 8.11, we can add $\mathfrak{q}$-multiples of the last entry to the other ones to ensure that the first $(n-1)$ entries of v form a unimodular vector in R^{n-1}. Using Lemma 8.3, we can thus find $a_1, \ldots, a_{n-1} \in R$ such that $a_1(1 + q_1) + a_2 q_2 + \cdots + a_{n-1} q_{n-1} = 1$. For each $1 \leq i \leq n-1$, add $(q_1 - q_n) a_i \in \mathfrak{q}$ times the i^{th} entry to the n^{th} one. This transforms v into $(1 + q_1, q_2, \ldots, q_{n-1}, q_1)$.

At this point, we will do something slightly tricky. Let $E \in \mathrm{EL}_n(R)$ be the elementary matrix that subtracts the n^{th} row from the first one (notice that E does not lie in $\mathrm{GL}_n(R, \mathfrak{q})$). We thus have

$$E \cdot v = E \cdot (1 + q_1, q_2, \ldots, q_{n-1}, q_1) = (1, q_2, \ldots, q_{n-1}, q_1).$$

We can then find a product $F \in \mathrm{EL}_n(R, \mathfrak{q})$ of elementary matrices such that

$$F \cdot (1, q_2, \ldots, q_{n-1}, q_1) = (1, 0, \ldots, 0).$$

Indeed, F first subtracts $q_2 \in \mathfrak{q}$ times the first entry from the 2^{nd}, then subtracts $q_3 \in \mathfrak{q}$ times the first entry from the 3^{rd}, etc., and finishes by subtracting q_1 times the first entry from the n^{th}. Since E fixes $(1, 0, \ldots, 0)$, it follows that

$$E^{-1} F E \cdot v = (1, 0, \ldots, 0).$$

Since $\mathrm{EL}_n(R, \mathfrak{q})$ is a normal subgroup of $\mathrm{EL}_n(R)$, we have $M = E^{-1} F E \in \mathrm{EL}_n(R, \mathfrak{q})$, as desired.

Step 2. *This is true in general.*

Set $w = (1, 0, \ldots, 0)$. Lemma 8.6 says that there exists some $A \in \mathrm{EL}_n(R)$ with $A \cdot v' = w$. Applying the previous step to $A \cdot v$, we can find some $B \in \mathrm{EL}_n(R, \mathfrak{q})$ with $B \cdot A \cdot v = w$, so $A^{-1} B A \cdot v = A^{-1} \cdot w = v'$. Since $\mathrm{EL}_n(R, \mathfrak{q})$ is a normal subgroup of $\mathrm{EL}_n(R)$, it follows that $M = A^{-1} B A$ lies in $\mathrm{EL}_n(R, \mathfrak{q})$. □

8.9. **Action of congruence subgroup on split unimodular vector.** The following lemma refines Lemma 8.12 in the same way that Lemma 8.8 refines Lemma 8.6:

Lemma 8.13. *Let R be a ring satisfying (SR_r), let $\mathfrak{q}$ be an ideal in R, and let $n \geq d$. Let $x, y \in R^n$ be unimodular vectors and $C, D \subset R^n$ be R-submodules such that $R^n = C \oplus x \cdot R$ and $R^n = D \oplus y \cdot R$. Assume that x and y (resp. C and D) map to the same vector (resp. submodule) in $(R/\mathfrak{q})^n$. Then there exists some $M \in \mathrm{EL}_n(R, \mathfrak{q})$ with $M \cdot x = y$ and $M \cdot C = D$.*

Proof. Using Lemma 8.12, we can assume without loss of generality that $x = y$. Let $\rho\colon R^n \to R$ be the projection with $D = \ker(\rho)$ and $\rho(x) = 1$. Lemma 8.9 implies that C and D are both free of rank $(n-1)$. Let $\{v_1, \ldots, v_{n-1}\}$ be a free basis for C and let $y_n = x = y$, so $\{v_1, \ldots, v_n\}$ is a free basis for R^n. For $1 \leq i \leq n-1$, let $\lambda_i = \rho(v_i)$. Since C and D map to the same submodule of $(R/\mathfrak{q})^n$, we have $\lambda_i \in \mathfrak{q}$. Define $M \in \mathrm{GL}_n(R, \mathfrak{q})$ via the formula

$$M \cdot v_i = v_i - v_n \cdot \lambda_i \quad \text{for } 1 \leq i \leq n-1, \text{ and } M \cdot v_n = v_n.$$

It is easy to write M as a product of elementary matrices, so $M \in \mathrm{EL}_n(R, \mathfrak{q})$. For $1 \leq i \leq n-1$, we have

$$\rho(M \cdot v_i) = \rho(v_i) - \rho(v_n)\lambda_i = \lambda_i - \lambda_i = 0,$$

so $M \cdot v_i \in D$. We conclude that $M \cdot C = D$, as desired. □

8.10. **Subgroups from K-theory.** We close this section with some K-theoretic constructions. Define

$$\mathrm{GL}(R) = \bigcup_{n=1}^{\infty} \mathrm{GL}_n(R) \quad \text{and} \quad \mathrm{EL}(R) = \bigcup_{n=1}^{\infty} \mathrm{EL}_n(R).$$

It turns out that $\mathrm{EL}(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$ (this is the "Whitehead Lemma"; see [21, Lemma 3.1]). By definition, the first algebraic K-theory group of R is the abelian group

$$\mathrm{K}_1(R) = \mathrm{H}_1(\mathrm{GL}(R)) = \mathrm{GL}(R)/\,\mathrm{EL}(R).$$

For a subgroup $\mathcal{K} \subset K_1(R)$, we define $\mathrm{GL}_n^{\mathcal{K}}(R)$ to be the preimage of $\mathcal{K}$ under the projection

$$\mathrm{GL}_n(R) \hookrightarrow \mathrm{GL}(R) \longrightarrow \mathrm{K}_1(R).$$

It follows that $\mathrm{EL}_n(R) \subset \mathrm{GL}_n^{\mathcal{K}}(R) \subset \mathrm{GL}_n(R)$.

Example 8.14. Let R be a ring. For $\mathcal{K} = \mathrm{K}_1(R)$, we have $\mathrm{GL}_n^{\mathcal{K}}(R) = \mathrm{GL}_n(R)$. □

Example 8.15. Let R be a commutative ring and let $\det\colon \mathrm{GL}(R) \to R^{\times}$ be the determinant map. Since the determinant of an elementary matrix is 1, the map det induces a map

$$\mathfrak{d}\colon\ \mathrm{K}_1(R) = \mathrm{GL}(R)/\,\mathrm{EL}(R) \to R^{\times}.$$

Letting $\mathcal{K} = \ker(\mathfrak{d})$, we then have $\mathrm{GL}_n^{\mathcal{K}}(R) = \mathrm{SL}_n(R)$. □

If α is a two-sided ideal of R, then we can also define

$$\mathrm{GL}(R,\alpha) = \bigcup_{n=1}^{\infty} \mathrm{GL}_n(R,\alpha) \quad \text{and} \quad \mathrm{EL}(R,\alpha) = \bigcup_{n=1}^{\infty} \mathrm{EL}_n(R,\alpha).$$

Just like in the absolute case, $\mathrm{EL}(R,\alpha)$ is a normal subgroup of $\mathrm{GL}(R,\alpha)$ with abelian quotient, and this abelian quotient is the first relative K-theory group of (R,α):

$$\mathrm{K}_1(R,\alpha) = \mathrm{GL}(R,\alpha)/\,\mathrm{EL}(R,\alpha).$$

See [21, Lemma 4.2]. For a subgroup $\mathcal{K} \subset \mathrm{K}_1(R,\alpha)$, we define $\mathrm{GL}_n^{\mathcal{K}}(R,\alpha)$ to be the preimage of $\mathcal{K}$ under the composition

$$\mathrm{GL}_n(R,\alpha) \hookrightarrow \mathrm{GL}(R,\alpha) \longrightarrow \mathrm{K}_1(R,\alpha).$$

Letting $\mathcal{K}' \subset \mathrm{K}_1(R)$ be the image of $\mathcal{K}$ under the map $\mathrm{K}_1(R,\alpha) \to \mathrm{K}_1(R)$, we have $\mathrm{GL}_n^{\mathcal{K}}(R,\alpha) \subset \mathrm{GL}_n^{\mathcal{K}'}(R)$. Just like in the above examples, by choosing $\mathcal{K}$ appropriately we can have $\mathrm{GL}_n^{\mathcal{K}}(R,\alpha) = \mathrm{GL}_n(R,\alpha)$ or (if R is commutative) $\mathrm{GL}_n^{\mathcal{K}}(R,\alpha) = \mathrm{SL}_n(R,\alpha)$.

To clarify the nature of these groups, we need the following deep theorem of Vaserstein ("injective stability for K_1"):

Theorem 8.16 (Vaserstein, [30])**.** *Let R be a ring satisfying* (SR_r) *and let $n \geq r$. Then $\mathrm{EL}_n(R)$ is a normal subgroup of $\mathrm{GL}_n(R)$, and the composition*

$$\mathrm{GL}_n(R)/\,\mathrm{EL}_n(R) \longrightarrow \mathrm{GL}(R)/\,\mathrm{EL}(R) = \mathrm{K}_1(R)$$

is an isomorphism. Moreover, if α is a two-sided ideal of R, then $\mathrm{EL}_n(R,\alpha)$ is a normal subgroup of $\mathrm{GL}_n(R,\alpha)$, and the composition

$$\mathrm{GL}_n(R,\alpha)/\,\mathrm{EL}_n(R,\alpha) \longrightarrow \mathrm{GL}(R,\alpha)/\,\mathrm{EL}(R,\alpha) = \mathrm{K}_1(R,\alpha)$$

is an isomorphism.

