# Math 444/539 : Geometric Topology
# Final Exam

This exam is pledged. The point values for each problem are as indicated. You are allowed six hours which begins once you start reading the problems on the next page (I expect that most people will not need the full 6 hours). During the exam, you may take one 30 minute break during which you are not to read, write, or talk about anything math-related (these 30 minutes do not count towards your five hours). You are allowed to use Massey's "Algebraic Topology : An Introduction", Hatcher's "Algebraic Topology" (available online – see the course webpage for a link), your course notes, and the notes I posted on the webpage. No other sources (including internet sources) or consultation with other people are allowed.

Please write your solutions on a separate sheet of paper and staple the exam pages (including this front page) and your solution pages together.

**Name** :

**Time Started** :

**Time break begins** :

**Time break ends** :

**Time Ended** :

**Write the honor pledge** :

**Problems** :

1. Let $X$ be the disk, annulus, or Möbius band, and let $\partial X \subset X$ be its boundary circle or circles.

   (a) (10 pts) For $x \in X$, show that the inclusion $X \setminus \{x\} \hookrightarrow X$ induces an isomorphism on $\pi_1$ iff $x \in \partial X$.

   (b) (10 pts) If $Y$ is also a disk, annulus, or Möbius band, show that a homeomorphism $f : X \to Y$ restricts to a homeomorphism from $\partial X$ to $\partial Y$.

   (c) (5 pts) Using part b, deduce that the Möbius band is not homeomorphic to an annulus.

2. (15 pts) Let $X$ be a space. The *suspension* of $X$, denoted $\Sigma X$, is the quotient of $X \times I$ that identifies $X \times \{0\}$ to a point and $X \times \{1\}$ to a point (these are two different points!). Prove that if $X$ is path-connected, then $\pi_1(\Sigma X) = 1$. Give an example to show that the path-connectedness is necessary.

3. (15 pts) Let $G$ be a finitely generated group. Let $H < G$ be a finite-index subgroup. Using topology, prove that $H$ is finitely generated.

4. (15 pts) Let $X$ be a path-connected Hausdorff space. Let $\rho : \tilde{X} \to X$ be a covering space of $X$ with $\tilde{X}$ compact. Prove that $\rho$ must be a finite-sheeted cover.

5. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $\phi(x, y) = (2x, y/2)$. This generates an action of $\mathbb{Z}$ on $X = \mathbb{R}^2 \setminus \{0\}$.

   (a) (5 pts) Show this action is a covering space action.

   (b) (5 pts) Show the orbit space $X/\mathbb{Z}$ is non-Hausdorff.

   (c) (5 pts) Show that $X/\mathbb{Z}$ is a union of four subspaces homeomorphic to $S^1 \times \mathbb{R}$ (hint : think about the complements of the x-axis and the y-axis).

   (d) (5 pts) Calculate the fundamental group of $X/\mathbb{Z}$.

6. (Extra Credit : 10 pts) Let $M^3$ be a compact triangulated 3-manifold. Assume that $M^3$ has $v$ vertices, $e$ edges, $f$ faces, and $t$ tetrahedra. Define

   $$\chi(M^3) = v - e + f - t.$$

   Prove that $\chi(M^3) = 0$ (without using (co)homology theory). Hint : This can be done with elementary combinatorial manipulations! The key observation will be that intersecting a small ball around a vertex results in a triangulation of a ball, the boundary of which is a triangulated 2-sphere (which has Euler characteristic 2).