Math 444/539 : Geometric Topology
Problem Set 1

1. This problem concerns the following theorem which was proven in the first lecture.

   Theorem. *Let X be a space and let $\mathcal{I} \subset \mathcal{P}(X)$. There then exists a unique space Y and a continuous map $\pi : X \to Y$ such that $\pi|_I$ is a constant function for all $I \in \mathcal{I}$ and such that the following holds. If $\phi : X \to Z$ is such that $\phi|_I$ is a constant function for all $I \in \mathcal{I}$, then there exists a unique $\phi' : Y \to Z$ such that $\phi = \phi' \circ \pi$.*

   (a) Prove that Y and $\pi : X \to Y$ are unique. Hints : Assume that Y' and $\pi' : X \to Y'$ also satisfy the universal mapping property. Use the universal mapping property to construct maps $f : Y \to Y'$ and $g : Y \to Y'$, and then use the universal mapping property again to show that $f \circ g$ and $g \circ f$ are the identity. Conclude that Y and Y' are homeomorphic.

   Remark. This proof will appear in different guises several times in this course.

   (b) Recall that in the proof of the above theorem, we defined

   $$\begin{aligned} E_p = \{p' \in X \mid {} & \text{exists } q_1, \ldots, q_n \in X \text{ and } I_1, \ldots, I_{n-1} \in \mathcal{I} \\ & \text{such that } p = q_1,\ p' = q_n, \text{ and } \{q_i, q_{i+1}\} \subset I_i \\ & \text{for } 1 \leq i < n\} \end{aligned}$$

   for $p \in X$. Prove that for $p, p' \in X$, either $E_p = E_{p'}$ or $E_p \cap E_{p'} = \emptyset$.

   (c) Recall that in the proof of the above theorem, we defined

   $$Y = \{E_p \mid p \in X\}$$

   and

   $$\mathcal{U} = \{U \subset Y \mid \cup_{E \in U} E \subset X \text{ is open}\}.$$

   Prove that $\mathcal{U}$ is a topology on Y.

2. Let X be an n-dimensional CW complex. Let the interiors of the n-cells of X be $U_1, \ldots, U_n$, and let $p_i \in U_i$ be arbitrary. Prove that there is a retract

   $$\pi : X \setminus \{p_1, \ldots, p_n\} \to X^{(n-1)}$$

   Recall that a retract of a space A onto a subspace B is a continuous map $f : A \to B$ such that $f|_B$ is the identity map.

3. (a) For $a \in \mathbb{R}$, define

   $$I_a = \{(x, y) \mid x + y^2 = a\} \subset \mathbb{R}^2$$

   and

   $$\mathcal{I} = \{I_a \mid a \in \mathbb{R}\} \subset \mathcal{P}(\mathbb{R}^2).$$

   Let Y be the quotient of $\mathbb{R}^2$ by $\mathcal{I}$. The space Y is a familar space : what is it?

   (b) Repeat part a for

   $$I_a = \{(x, y) \mid x^2 + y^2 = a\} \subset \mathbb{R}^2$$

   and

   $$\mathcal{I} = \{I_a \mid a \in \mathbb{R},\ a \geq 0\} \subset \mathcal{P}(\mathbb{R}^2).$$