This allows us to deal with another example:

Example 8.17. Let R be a ring satisfying (SR_r) and let $n \geq r$. Set $\mathcal{K} = 0$. Then by Theorem 8.16 we have $\mathrm{GL}_n^{\mathcal{K}}(R) = \mathrm{EL}_n(R)$, and for a two-sided ideal α of R we have $\mathrm{GL}_n^{\mathcal{K}}(R,\alpha) = \mathrm{EL}_n(R,\alpha)$. □

It also allows us to prove the following useful result:

Lemma 8.18. *Let R be a ring satisfying* (SR_r) *and let $n \geq r$. For all subgroups $\mathcal{K} \subset \mathrm{K}_1(R)$, we have*

$$\mathrm{GL}_{n+1}^{\mathcal{K}}(R) \cap \mathrm{GL}_n(R) = \mathrm{GL}_n^{\mathcal{K}}(R).$$

Similarly, for all two-sided ideals α of R and for all subgroups $\mathcal{K} \subset \mathrm{K}_1(R,\alpha)$, we have

$$\mathrm{GL}_{n+1}^{\mathcal{K}}(R,\alpha) \cap \mathrm{GL}_n(R,\alpha) = \mathrm{GL}_n^{\mathcal{K}}(R,\alpha).$$

Proof. By Theorem 8.16, the maps

$$\mathrm{GL}_n(R)/\mathrm{EL}_n(R) \to \mathrm{GL}_{n+1}(R)/\mathrm{EL}_{n+1}(R) \to \mathrm{K}_1(R)$$

are both isomorphisms. This implies that

$$\mathrm{GL}_{n+1}^{\mathcal{K}}(R) \cap \mathrm{GL}_n(R) = \mathrm{GL}_n^{\mathcal{K}}(R).$$

The relative statement is proved similarly. □

9. Complex of split partial bases

We now introduce a simplicial complex called the complex of split partial bases, which is a tiny variant on one introduced by Charney [8].

Remark 9.1. The lemmas we prove in this section are all essentially due to Charney [8]. Many of them are also proved in [28, §5.3]. We include proofs because they are short, and also because it is annoyingly nontrivial to match up our notation and indexing conventions with those papers. □

9.1. **Definition.** Let R be a ring and let M be a right R-module. The *complex of split partial bases* for M, denoted $\mathcal{B}(M)$, is the following simplicial complex:

- The vertices are pairs $(x; C)$ with $x \in M$ a nonzero element and $C \subseteq M$ a submodule such that $M = C \oplus x \cdot R$.
- A collection $\{(x_0; C_0), \ldots, (x_k; C_k)\}$ of vertices forms a k-simplex if $x_i \in C_j$ for all distinct $0 \leq i, j \leq k$.

The group $\mathrm{Aut}_R(M)$ acts on $\mathcal{B}(M)$ on the left. In particular, $\mathrm{GL}_n(R)$ acts on $\mathcal{B}(R^n)$.

9.2. **Splitting from a simplex.** The following lemma might clarify the definition of a simplex in $\mathcal{B}(M)$.

Lemma 9.2. *Let R be a ring, let M be a right R-module, and let $\{(x_0; C_0), \ldots, (x_k; C_k)\}$ be a k-simplex of $\mathcal{B}(M)$. Setting $C = C_0 \cap \cdots \cap C_k$, we then have*

$$M = C \oplus \bigoplus_{i=0}^{k} x_i \cdot R. \tag{9.1}$$

Proof. The proof is by induction on k. The base case $k = 0$ is trivial, so assume that $k > 0$ and that the lemma is true for all smaller k. For $0 \leq i \leq k-1$, let $C_i' = C_i \cap C_k$. For these i, we claim that $C_k = C_i' \oplus x_i \cdot R$. Since $x_i \in C_k$ and $M = C_i \oplus x_i \cdot R$, it is enough to show that for all $z \in C_k$, we can write $z = z' + x_i \cdot a$ for some $z' \in C_i'$ and $a \in R$. Since $M = C_i \oplus x_i \cdot R$, we can write $z = z' + x_i \cdot a$ for some $z' \in C_i$ and $a \in R$, and what we must show is that $z' \in C_i'$. But since $z, x_i \in C_k$ we have $z' = z - x_i \cdot a \in C_k$, so $z' \in C_i'$, as desired.

It follows that $\{(x_0; C_0'), \ldots, (x_{k-1}; C_{k-1}')\}$ is a $(k-1)$-simplex of $\mathcal{B}(C_k)$. Since

$$C = C_0 \cap \cdots \cap C_k = C_0' \cap \cdots \cap C_{k-1}',$$

our inductive hypothesis implies that

$$C_k = C \oplus \bigoplus_{i=0}^{k-1} x_i \cdot R.$$

Since $M = C_k \oplus x_k \cdot R$, equation (9.1) follows. □

This has the following useful consequence:

Lemma 9.3. *Let R be a ring and let M be a right R-module. The k-simplices of $\mathcal{B}(M)$ are in bijection with tuples $(\{x_0, \ldots, x_k\}; C)$, where $\{x_0, \ldots, x_k\}$ is an unordered collection of $(k+1)$ elements of M and $C \subset M$ is a submodule such that*

$$M = C \oplus \bigoplus_{i=0}^{k} x_i \cdot R.$$

Proof. For a k-simplex $\{(x_0; C_0), \ldots, (x_k, C_k)\}$ of $\mathcal{B}(M)$, by Lemma 9.2 we can associate the tuple

$$(\{x_0, \ldots, x_k\}; \bigcap_{i=0}^{k} C_i).$$

Conversely, consider such a tuple $(\{x_0, \ldots, x_k\}; C)$. For $0 \leq j \leq k$, define

$$C_j = \bigoplus_{\substack{0 \leq i \leq k \\ i \neq j}} x_i \cdot R.$$

Then $\{(x_0; C_0), \ldots, (x_k; C_k)\}$ is a k-simplex of $\mathcal{B}(M)$. It is clear that these two operations are inverses to one another. □

Subsequently, we will freely move between the notations $\{(x_0; C_0), \ldots, (x_k; C_k)\}$ and $(\{x_0, \ldots, x_k\}; C)$ for the k-simplices of $\mathcal{B}(M)$.

9.3. **Links in complex of split partial bases.** We now focus our attention on $\mathcal{B}(R^n)$. If R satisfies a stable rank condition, then the links in this complex are well-behaved:

Lemma 9.4. *Let R be a ring satisfying* (SR_r). *For some n and k with $k \leq n - r + 1$, let σ be a k-simplex of $\mathcal{B}(R^n)$. We then have $\mathrm{Link}_{\mathcal{B}(M)}(\sigma) \cong \mathcal{B}(R^{n-k-1})$.*

Proof. Let $\sigma = (\{x_0, \ldots, x_k\}; C)$. Since

$$R^n = C \oplus \bigoplus_{i=0}^{k} x_i \cdot R,$$

Lemma 8.9 implies that $C \cong R^{n-k-1}$, so it is enough to prove that $\mathrm{Link}_{\mathcal{B}(R^n)}(\sigma)$ and $\mathcal{B}(C)$ are isomorphic. For this, note that the maps $\phi\colon \mathrm{Link}_{\mathcal{B}(M)}(\sigma) \to \mathcal{B}(C)$ and $\psi\colon \mathcal{B}(C) \to \mathrm{Link}_{\mathcal{B}(M)}(\sigma)$ defined on ℓ-simplices via the formulas

$$\phi((\{y_0, \ldots, y_\ell\}; D)) = (\{y_0, \ldots, y_\ell\}; D \cap C)$$

and

$$\psi((\{z_0, \ldots, z_\ell\}; E)) = (\{z_0, \ldots, z_\ell\}; E \oplus \bigoplus_{i=0}^{k} x_i \cdot R)$$

are inverse isomorphisms. □

9.4. **General linear group action.** We now turn to the action of $\mathrm{GL}_n(R)$ and its elementary subgroup $\mathrm{EL}_n(R)$ on $\mathcal{B}(R^n)$. Our main result will be as follows:

Lemma 9.5. *Let R be a ring satisfying (SR_r). Let*

$$\{(x_0; C_0), \ldots, (x_k; C_k)\} \quad \textit{and} \quad \{(y_0; D_0), \ldots, (y_k; D_k)\}$$

be k-simplices of $\mathcal{B}(R^n)$. Assume that $k \leq n - r$. The following then hold:

- *(a) There exists $M \in \mathrm{EL}_n(R)$ such that $M \cdot (x_i; C_i) = (y_i; D_i)$ for $0 \leq i \leq k$.*
- *(b) Let $\mathfrak{q}$ be a two-sided ideal of R and let $\pi\colon R^n \to (R/\mathfrak{q})^n$ be the projection. Assume that $\pi(x_i) = \pi(y_i)$ and $\pi(C_i) = \pi(D_i)$ for $0 \leq i \leq k$. Then the M in part (a) can be chosen to lie in $\mathrm{EL}_n(R, \mathfrak{q})$.*

Proof. For $k = 0$, this is immediate from Lemmas 8.8 and 8.13. The general case can be deduced from this by induction using Lemma 9.4. □

9.5. **Topology.** Recall that we defined what it means for a simplicial complex to be weakly Cohen-Macaulay in §2.2. The complex of split partial bases has this property:

Theorem 9.6. *Let R be a ring satisfying (SR_r) and let $n \geq r$. Then $\mathcal{B}(R^n)$ is weakly Cohen-Macaulay of dimension $\left(\frac{n-r+1}{2}\right)$.*

The key to the proof of Theorem 9.6 is the following theorem. It is essentially due to Charney [8, Theorem 3.5], though she works with a slightly different complex, so one needs to adapt her proof. For a complete proof of exactly the statement below, see [28, Lemma 5.10] (or rather its proof – the lemma in the reference deals with the large ordering of $\mathcal{B}(R^n)$, but proves this theorem along the way):

Theorem 9.7. *Let R be a ring satisfying (SR_r). Then for all $n \geq 1$, the complex $\mathcal{B}(R^n)$ is $\left(\frac{n-r-1}{2}\right)$-connected.*

Proof of Theorem 9.6. Theorem 9.7 says that $\mathcal{B}(R^n)$ is $\left(\frac{n-r+1}{2}\right) - 1 = \left(\frac{n-r-1}{2}\right)$ connected, so what we must prove is that for all k-simplices σ of $\mathcal{B}(R^n)$, the complex $\mathrm{Link}_{\mathcal{B}(R^n)}(\sigma)$ is $\left(\frac{n-r-1}{2} - k - 1\right)$-connected. This is a non-tautological condition precisely when

$$\frac{n-r-1}{2} - k - 1 \geq -1, \quad \text{i.e., when } k \leq \frac{n-r-1}{2}. \tag{9.2}$$

The connectivity condition we want will follow from Theorem 9.7 if $\mathrm{Link}_{\mathcal{B}(R^n)}(\sigma) \cong \mathcal{B}(R^{n-k-1})$, which by Lemma 9.4 holds if

$$k \leq n - r + 1. \tag{9.3}$$

So we must prove that (9.2) implies (9.3), i.e., that

$$\frac{n-r-1}{2} \leq n - r + 1.$$

This holds precisely when $n - r \geq -3$, which follows from our assumption that $n \geq r$. □

10. The large ordering of the complex of split partial bases

Let R be a ring. We now discuss the large ordering of $\mathcal{B}(R^n)$ and show how to use a $\mathtt{VIC}(R)$-module M to define a $\mathrm{GL}_n(R)$-equivariant coefficient system on it (or, rather, on an appropriate subcomplex).

10.1. **Large ordering and shifted large ordering.** For $n \geq 0$, define $\mathbb{B}(R^n)$ to be the large ordering of $\mathcal{B}(R^n)$. As we discussed in Lemma 9.3, the k-simplices of $\mathcal{B}(R^n)$ can be identified with tuples $(\{x_0, \ldots, x_k\}; C)$, where $x_0, \ldots, x_k \in R^n$ are elements and $C \subset R^n$ is a submodule such that

$$R^n = C \oplus \bigoplus_{i=0}^{k} x_i \cdot R.$$

The k-simplices of $\mathbb{B}(R^n)$ are obtained by imposing an ordering on the x_i, so can be identified with ordered tuples $(x_0, \ldots, x_k; C)$ where the x_i and C are as above. It will be technically annoying that sometimes C is not a free module. To fix this, assume that R satisfies (SR_r), and define $\mathbb{B}(R^{n,r})$ to be the semisimplicial subset of $\mathbb{B}(R^{n+r})$ consisting of simplices of dimension at most n. The group $\mathrm{GL}_{n+r}(R)$ acts on $\mathbb{B}(R^{n,r})$, and the following lemma shows that $\mathbb{B}(R^{n,r})$ avoids some of the annoying features of $\mathbb{B}(R^n)$:

Lemma 10.1. *Let R be a ring satisfying (SR_r) and let $n \geq 1$. The following hold:*

(i) For each k-simplex $(x_0, \ldots, x_k; C)$ of $\mathbb{B}(R^{n,r})$, we have $C \cong R^{n+r-k-1}$.

(ii) For all k and all subgroups $\mathcal{K} \subset \mathrm{K}_1(R)$, the group $\mathrm{GL}_{n+r}^{\mathcal{K}}(R)$ acts transitively on the k-simplices of $\mathbb{B}(R^{n,r})$.

(iii) For all two-sided ideals α of R, we have

$$\mathbb{B}(R^{n,r})/\operatorname{EL}_{n+r}(R, \alpha) = \mathbb{B}((R/\alpha)^{n,r}).$$

(iv) The semisimplicial set $\mathbb{B}(R^{n,r})$ is a large ordering of a weakly Cohen–Macaulay complex of dimension $\frac{n+1}{2}$.

Proof. Conclusions (i) and (ii) follow from Lemmas 8.9 and 9.5 along with the fact that we have restricted the simplices of $\mathbb{B}(R^{n,r})$ to have dimension at most n. Here for (ii) we are using the fact that $\mathrm{EL}_{n+r}(R) \subset \mathrm{GL}_{n+r}^{\mathcal{K}}(R)$.

For (iii), the projection

$$\pi \colon \mathbb{B}(R^{n,r}) \longrightarrow \mathbb{B}((R/\alpha)^{n,r})$$

is $\mathrm{GL}_{n+r}(R)$-equivariant, where the group $\mathrm{GL}_{n+r}(R)$ acts on $\mathbb{B}((R/\alpha)^{n,r})$ via the projection $\mathrm{GL}_{n+r}(R) \to \mathrm{GL}_{n+r}(R/\alpha)$. Lemma 8.10 says that R/α satisfies (SR_r), so by Lemma 9.5 the group $\mathrm{EL}_{n+r}(R/\alpha)$ acts transitively on the k-simplices of $\mathbb{B}((R/\alpha)^{n,r})$ for all k. Since $\mathrm{EL}_{n+r}(R)$ maps surjectively onto $\mathrm{EL}_{n+r}(R/\alpha)$ and π is $\mathrm{EL}_{n+r}(R)$-equivariant, it follows that π is surjective. The map π descends to a map

$$\overline{\pi} \colon \mathbb{B}(R^{n,r})/\operatorname{EL}_{n+r}(R, \alpha) \to \mathbb{B}((R/\alpha)^{n,r}).$$

The map $\overline{\pi}$ is surjective since π is. Lemma 9.5 implies that two simplices of $\mathbb{B}(R^{n,r})$ that map to the same simplex of $\mathbb{B}((R/\alpha)^{n,r})$ differ by an element of $\mathrm{EL}_{n+r}(R)$, and thus are identified with the same simplex of $\mathbb{B}(R^{n,r})/\operatorname{EL}_{n+r}(R, \alpha)$. It follows that $\overline{\pi}$ is injective, and hence an isomorphism, as desired.

For (iv), Theorem 9.6 says that $\mathcal{B}(R^{n+r})$ is weakly Cohen–Macaulay of dimension $\frac{n+1}{2}$. Since $n \geq 1$, we have $n \geq \frac{n+1}{2}$, so the n-skeleton of $\mathcal{B}(R^{n+r})$ is also weakly Cohen–Macaulay of dimension $\frac{n+1}{2}$, as desired. □

10.2. **Coefficient system.** Let R be a ring satisfying (SR_r), let $\Bbbk$ be a commutative ring, and let M be a $\mathtt{VIC}(R)$-module over $\Bbbk$. For each $n \geq 1$, let $\mathcal{G}_{M,n,r}$ be the augmented coefficient system on $\mathbb{B}(R^{n,r})$ defined via the formula

$$\mathcal{G}_{M,n,r}(x_0, \ldots, x_k; C) = M(C) \quad \text{for a simplex } (x_0, \ldots, x_k; C) \text{ of } \mathbb{B}(R^{n,r}),$$

where our convention is that the -1-simplex of $\mathbb{B}(R^{n,r})$ is $(; R^{n+r})$. For this to make sense, we need C to be a free R-module, which is ensured by Lemma 10.1.(i). To see how $\mathcal{G}_{M,n,r}$

behaves under face maps, note that for a simplex $\sigma = (x_0, \ldots, x_k; C)$ of $\mathbb{B}(R^{n,r})$, the faces of σ are of the form

$$\sigma' = (x_{i_0}, \ldots, x_{i_\ell}; C') \quad \text{with} \quad C' = C \oplus \bigoplus_{j \notin \{i_0, \ldots, i_\ell\}} x_j \cdot R$$

for increasing sequences $0 \leq i_0 < \cdots < i_\ell \leq k$. Letting $\iota \colon C \to C'$ be the natural inclusion and letting

$$D = \bigoplus_{j \notin \{i_0, \ldots, i_\ell\}} x_j \cdot R,$$

the induced map $\mathcal{G}_{M,n,r}(\sigma) \to \mathcal{G}_{M,n,r}(\sigma')$ is the one induced by the following $\mathtt{VIC}(R)$-morphism:

$$\mathcal{G}_{M,n,r}(\sigma) = M(C) \xrightarrow{(\iota,D)_*} M(C') = \mathcal{G}_{M,n,r}(\sigma').$$

Just like in Example 4.3, the augmented coefficient system $\mathcal{G}_{M,n,r}$ is $\mathrm{GL}_{n+r}(R)$-equivariant. It satisfies the following lemma:

Lemma 10.2 (c.f. Lemma 6.2)**.** *Let R be a ring satisfying* (SR_r)*, let $\Bbbk$ be a commutative ring, and let M be a $\mathtt{VIC}(R)$-module over $\Bbbk$ that is polynomial of degree d starting at $m \geq 0$. For each $n \geq m$, the coefficient system $\mathcal{G}_{M,n,r}$ on $\mathbb{B}(R^{n,r})$ is polynomial of degree d up to dimension $n + r - m - 1$.*

Proof. The proof will be by induction on d. If $d = -1$, then consider a simplex $\sigma = (x_0, \ldots, x_k; C)$ with $k \leq n + r - m - 1$. Lemma 10.1.(i) says that $C \cong R^{n+r-k-1}$, and since

$$n + r - k - 1 \geq n + r - (n + r - m - 1) - 1 = m \tag{10.1}$$

the fact that M is polynomial of degree -1 starting at m implies that $\mathcal{G}_{M,n,r}(\sigma) = M(C) = 0$, as desired.

Now assume that $d \geq 0$. There are two things to check. For the first, let $\sigma = (x_0, \ldots, x_k; C)$ be a simplex with $k \leq n + r - m - 1$. We must prove that the map $\mathcal{G}_{M,n,r}(\sigma) \to \mathcal{G}_{M,n,r}(\emptyset)$ is injective, i.e., that the map

$$M(C) \to M(R^{n+r})$$

is injective. The calculation (10.1) shows that this injectivity follows from the fact that M is polynomial of degree d starting at m.

For the second, let $\tau = (y_0, \ldots, y_\ell; D)$ be any simplex with $\ell \leq n + r - m - 1$ and let $\mathbb{Y}$ be the forward link $\overrightarrow{Link}_{\mathbb{B}(R^{n,r})}(\tau)$. Set

$$D' = D \oplus \bigoplus_{i=0}^{\ell-1} y_i \cdot R,$$

so $(y_\ell; D')$ is the last vertex of τ. Let $\mathcal{H}$ be the coefficient system on $\mathbb{Y}$ defined by the formula

$$\mathcal{H}(\sigma) = \frac{\mathcal{G}_{M,n,r}(\sigma)}{\mathrm{Im}\left(\mathcal{G}_{M,n,r}\left((y_\ell; D') \cdot \sigma\right) \to \mathcal{G}_{M,n,r}\left(\sigma\right)\right)} \quad \text{for a simplex } \sigma \text{ of } \mathbb{Y}.$$

We must prove that $\mathcal{H}$ is polynomial of degree $d-1$ up to dimension $n+r-m-1-\ell$. This condition is invariant under the action of $\mathrm{GL}_{n+r}(R)$. Letting $\{v_1, \ldots, v_{n+r}\}$ be the standard basis for R^{n+r}, Lemma 10.1.(ii) says that by applying an appropriate element of $\mathrm{GL}_{n+r}(R)$ we can assume without loss of generality that

$$\tau = (v_{n+r-\ell}, v_{n+r-\ell+1}, \ldots, v_{n+r}; D) \quad \text{with} \quad D = \bigoplus_{i=1}^{n+r-\ell-1} v_i \cdot R.$$

This implies that

$$\mathbb{Y} = \overrightarrow{Link}_{\mathbb{B}(R^{n,r})}(\tau) = \mathbb{B}^{n-\ell-1,r}.$$

Recall that we defined the derived $\texttt{VIC}(R)$-module DM in Definition 1.16. By construction, there is an isomorphism between the coefficient systems $\mathcal{H}$ and $\mathcal{G}_{DM,n-\ell-1,r}$ on $\mathbb{B}^{n-\ell-1,r}$. Since M is polynomial of degree d starting at m, the $\texttt{VIC}(R)$-module DM is polynomial of degree $d-1$ starting at $m-1$. By induction, $\mathcal{H}$ is polynomial of degree $d-1$ starting at

$$(n-\ell-1+r)-(m-1)-1 = n+r-m-1-\ell,$$

as desired. $\square$

11. Stability for general linear groups

We now are in a position to prove Theorems B and B′. In fact, we will prove more general results that also deal with the groups $\mathrm{GL}_n^{\mathcal{K}}(R)$. The following generalizes Theorem B.

Theorem 11.1. *Let R be a ring satisfying (SR_r), let $\mathcal{K} \subset \mathrm{K}_1(R)$ be a subgroup, let $\Bbbk$ be a commutative ring, and let M be a $\texttt{VIC}(R)$-module over $\Bbbk$ that is polynomial of degree d starting at $m \geq 0$. For each $k \geq 0$, the map*

$$\mathrm{H}_k(\mathrm{GL}_n^{\mathcal{K}}(R); M(R^n)) \to \mathrm{H}_k(\mathrm{GL}_{n+1}^{\mathcal{K}}(R); M(R^{n+1}))$$

is an isomorphism for $n \geq 2k + \max(2d+r, m+1)$ and a surjection for $n = 2k + \max(2d+r-1, m)$.

Proof. The group $\mathrm{GL}_{n+r}^{\mathcal{K}}(R)$ acts on $\mathbb{B}(R^{n,r})$. Let $\mathcal{G}_{M,n,r}$ be the $\mathrm{GL}_{n+r}^{\mathcal{K}}(R)$-equivariant augmented system of coefficients on $\mathbb{B}(R^{n,r})$ discussed in §10.2, so

$$\mathcal{G}_{M,n,r}(x_0,\ldots,x_k;C) = M(C) \quad \text{for a simplex } (x_0,\ldots,x_k;C) \text{ of } \mathbb{B}(R^{n,r}).$$

The following claim will be used to show that with an appropriate degree shift, this all satisfies the hypotheses of Theorem 5.2.

Claim. *The following hold:*

- *(a) For all $-1 \leq k \leq \min(\frac{n-2d-1}{2}, n+r-m-1)$, we have $\widetilde{\mathrm{H}}_k(\mathbb{B}(R^{n,r}); \mathcal{G}_{M,n,r}) = 0$.*
- *(b) For all $-1 \leq k \leq n$, the group $\mathrm{GL}_{n+r-k-1}^{\mathcal{K}}(R)$ is the $\mathrm{GL}_{n+r}^{\mathcal{K}}(R)$-stabilizer of a k-simplex σ_k of $\mathbb{B}(R^{n,r})$ with $\mathcal{G}_{M,n,r}(\sigma_k) = M(R^{n+r-k-1})$.*
- *(c) For all $0 \leq k \leq n$, the group $\mathrm{GL}_{n+r}^{\mathcal{K}}(R)$ acts transitively on the k-simplices of $\mathbb{B}(R^{n,r})$.*
- *(d) For all $n \geq 2$ and all 1-simplices e of $\mathbb{B}(R^{n,r})$ of the form $e = ((x_0;C_0),(x_1;C_1))$, there exists some $\lambda \in \mathrm{GL}_{n+r}^{\mathcal{K}}(R)$ with $\lambda(x_0;C_0) = (x_1;C_1)$ such that λ commutes with all elements of $(\mathrm{GL}_{n+r}^{\mathcal{K}}(R))_e$ and fixes all elements of $\mathcal{G}_{M,n,r}(e)$.*

Proof of claim. For (a), Lemma 10.2 says that $\mathcal{G}_{M,n,r}$ is a polynomial coefficient system of degree d up to dimension $n+r-m-1$. Also, by Lemma 10.1.(iv) the semisimplicial set $\mathbb{B}(R^{n,r})$ is the large ordering of the simplicial complex that is weakly Cohen–Macaulay of dimension $\frac{n+1}{2}$. Letting

$$N = \min(\frac{n+1}{2} - d - 1, n+r-m-1) = \min(\frac{n-2d-1}{2}, n+r-m-1),$$

the complex $\mathbb{B}(R^{n,r})$ is weakly Cohen–Macaulay of dimension $N+d+1$ and $\mathcal{G}_{M,n,r}$ is a polynomial coefficient system of degree d up to dimension N. Theorem 6.3 thus implies that $\widetilde{\mathrm{H}}_k(\mathbb{B}(R^{n,r}); \mathcal{G}_{M,n,r}) = 0$ for $-1 \leq k \leq N$.

For (b), let $\{v_1,\ldots,v_{n+r}\}$ be the standard basis for R^{n+r}. By Lemma 8.18, the group $\mathrm{GL}_{n+r-k-1}^{\mathcal{K}}(R)$ is the $\mathrm{GL}_{n+r}^{\mathcal{K}}(R)$-stabilizer of the k-simplex

$$\sigma_k = \begin{cases} (R^{n+r}) & \text{if } k = -1, \\ (v_{n+r-k}, v_{n+r-k+1}, \ldots, v_{n+r}; R^{n+r-k-1}) & \text{if } 0 \leq k \leq n \end{cases}$$

of $\mathbb{B}(R^{n,r})$, and by definition

$$\mathcal{G}_{M,n,r}(\sigma_k) = M(R^{n+r-k-1}).$$

Condition (c) is Lemma 10.1.(ii).

For (d), define $\lambda\colon R^{n+r} \to R^{n+r}$ to be the R-module homomorphism defined via the formulas

$$\lambda(x_0) = x_1 \quad \text{and} \quad \lambda(x_1) = -x_0 \quad \text{and} \quad \lambda|_{C_0 \cap C_1} = \text{id}.$$

This lies in $\text{EL}_{n+r}(R) \subset \text{GL}^{\mathcal{K}}_{n+r}(R)$; indeed, the group $\text{SL}_2(\mathbb{Z})$ is generated by elementary matrices and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \text{SL}_2(\mathbb{Z}),$$

so λ can be written as a product of elementary matrices. Here we are using the fact that $n + r \geq r$, so by Theorem 8.16 the group $\text{EL}_{n+r}(R)$ is a normal subgroup of $\text{GL}_{n+r}(R)$. In particular, the fact that something can be written as a product of elementary matrices is independent of the choice of basis. The map λ acts trivially on

$$\mathcal{G}_{M,n,r}(e) = M(C_0 \cap C_1)$$

by basic properties of $\texttt{VIC}(R)$-modules. $\square$

Letting $e = \max(2d-1, m-r)$, the Claim verifies that the conditions of Theorem 5.2 are satisfied for

$$G_n = \text{GL}^{\mathcal{K}}_{n+e+r}(R) \quad \text{and} \quad M_n = M(R^{n+e+r}) \quad \text{and} \quad \mathbb{X}_n = \mathbb{B}(R^{n+e,r}) \quad \text{and} \quad \mathcal{M}_n = \mathcal{G}_{M,n+e,r}.$$

with $c = 2$. The shift by $e + r$ is needed for condition (a) of Theorem 5.2, which requires that $\widetilde{\text{H}}_k(\mathbb{X}_n; \mathcal{M}_n) = 0$ for all $-1 \leq k \leq \frac{n-2}{2}$. Conclusion (a) of the Claim says that $\widetilde{\text{H}}_k(\mathbb{B}(R^{n+e,r}); \mathcal{G}_{M,n+e,r}) = 0$ for $-1 \leq k \leq \min(\frac{(n+e)-2d-1}{2}, (n+e)+r-m-1)$, which implies the desired range of vanishing for $\widetilde{\text{H}}_k(\mathbb{X}_n; \mathcal{M}_n)$ since

$$\frac{(n+e)-2d-1}{2} \geq \frac{(n+2d-1)-2d-1}{2} = \frac{n-2}{2} \quad \text{for all } n$$

and

$$(n+e)+r-m-1 \geq n+(m-r)+r-m-1 = n-1 \geq \frac{n-2}{2} \quad \text{for all } n \geq 0,$$

so

$$\frac{n-2}{2} \leq \min(\frac{(n+e)-2d-1}{2}, (n+e)+r-m-1) \quad \text{for all } n \geq 0.$$

Applying Theorem 5.2, we deduce that the map

$$\text{H}_k(\text{GL}^{\mathcal{K}}_{n+e+r-1}(R); M(R^{n+e+r-1})) \to \text{H}_k(\text{GL}^{\mathcal{K}}_{n+e+r}(R); M(R^{n+e+r}))$$

is an isomorphism for $n \geq 2k+2$ and a surjection for $n = 2k+1$, which implies that

$$\text{H}_k(\text{GL}^{\mathcal{K}}_n(R); M(R^n)) \to \text{H}_k(\text{GL}^{\mathcal{K}}_{n+1}(R); M(R^{n+1}))$$

is an isomorphism for

$$n \geq 2k + (e+r-1) + 2 = 2k + \max(2d-1, m-r) + r + 1 = 2k + \max(2d+r, m+1)$$

and a surjection for

$$n = 2k + \max(2d+r-1, m). \qquad \square$$

The following theorem generalizes Theorem B′:

Theorem 11.2. *Let R be a ring satisfying* (SR_r), *, let $\mathcal{K} \subset \mathrm{K}_1(R)$ be a subgroup, let $\Bbbk$ be a commutative ring, and let M be a* $\mathtt{VIC}(R)$*-module over $\Bbbk$ that is polynomial of degree d starting at $m \geq 0$. For each $k \geq 0$, the map*

$$\mathrm{H}_k(\mathrm{GL}_n^{\mathcal{K}}(R); M(R^n)) \to \mathrm{H}_k(\mathrm{GL}_{n+1}^{\mathcal{K}}(R); M(R^{n+1})) \tag{11.1}$$

is an isomorphism for $n \geq \max(m, 2k + 2d + r + 1)$ and a surjection for $n \geq \max(m, 2k + 2d + r - 1)$.

Proof. The proof will be by double induction on d and m. There are two base cases:

- The first is where $m = 0$ and $d \geq 0$. Theorem 11.1 says in this case that (11.1) is an isomorphism for
$$n \geq 2k + \max(2d + r, m + 1) = 2k + \max(2d + r, 1) = 2k + 2d + r$$
and a surjection for
$$n = 2k + \max(2d + r - 1, m) = 2k + \max(2d + r - 1, 0) = 2k + 2d + r - 1.$$
In both of these we use the fact that $r \geq 2$, which holds since the condition (SR_r) only makes sense for $r \geq 2$ (see Definition 8.4). These bounds are even stronger than our purported bounds of
$$n \geq \max(m, 2k + 2d + r + 1) = \max(0, 2k + 2d + r + 1) = 2k + 2d + r + 1$$
for (11.1) to be an isomorphism and
$$n \geq \max(m, 2k + 2d + r - 1) = \max(0, 2k + 2d + r - 1) = 2k + 2d + r - 1$$
for (11.1) to be a surjection.
- The second is where $m \geq 0$ and $d = -1$. In this case, by the definition of a $\mathtt{VIC}(R)$-module being polynomial of degree -1 starting at m we have for $n \geq m$ that $M(R^n) = 0$ and hence $\mathrm{H}_k(\mathrm{GL}_n^{\mathcal{K}}(R); M(R^n)) = 0$. In other words, for $n \geq m$ the domain and codomain of (11.1) are both 0, so it is trivially an isomorphism.

Assume now that $m \geq 1$ and $d \geq 0$, and that the theorem is true for all smaller m and d. As in Definition 1.17, let ΣM be the shifted $\mathtt{FI}$-module and DM be the derived $\mathtt{VIC}(R)$-module. For $n \geq m$, we have a short exact sequence

$$0 \longrightarrow M(R^n) \longrightarrow \Sigma M(R^n) \longrightarrow DM(R^n) \longrightarrow 0 \tag{11.2}$$

of $\Bbbk[\mathrm{GL}_n^{\mathcal{K}}(R)]$-modules. The $\mathtt{VIC}(R)$-module ΣM is polynomial of degree d starting at $(m-1)$, and the $\mathtt{VIC}(R)$-module DM is polynomial of degree $(d-1)$ starting at $(m-1)$.

To simplify our notation, for all $s \geq 0$ and all $\Bbbk[\mathrm{GL}_s^{\mathcal{K}}(R)]$-modules N, we will denote $\mathrm{H}_k(\mathrm{GL}_s^{\mathcal{K}}(R); N)$ by $\mathrm{H}_k(N)$. The long exact sequence in $\mathrm{GL}_n^{\mathcal{K}}(R)$-homology associated to (11.2) maps to the one in $\mathrm{GL}_{n+1}^{\mathcal{K}}(R)$-homology, so for $n \geq m$ and all k we have a commutative diagram

$$\begin{tikzcd}
\mathrm{H}_{k+1}(\Sigma M(R^n)) \arrow[r] \arrow[d, "g_1"] & \mathrm{H}_{k+1}(DM(R^n)) \arrow[r] \arrow[d, "g_2"] & \mathrm{H}_k(M(R^n)) \arrow[r] \arrow[d, "f_1"] & \mathrm{H}_k(\Sigma M(R^n)) \arrow[r] \arrow[d, "f_2"] & \mathrm{H}_k(DM(R^n)) \arrow[d, "f_3"] \\
\mathrm{H}_{k+1}(\Sigma M(R^{n+1})) \arrow[r] & \mathrm{H}_{k+1}(DM(R^{n+1})) \arrow[r] & \mathrm{H}_k(M(R^{n+1})) \arrow[r] & \mathrm{H}_k(\Sigma M(R^{n+1})) \arrow[r] & \mathrm{H}_k(DM(R^{n+1}))
\end{tikzcd}$$

with exact rows. Our inductive hypothesis says the following about the g_i and f_i:

- Since ΣM is polynomial of degree d starting at $(m-1)$, the map f_2 is an isomorphism for $n \geq \max(m-1, 2k+2d+r+1)$ and a surjection for $n \geq \max(m-1, 2k+2d+r-1)$. Also, the map g_1 is an isomorphism for
$$n \geq \max(m - 1, 2(k + 1) + 2d + r + 1) = \max(m - 1, 2k + 2d + r + 3)$$
and a surjection for $n \geq \max(m - 1, 2k + 2d + r + 1)$.

- Since DM is polynomial of degree $(d-1)$ starting at $(m-1)$, the map f_3 is an isomorphism for
$$n \geq \max(m-1, 2k+2(d-1)+r+1) = \max(m-1, 2k+2d+r-1)$$
and a surjection for $n \geq \max(m-1, 2k+2d+r-3)$. Also, the map g_2 is an isomorphism for
$$n \geq \max(m-1, 2(k+1)+2(d-1)+r+1) = \max(m-1, 2k+2d+r+1)$$
and a surjection for $n \geq \max(m-1, 2k+2d+r-1)$.

For $n \geq \max(m, 2k+2d+r+1)$, the maps g_2 and f_2 and f_3 are isomorphisms and the map g_1 is a surjection, so by the five-lemma the map f_1 is an isomorphism. For $n \geq \max(m, 2k+2d+r-1)$, the maps g_2 and f_2 are surjections and the map f_3 is an isomorphism, so by the five-lemma[16] the map f_1 is a surjection. The claim follows. □

12. Unipotence and its consequences

We now turn our attention to congruence subgroups. Before we can prove Theorem C, we need some preliminary results about unipotent representations.

12.1. **Unipotent representations.** Let $\Bbbk$ be a field and let V be a vector space over $\Bbbk$. A *unipotent operator* on V is a linear map $f\colon V \to V$ that can be written as $f = \mathrm{id}_V + \phi$ where $\phi\colon V \to V$ is nilpotent, i.e., there exists some $k \geq 1$ such that $\phi^k = 0$. If G is a group and V is a $\Bbbk[G]$-module, then we say that V is a *unipotent representation* of G if all elements of G act on V via unipotent operators. We will mostly be interested in abelian G, where this can be checked on generators:

Lemma 12.1. *Let G be an abelian group generated by a set S, let $\Bbbk$ be a field, and let V be a $\Bbbk[G]$-module. Assume that each $s \in S$ acts on V via a unipotent operator. Then V is a unipotent representation of G.*

Proof. It is enough to prove that if $g_1, g_2 \in G$ are elements that both act on V via unipotent operators, then $g_1 g_2$ also acts via a unipotent operator. Let g_i act on V via the linear map $f_i\colon V \to V$, and write $f_i = \mathrm{id}_V + \phi_i$ with ϕ_i nilpotent. We thus have
$$f_1 f_2 = \mathrm{id}_V + \phi_1 + \phi_2 + \phi_1\phi_2.$$
Since the g_i commute, the ϕ_i also commute. This implies that $\phi_1 + \phi_2 + \phi_1\phi_2$ is nilpotent, so $f_1 f_2$ is a unipotent operator. □

12.2. **Unipotence and VIC(R)-modules.** The following shows how to find many unipotent operators within a $\mathtt{VIC}(R)$-module. We thank Harman for explaining its proof to us.

Lemma 12.2. *Let R be a ring, let $\Bbbk$ be a field of characteristic 0, and let M be a $\mathtt{VIC}(R)$-module over $\Bbbk$ that is polynomial of degree d starting at m. Assume that $M(R^n)$ is a finite-dimensional vector space over $\Bbbk$ for all n. Then there exists some $u \geq 0$ such that for $n \geq u$, all elementary matrices in $\mathrm{GL}_n(R)$ act on $M(R^n)$ via unipotent operators.*

Proof. Whether or not an operator is unipotent is unchanged by field extensions, so without loss of generality we can assume that $\Bbbk$ is algebraically closed. Via the ring homomorphism $\mathbb{Z} \to R$, regard M as a $\mathtt{VIC}(\mathbb{Z})$-module. This does not change the fact that it is polynomial of degree d starting at m. We can thus appeal to a theorem of Harman [17, Proposition

[16] Or, more precisely, one of the four-lemmas.

4.4][17] saying that there exists some $u \geq 0$ such that for all $n \geq u$, the action of $\mathrm{SL}_n(\mathbb{Z})$ on $M(\mathbb{Z}^n) = M(R^n)$ extends to a rational representation of the algebraic group SL_n.

Increasing u if necessary, we can assume that $u \geq 3$. Consider some $n \geq u$. For distinct $1 \leq i, j \leq n$ and $r \in R$, let $e_{ij}^r \in \mathrm{GL}_n(R)$ be the elementary matrix obtained from the identity by putting r at position (i,j). We must check that each e_{ij}^r acts on $M(R^n)$ as a unipotent operator.

Since the action of $\mathrm{SL}_n(\mathbb{Z})$ on $M(R^n)$ extends to a rational representation of SL_n, each elementary matrix in $\mathrm{SL}_n(\mathbb{Z})$ acts as a unipotent operator [4, Theorem I.4.4]. For distinct $1 \leq i, j, k \leq n$, we have the matrix identity

$$e_{ij}^r = [e_{ik}^1, e_{kj}^r] = e_{ik}^1 e_{kj}^r \left(e_{ik}^1\right)^{-1} \left(e_{kj}^r\right)^{-1}.$$

Manipulating this, we get

$$\left(e_{ik}^1\right)^{-1} e_{ij}^r = e_{kj}^r \left(e_{ik}^1\right)^{-1} \left(e_{kj}^r\right)^{-1}.$$

Since e_{ik}^1 acts on $M(R^n)$ as a unipotent operator and the class of unipotent operators is closed under conjugation and inversion, the right hand side of this expression acts on $M(R^n)$ as a unipotent operator. The matrices

$$e_{ik}^1 \quad \text{and} \quad \left(e_{ik}^1\right)^{-1} e_{ij}^r$$

commute and act on $M(R^n)$ as unipotent operators, so by Lemma 12.1 their product e_{ij}^r acts on $M(R^n)$ as a unipotent operator, as desired. □

12.3. **Stability of invariants.** If V is a module over a commutative ring $\Bbbk$ and $f\colon V \to V$ is a module homomorphism, then let

$$V^f = \{v \in V \mid f(v) = v\}$$

denote the submodule of invariants. One basic property of unipotent operators on vector spaces of characteristic 0 is as follows:

Lemma 12.3. *Let V be a finite-dimensional vector space over a field $\Bbbk$ of characteristic 0 and let $f\colon V \to V$ be a unipotent operator. Then for all $n \geq 1$ we have $V^f = V^{f^n}$.*

Proof. Since $V^f \subset V^{f^n}$, it is enough to prove that $\dim(V^f) = \dim(V^{f^n})$. Write $f = \mathrm{id}_V + \phi$ with ϕ nilpotent. Set $c_i = \binom{n}{i}$, so

$$f^n = \mathrm{id}_V + \phi' \quad \text{with} \quad \phi' = \sum_{i=1}^{n} c_i \phi^i.$$

We have

$$V^f = \ker(\phi) \quad \text{and} \quad V^{f^n} = \ker(\phi'),$$

so our goal is to prove that $\ker(\phi)$ and $\ker(\phi')$ have the same dimension. Since $\Bbbk$ has characteristic 0, we have $c_1 = n \neq 0$. This allows us to write

$$\phi' = c_1\phi \circ \left(\mathrm{id}_V + \sum_{i=1}^{n-1} \frac{c_{i+1}}{c_1}\phi^i\right) = c_1\phi \circ \left(\mathrm{id}_V + \phi''\right) \quad \text{with} \quad \phi'' = \sum_{i=1}^{n-1} \frac{c_{i+1}}{c_1}\phi^i.$$

The linear map ϕ'' is nilpotent, so $\mathrm{id}_V + \phi''$ is invertible. It follows that

$$\dim\ker(\phi') = \dim\ker(c_1\phi \circ (\mathrm{id}_V + \phi'')) = \dim\ker(c_1\phi) = \dim\ker(\phi),$$

as desired. □

[17] The statement of Harman's theorem requires $\Bbbk = \mathbb{C}$, but the proof just uses the fact that $\Bbbk$ is algebraically closed.

12.4. **Stability of homology.** Lemma 12.3 is the key input to the following result.

Lemma 12.4. *Let G be an abelian group and let V be a finite-dimensional unipotent representation of G over a field $\Bbbk$ of characteristic 0. Then for all finite-index subgroups $G' < G$, the inclusion map $G' \hookrightarrow G$ induces an isomorphism $\mathrm{H}_k(G';V) \cong \mathrm{H}_k(G;V)$ for all $k \geq 0$.*

Proof. We divide the proof into several steps.

Step 1. *This holds if G is an infinite cyclic group.*

We thus have $G = \mathbb{Z}$ and $G' = n\mathbb{Z}$ for some $n \geq 1$. Recall that the zeroth homology group is the coinvariants, which is isomorphic to the invariants of the dual. The dual representation V^* is also a unipotent representation of G, so by Lemma 12.3 we have

$$\mathrm{H}_0(G;V) = (V^*)^G = (V^*)^{G'} = \mathrm{H}_0(G';V).$$

For the first homology group, since $G = \mathbb{Z}$ is the fundamental group of the circle we can apply Poincaré duality and use Lemma 12.3 to see that

$$\mathrm{H}_1(G;V) \cong \mathrm{H}^0(G;V) = V^G = V^{G'} = \mathrm{H}^0(G';V) \cong \mathrm{H}_1(G';V).$$

Finally, for $k \geq 2$, we have $\mathrm{H}_k(G;V) = \mathrm{H}_k(G';V) = 0$.

Step 2. *This holds if G is a finite abelian group.*

In this case, we claim that V is a trivial representation of G. Indeed, since G is finite it is enough to prove that every nontrivial unipotent operator $f\colon V \to V$ has infinite order. Write $f = \mathrm{id}_V + \phi$ with $\phi \neq 0$ a nilpotent operator. For all $n \geq 1$, we then have

$$f^n = \mathrm{id}_V + \sum_{i=1}^{n} \binom{n}{i} \phi^i.$$

Since $\Bbbk$ has characteristic 0, we have $\binom{n}{1} = n \neq 0$, so the coefficient of ϕ in this expression is nonzero. Letting r be the order of ϕ, the operators $\{\mathrm{id}_V, \phi, \phi^2, \ldots \phi^{r-1}\}$ are linearly independent in the vector space of linear operators, so $f^n \neq \mathrm{id}_V$, as claimed.

From this, we see that

$$\mathrm{H}_k(G';V) = \mathrm{H}_k(G;V) = \begin{cases} V & \text{if } k = 0, \\ 0 & \text{if } k \geq 1. \end{cases}$$

Step 3. *This holds if G is a finitely generated abelian group.*

We can find a chain of subgroups

$$G = G_1 \supset G_2 \supset \cdots \supset G_n = G'$$

where for each $1 \leq i < n$ the group G_{i+1} is an index-p_i subgroup of G_i for some prime p_i. From this, we see that we can assume without loss of generality that G' is an index-p subgroup of G for some prime p.

Since G' is an index-p subgroup of the finitely generated abelian group G, we can write $G = C \oplus G''$ and $G' = C' \oplus G''$ with C a cyclic subgroup of G and C' an index-p subgroup of C. We thus have a commutative diagram of short exact sequences

$$\begin{tikzcd}
1 \arrow[r] & C' \arrow[r] \arrow[d] & G' \arrow[r] \arrow[d] & G'' \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & C \arrow[r] & G \arrow[r] & G'' \arrow[r] & 1.
\end{tikzcd}$$

This induces a map between the Hochschild–Serre spectral sequences computing $\mathrm{H}_\bullet(G';V)$ and $\mathrm{H}_\bullet(G;V)$. On the E^2-page, this morphism takes the form

$$\mathrm{H}_p(G'';\mathrm{H}_q(C';V)) \to \mathrm{H}_p(G'';\mathrm{H}_q(C;V)). \tag{12.1}$$

The previous two steps imply that the inclusion $C' \hookrightarrow C$ induces isomorphisms $\mathrm{H}_q(C';V) \cong \mathrm{H}_q(C;V)$ for all q, so the map (12.1) is an isomorphism for all p and q. The spectral sequence comparison theorem now implies that $\mathrm{H}_k(G';V) \cong \mathrm{H}_k(G;V)$ for all k, as desired.

Step 4. *This holds if G is an arbitrary abelian group.*

Let $\mathfrak{F}$ be the set of finitely generated subgroups of G. We thus have

$$G = \varinjlim_{H\in\mathfrak{F}} H \quad \text{and} \quad G' = \varinjlim_{H\in\mathfrak{F}} H \cap G'.$$

Since homology commutes with direct limits, this reduces us to the previous case. □

13. Stability for congruence subgroups

We now turn to the proof of Theorem C. We will actually prove three increasingly stronger results, with the third a generalization of Theorem C.

13.1. **Elementary matrices and unipotence.** The first is as follows, which we regard as the heart of the whole proof. The differences between this result and Theorem C are as follows:

- It involves a $\mathtt{VIC}(R)$-module that is polynomial of degree $d \geq 0$ starting at 0.
- It assumes that elementary matrices act unipotently on $M(R^n)$.
- It concerns the elementary congruence subgroup $\mathrm{EL}_n(R,\alpha)$ rather than $\mathrm{GL}_n(R,\alpha)$.
- Finally, it has a better range of stability.

Theorem 13.1. *Let R be a ring satisfying* (SR_r)*, let $\Bbbk$ be a field of characteristic 0, and let M be a $\mathtt{VIC}(R)$-module over $\Bbbk$ that is polynomial of degree $d \geq 0$ starting at 0. For each $n \geq 0$, assume that $M(R^n)$ is a finite-dimensional vector space over $\Bbbk$ and that each elementary matrix in $\mathrm{GL}_n(R)$ acts unipotently on $M(R^n)$. Then for all 2-sided ideals α of R such that R/α is finite, the map*

$$\mathrm{H}_k(\mathrm{EL}_n(R,\alpha);M(R^n)) \to \mathrm{H}_k(\mathrm{EL}_n(R);M(R^n))$$

is an isomorphism for $n \geq 2k+2d+r+1$.

Proof. Theorem 11.1 says that the stabilization map

$$\mathrm{H}_k(\mathrm{EL}_n(R);M(R^n)) \to \mathrm{H}_k(\mathrm{EL}_{n+1}(R);M(R^{n+1}))$$

is an isomorphism for $n \geq 2k+2d+r+1$. We proved this by verifying the conditions of our stability machine (Theorem 5.2) with the parameter $c = 2$ (corresponding to the multiple 2 in front of k in $n \geq 2k+2d+r+1$). Letting $e = \max(2d-1, m-r)$, the inputs to Theorem 5.2 were

$$G_n = \mathrm{EL}_{n+e+r}(R) \quad \text{and} \quad M_n = M(R^{n+e+r}) \quad \text{and} \quad \mathbb{X}_n = \mathbb{B}(R^{n+e,r}) \quad \text{and} \quad \mathcal{M}_n = \mathcal{G}_{M,n+e,r}.$$

To prove that furthermore the map

$$\mathrm{H}_k(\mathrm{EL}_n(R,\alpha);M(R^n)) \to \mathrm{H}_k(\mathrm{EL}_n(R);M(R^n))$$

is an isomorphism for $n \geq 2k+2d+r+1$, it is enough to verify the additional hypotheses of Theorem 5.7 for

$$G'_n = \mathrm{EL}_{n+e+r}(R,\alpha).$$

These additional hypotheses are numbered (e)-(h), and we verify each of them in turn.

Condition (e) says that each M_n is a vector space over a field $\mathbb{k}$ of characteristic 0, which is one of our hypotheses.

Condition (f) says that for the k-simplex σ_k of $\mathbb{X}_n$ from condition (b) of Theorem 5.2 whose G_n-stabilizer is G_{n-k-1}, the G'_n-stabilizer of σ_k is G'_{n-k-1}. Looking back at our proof of Theorem 11.1, the simplex σ_k is as follows. Let $\{v_1, \ldots, v_{n+r}\}$ be the standard basis for R^{n+e+r}. We then have

$$\sigma_k = \begin{cases} (R^{n+e+r}) & \text{if } k = -1, \\ (v_{n+e+r-k}, v_{n+e+r-k+1}, \ldots, v_{n+e+r}; R^{n+e+r-k-1}) & \text{if } 0 \leq k \leq n. \end{cases}$$

Thus the $\mathrm{GL}_{n+e+r}(R)$-stabilizer of σ_k is $\mathrm{GL}_{n+e+r-k-1}(R)$, and by Lemma 8.18 the $G_n = \mathrm{EL}_{n+e+r}(R)$ stabilizer of σ_k is $G_{n-k-1} = \mathrm{EL}_{n+e+r-k-1}(R)$. Another application of Lemma 8.18 says that the $G'_n = \mathrm{EL}_{n+e+r}(R, \alpha)$ stabilizer of σ_k is $G'_{n-k-1} = \mathrm{EL}_{n+e+r-k-1}(R, \alpha)$, as desired.

Condition (g) says that the quotient $\mathbb{X}_n/G'_n$ is $\frac{n-2}{2}$-connected. Conclusion (iii) of Lemma 10.1 says that

$$\mathbb{X}_n/G'_n = \mathbb{B}(R^{n+e,r})/\, \mathrm{EL}_{n+e+r}(R, \alpha) \cong \mathbb{B}((R/\alpha)^{n+e,r}).$$

Lemma 8.10 says that R/α satisfies (SR_r), so we can apply Conclusion (iv) of Lemma 10.1 to see that $\mathbb{B}((R/\alpha)^{n+e,r})$ is a large ordering of a weakly Cohen–Macaulay complex of dimension $\frac{n+e+1}{2}$. By Theorem 3.5, this implies that $\mathbb{B}((R/\alpha)^{n+e,r})$ is

$$\frac{n+e+1}{2} - 1 = \frac{n+e-1}{2} \geq \frac{n-2}{2}$$

connected. Here we use the fact that $e = \max(2d-1, m-r)$ is at least -1, which follows from our assumption that $d \geq 0$.

Finally, the key condition (h) says that for $k \geq 0$ and $n \geq 2k+2$, the action of G_n on $\mathrm{H}_k(G'_n; M(R^n))$ induced by the conjugation action of G_n on G'_n fixes the image of the stabilization map

$$\mathrm{H}_k(G'_{n-1}; M(R^{n-1})) \to \mathrm{H}_k(G'_n; M(R^n)).$$

This is the content of the following claim, which is the heart of our proof. Note that this claim is even stronger since it holds for all $G_n = \mathrm{EL}_{n+e+r}(R)$, not just those where $n \geq 2k+2$.

Claim. *The group* $\mathrm{EL}_n(R)$ *acts trivially on the image of the stabilization map*

$$\mathrm{H}_k(\mathrm{EL}_{n-1}(R, \alpha); M(R^{n-1})) \to \mathrm{H}_k(\mathrm{EL}_n(R, \alpha); M(R^n)). \tag{13.1}$$

The proof of this claim generalizes a beautiful argument of Charney from [8]. For distinct $1 \leq i, j \leq n$ and $r \in R$, let $e^r_{ij} \in \mathrm{EL}_n(R)$ denote the elementary matrix obtained from the identity by putting r at position (i, j). Define A to be the subgroup of $\mathrm{EL}_n(R)$ generated by $\left\{e^r_{nj} \mid 1 \leq j \leq n-1,\ r \in R\right\}$ and let B be the subgroup generated by $\{e^r_{in} \mid 1 \leq i \leq n-1,\ r \in R\}$. Using the matrix identity

$$e^r_{ij} = [e^r_{ik}, e^1_{kj}] \quad \text{for distinct } 1 \leq i, j, k \leq n \text{ and } r \in R,$$

we see that $\mathrm{EL}_n(R)$ is generated by $A \cup B$. It is thus enough to prove that A and B act trivially on the image of (13.1). The arguments for A and B are similar, so we will give the details for A and leave B to the reader.

The group A is the abelian subgroup of $\mathrm{GL}_n(R)$ consisting of matrices that differ from the identity only in their n^{th} row. In particular, A is isomorphic to the abelian group R^{n-1}. Moreover, letting Γ be the subgroup of $\mathrm{EL}_n(R)$ generated by A and $\mathrm{EL}_{n-1}(R)$, the group Γ is a sort of "affine group" and in particular

$$\Gamma = A \rtimes \mathrm{EL}_{n-1}(R). \tag{13.2}$$

We would like to imitate this for $\mathrm{EL}_n(R, \alpha)$.

Define A_α to be the subgroup of $\mathrm{EL}_n(R,\alpha)$ generated by $\left\{e_{nj}^r \mid 1 \leq j \leq n-1,\, r \in \alpha\right\}$. As an abelian group, we have $A_\alpha \cong \alpha^{n-1}$. Define Γ_α to be the subgroup of $\mathrm{EL}_n(R,\alpha)$ generated by A_α and $\mathrm{EL}_{n-1}(R,\alpha)$, so just like (13.2) we have

$$\Gamma_\alpha = A_\alpha \rtimes \mathrm{EL}_{n-1}(R,\alpha). \tag{13.3}$$

The stabilization map (13.1) factors through the map

$$\mathrm{H}_k(\Gamma_\alpha; M(R^n)) \to \mathrm{H}_k(\mathrm{EL}_n(R,\alpha); M(R^n))$$

induced by the inclusion $\Gamma_\alpha \hookrightarrow \mathrm{EL}_n(R,\alpha)$.

The conjugation action of A on $\mathrm{EL}_n(R,\alpha)$ takes Γ_α to itself. It is thus enough to prove that the conjugation action of A on $\mathrm{H}_k(\Gamma_\alpha; M(R^n))$ is trivial. Define Γ'_α to be the subgroup of $\mathrm{EL}_n(R)$ generated by A and $\mathrm{EL}_{n-1}(R,\alpha)$. Since inner automorphisms act trivially on homology (even with twisted coefficients; see [7, Proposition III.8.1]), the conjugation action of A on $\mathrm{H}_k(\Gamma'_\alpha; M(R^n))$ is trivial. It is thus enough to prove that the inclusion $\Gamma_\alpha \hookrightarrow \Gamma'_\alpha$ induces an isomorphism $\mathrm{H}_k(\Gamma_\alpha; M(R^n)) \cong \mathrm{H}_k(\Gamma'_\alpha; M(R^n))$.

For this, observe that we have a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & A_\alpha \arrow[r] \arrow[d] & \Gamma_\alpha \arrow[r] \arrow[d] & \mathrm{EL}_{n-1}(R,\alpha) \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & A \arrow[r] & \Gamma'_\alpha \arrow[r] & \mathrm{EL}_{n-1}(R,\alpha) \arrow[r] & 1
\end{tikzcd}$$

with exact rows. The first row is the split exact sequence corresponding to (13.3), and the second row is the split exact sequence corresponding to the similar semidirect product decomposition of Γ'_α. This induces a morphism between the Hochschild–Serre spectral sequences computing the homology of Γ_α and Γ'_α with coefficients in $M(R^n)$. On the E^2-page, this map of spectral sequences takes the form

$$\mathrm{H}_p(\mathrm{EL}_{n-1}(R,\alpha); \mathrm{H}_q(A_\alpha; M(R^n))) \to \mathrm{H}_p(\mathrm{EL}_{n-1}(R,\alpha); \mathrm{H}_q(A; M(R^n))). \tag{13.4}$$

The abelian group A_α is a finite-index subgroup of the abelian group A, and since we assumed that elementary matrices act unipotently on $M(R^n)$ we can appeal to Lemma 12.1 to see that the action of A on $M(R^n)$ is a unipotent action. Lemma 12.4 therefore implies that the inclusion $A_\alpha \hookrightarrow A$ induces isomorphisms

$$\mathrm{H}_q(A_\alpha; M(R^n)) \cong \mathrm{H}_q(A; M(R^n)) \quad \text{for all } q.$$

We deduce that (13.4) is an isomorphism for all p and q. The spectral sequence comparison theorem therefore implies that the inclusion $\Gamma_\alpha \hookrightarrow \Gamma'_\alpha$ induces an isomorphism

$$\mathrm{H}_k(\Gamma_\alpha; M(R^n)) \cong \mathrm{H}_k(\Gamma'_\alpha; M(R^n)),$$

as desired. □

13.2. **Non-elementary subgroups.** We now generalize Theorem 13.1 by extending it to subgroups other than the elementary ones. This causes us to have a worse range of stability.

Theorem 13.2. *Let R be a ring satisfying (SR_r), let $\mathcal{K} \subset \mathrm{K}_1(R)$ be a subgroup, let $\Bbbk$ be a field of characteristic 0, and let M be a $\mathtt{VIC}(R)$-module over $\Bbbk$ that is polynomial of degree $d \geq 0$ starting at 0. Assume furthermore that $M(R^n)$ is a finite-dimensional vector space over $\Bbbk$ for all $n \geq 0$ and that each elementary matrix in $\mathrm{GL}_n(R)$ acts unipotently on $M(R^n)$. Then for all 2-sided ideals α of R such that R/α is finite and all subgroups $\mathcal{K}' \subset \mathrm{K}_1(R,\alpha)$ that map onto $\mathcal{K}$ under the map $\mathrm{K}_1(R,\alpha) \to \mathrm{K}_1(R)$, the map*

$$\mathrm{H}_k(\mathrm{GL}_n^{\mathcal{K}'}(R,\alpha); M(R^n)) \to \mathrm{H}_k(\mathrm{GL}_n^{\mathcal{K}}(R); M(R^n))$$

is an isomorphism for $n \geq 2k+2d+2r$.

Proof. Consider some $n \geq 2k + 2d + 2r$. Using Theorem 8.16, we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{EL}_n(R,\alpha) & \longrightarrow & \mathrm{GL}_n^{\mathcal{K}'}(R,\alpha) & \longrightarrow & \mathcal{K}' & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{EL}_n(R) & \longrightarrow & \mathrm{GL}_n^{\mathcal{K}}(R) & \longrightarrow & \mathcal{K} & \longrightarrow & 1 \end{array}$$

with exact rows. This induces a map between the Hochschild–Serre spectral sequences computing the homology of $\mathrm{GL}_n^{\mathcal{K}'}(R,\alpha)$ and $\mathrm{GL}_n^{\mathcal{K}}(R)$ with coefficients in $M(R^n)$. On the E^2-page, this map of spectral sequences is of the form

$$\mathrm{H}_p(\mathcal{K}';\mathrm{H}_q(\mathrm{EL}_n(R,\alpha);M(R^n))) \to \mathrm{H}_p(\mathcal{K};\mathrm{H}_q(\mathrm{EL}_n(R);M(R^n))). \tag{13.5}$$

We analyze this via a sequence of two claims.

Claim. *For all $q \leq k$, the map $\mathrm{H}_q(\mathrm{EL}_n(R,\alpha);M(R^n)) \to \mathrm{H}_q(\mathrm{EL}_n(R);M(R^n))$ is an isomorphism.*

Proof of claim. Immediate from Theorem C. □

Claim. *For all $q \leq k$, the action of the group $\mathcal{K}$ (resp. $\mathcal{K}'$) on $\mathrm{H}_q(\mathrm{EL}_n(R);M(R^n))$ (resp. $\mathrm{H}_q(\mathrm{EL}_n(R,\alpha);M(R^n))$) is trivial.*

Proof of claim. The previous claim implies that it is enough to prove that $\mathcal{K}$ acts trivially on $\mathrm{H}_q(\mathrm{EL}_n(R);M(R^n))$. In fact, we will prove that all of $\mathrm{K}_1(R)$ acts trivially on $\mathrm{H}_q(\mathrm{EL}_n(R);M(R^n))$. Since

$$n-(r-1) \geq 2k+2d+r+1,$$

Theorem 11.2 implies that the map

$$\mathrm{H}_q(\mathrm{EL}_{n-(r-1)}(R);M(R^{n-(r-1)})) \to \mathrm{H}_q(\mathrm{EL}_n(R);M(R^n)) \tag{13.6}$$

is surjective. Theorem 8.16 together with Theorem 8.7 implies that the subgroup $\mathrm{GL}_{r-1}(R)$ of $\mathrm{GL}_n(R)$ surjects onto $\mathrm{K}_1(R)$ under the map $\mathrm{GL}_n(R) \to \mathrm{K}_1(R)$ whose kernel is $\mathrm{EL}_n(R)$. The group $\mathrm{GL}_{r-1}(R)$ is embedded in $\mathrm{GL}_n(R)$ using the upper-left-hand matrix embedding, but we can conjugate it by any element of $\mathrm{GL}_n(R)$ and it will still surject onto $\mathrm{K}_1(R)$. We can thus use the lower-right-hand embedding, which makes $\mathrm{GL}_{r-1}(R)$ commute with $\mathrm{EL}_{n-(r-1)}(R)$. This implies that $\mathrm{K}_1(R)$ acts trivially on the image of (13.6), and hence on $\mathrm{H}_q(\mathrm{EL}_n(R);M(R^n))$. □

By the second Claim, for $q \leq k$ we can rewrite our map of spectral sequences (13.5) as

$$\mathrm{H}_p(\mathcal{K}';\Bbbk) \otimes \mathrm{H}_q(\mathrm{EL}_n(R,\alpha);M(R^n)) \to \mathrm{H}_p(\mathcal{K};\Bbbk) \otimes \mathrm{H}_q(\mathrm{EL}_n(R);M(R^n)). \tag{13.7}$$

The map $\mathcal{K}' \to \mathcal{K}$ of abelian groups has a finite kernel and cokernel.[18] Since $\Bbbk$ is a field of characteristic 0, this implies that the map $\mathrm{H}_p(\mathcal{K}';\Bbbk) \to \mathrm{H}_p(\mathcal{K};\Bbbk)$ is an isomorphism for all p. Combining this with the first Claim, we see that (13.7) is an isomorphism for all p and q with $q \leq k$. By the spectral sequence comparison theorem, we deduce that the map

$$\mathrm{H}_k(\mathrm{GL}_n^{\mathcal{K}'}(R,\alpha);M(R^n)) \to \mathrm{H}_k(\mathrm{GL}_n^{\mathcal{K}}(R);M(R^n))$$

is an isomorphism, as desired. □

[18]This follows from the fact that the map $\mathrm{K}_1(R,\alpha) \to \mathrm{K}_1(R)$ has a finite kernel and cokernel, which follows from the fact that it fits into an exact sequence [21, Theorem 6.2]

$$\mathrm{K}_2(R/\alpha) \to \mathrm{K}_1(R,\alpha) \to \mathrm{K}_1(R) \to \mathrm{K}_1(R/\alpha)$$

along with the fact that R/α is finite.

13.3. **The general case: removing unipotence.** We now prove the following theorem, which generalizes Theorem C.

Theorem 13.3. *Let R be a ring satisfying* (SR_r)*, let $\mathcal{K} \subset \mathrm{K}_1(R)$ be a subgroup, let $\Bbbk$ be a field of characteristic* 0*, and let M be a* $\mathtt{VIC}(R)$*-module over $\Bbbk$ that is polynomial of degree d starting at m. Assume furthermore that $M(R^n)$ is a finite-dimensional vector space over $\Bbbk$ for all $n \geq 0$. Then for all 2-sided ideals α of R such that R/α is finite and all subgroups $\mathcal{K}' \subset \mathrm{K}_1(R,\alpha)$ that map onto $\mathcal{K}$ under the map $\mathrm{K}_1(R,\alpha) \to \mathrm{K}_1(R)$, the map*

$$\mathrm{H}_k(\mathrm{GL}_n^{\mathcal{K}'}(R,\alpha); M(R^n)) \to \mathrm{H}_k(\mathrm{GL}_n^{\mathcal{K}}(R); M(R^n)) \tag{13.8}$$

is an isomorphism for $n \geq \max(m, 2k+2d+2r)$.

Proof. Lemma 12.2 says that there exists some u such that for $n \geq u$, all elementary matrices in $\mathrm{GL}_n(R)$ act unipotently on $M(R^n)$. The *parameters* of M are the triple (d,m,u). They satisfy $d \geq -1$ and $m, u \in \mathbb{Z}$.

We will prove the theorem by triple induction on the parameters (d,m,u). There are two base cases:

- The first is where $d \geq 0$ and $m, u \leq 0$. In this case, Theorem 13.2 says that (13.8) is an isomorphism for
$$n \geq 2k+2d+2r = \max(m, 2k+2d+2r),$$
as desired.
- The second is where $d = -1$ and $m \geq 0$ and $u \geq 0$. In this case, by the definition of a $\mathtt{VIC}(R)$-module being polynomial of degree -1 starting at m we have for $n \geq m$ that $M(R^n) = 0$ and hence
$$\mathrm{H}_k(\mathrm{GL}_n^{\mathcal{K}'}(R,\alpha); M(R^n)) = \mathrm{H}_k(\mathrm{GL}_n^{\mathcal{K}}(R); M(R^n)) = 0.$$
In other words, for $n \geq m$ the domain and codomain of (13.8) are both 0, so it is trivially an isomorphism.

We can therefore assume that the parameters (d,m,u) satisfy the following:

- $d \geq 0$, and either $m \geq 1$ or $u \geq 1$ (possibly both hold: $m \geq 1$ and $u \geq 1$).

Moreover, we can assume as an inductive hypothesis that the theorem is true for all M with parameters (d', m', u') satisfying the following:

- Either $d' \leq d-1$, or both $m' \leq m-1$ and $u' \leq u-1$.

Note that it is *not* enough to shrink just one of m and u since otherwise we might never reach one of our base cases.

As in Definition 1.17, let ΣM be the shifted $\mathtt{FI}$-module and DM be the derived $\mathtt{VIC}(R)$-module. For $n \geq m$, we have a short exact sequence

$$0 \longrightarrow M(R^n) \longrightarrow \Sigma M(R^n) \longrightarrow DM(R^n) \longrightarrow 0 \tag{13.9}$$

of $\Bbbk[\mathrm{GL}_n(R)]$-modules. The $\mathtt{VIC}(R)$-module ΣM has parameters $(d, m-1, u-1)$, and the $\mathtt{VIC}(R)$-module DM has parameters $(d-1, m-1, u-1)$.

To simplify our notation, for all all $\Bbbk[\mathrm{GL}_n(R)]$-modules N, we will denote

- $\mathrm{H}_k(\mathrm{GL}_n^{\mathcal{K}}(R); N)$ by $\mathrm{H}_k(N)$.
- $\mathrm{H}_k(\mathrm{GL}_n^{\mathcal{K}'}(R,\alpha); N)$ by $\mathrm{H}_k(\alpha, N)$.

The long exact sequence in $\mathrm{GL}_n^{\mathcal{K}'}(R,\alpha)$-homology associated to (13.9) maps to the one in $\mathrm{GL}_n^{\mathcal{K}}(R)$-homology, so for $n \geq m$ and all k we have a commutative diagram

$$\begin{tikzcd} \mathrm{H}_{k+1}(\alpha, \Sigma M(R^n)) \arrow[r] \arrow[d, "g_1"] & \mathrm{H}_{k+1}(\alpha, DM(R^n)) \arrow[r] \arrow[d, "g_2"] & \mathrm{H}_k(\alpha, M(R^n)) \arrow[r] \arrow[d, "f_1"] & \mathrm{H}_k(\alpha, \Sigma M(R^n)) \arrow[r] \arrow[d, "f_2"] & \mathrm{H}_k(\alpha, DM(R^n)) \arrow[d, "f_3"] \\ \mathrm{H}_{k+1}(\Sigma M(R^n)) \arrow[r] & \mathrm{H}_{k+1}(DM(R^n)) \arrow[r] & \mathrm{H}_k(M(R^n)) \arrow[r] & \mathrm{H}_k(\Sigma M(R^n)) \arrow[r] & \mathrm{H}_k(DM(R^n)) \end{tikzcd}$$

with exact rows. Since $\mathrm{GL}_n^{\mathcal{K}'}(R, \alpha)$ is a finite-index subgroup of $\mathrm{GL}_n^{\mathcal{K}}(R)$ and all our coefficients are vector spaces over a field $\Bbbk$ of characteristic 0, the transfer map (see [7, Chapter III.9]) implies that all the f_i and g_i are surjections. Our inductive hypothesis says the following about them:

- Since ΣM has parameters $(d, m-1, u-1)$, the map f_2 is an isomorphism for $n \geq \max(m-1, 2k+2d+2r)$ and the map g_1 is an isomorphism for
$$n \geq \max(m-1, 2(k+1)+2d+2r) = \max(m-1, 2k+2d+2r+2).$$
- Since DM has parameters $(d-1, m-1, u-1)$, the map f_3 is an isomorphism for
$$n \geq \max(m-1, 2k+2(d-1)+2r) = \max(m-1, 2k+2d+2r-2)$$
and the map g_2 is an isomorphism for
$$n \geq \max(m-1, 2(k+1)+2(d-1)+2r) = \max(m-1, 2k+2d+2r).$$

For $n \geq \max(m, 2k+2d+2r)$, the maps g_2 and f_2 and f_3 are isomorphisms and the map g_1 is a surjection (remember, it is always a surjection!), so by the five-lemma the map f_1 is an isomorphism, as desired. □

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