1. This problem concerns the following theorem which was proven in the first lecture.

**Theorem.** Let \( X \) be a space and let \( \mathcal{I} \subset \mathcal{P}(X) \). There then exists a unique space \( Y \) and a continuous map \( \pi : X \to Y \) such that \( \pi|_I \) is a constant function for all \( I \in \mathcal{I} \) and such that the following holds. If \( \phi : X \to Z \) is such that \( \phi|_I \) is a constant function for all \( I \in \mathcal{I} \), then there exists a unique \( \phi' : Y \to Z \) such that \( \phi = \phi' \circ \pi \).

(a) Prove that \( Y \) and \( \pi : X \to Y \) are unique. Hints: Assume that \( Y' \) and \( \pi' : X \to Y' \) also satisfy the universal mapping property. Use the universal mapping property to construct maps \( f : Y \to Y' \) and \( g : Y' \to Y \), and then use the universal mapping property again to show that \( f \circ g \) and \( g \circ f \) are the identity. Conclude that \( Y \) and \( Y' \) are homeomorphic.

*Remark.* This proof will appear in different guises several times in this course.

(b) Recall that in the proof of the above theorem, we defined \( E_p = \{ p' \in X \mid \exists q_1, \ldots, q_n \in X \text{ and } I_1, \ldots, I_{n-1} \in \mathcal{I} \text{ such that } p = q_1, p' = q_n, \text{ and } \{q_i, q_{i+1}\} \subset I_i \text{ for } 1 \leq i < n \} \) for \( p \in X \). Prove that for \( p, p' \in X \), either \( E_p = E_{p'} \) or \( E_p \cap E_{p'} = \emptyset \).

(c) Recall that in the proof of the above theorem, we defined \( Y = \{ E_p \mid p \in X \} \) and \( \mathcal{U} = \{ U \subset Y \mid \cup_{E \in U} E \subset X \text{ is open} \} \). Prove that \( \mathcal{U} \) is a topology on \( Y \).

2. Let \( X \) be an \( n \)-dimensional CW complex. Let the interiors of the \( n \)-cells of \( X \) be \( U_1, \ldots, U_n \), and let \( p_i \in U_i \) be arbitrary. Prove that there is a retract \( \pi : X \setminus \{p_1, \ldots, p_n\} \to X^{(n-1)} \).

Recall that a retract of a space \( A \) onto a subspace \( B \) is a continuous map \( f : A \to B \) such that \( f|_B \) is the identity map.

3. (a) For \( a \in \mathbb{R} \), define \( I_a = \{ (x, y) \mid x + y^2 = a \} \subset \mathbb{R}^2 \) and \( \mathcal{I} = \{ I_a \mid a \in \mathbb{R} \} \subset \mathcal{P}(\mathbb{R}^2) \).

Let \( Y \) be the quotient of \( \mathbb{R}^2 \) by \( \mathcal{I} \). The space \( Y \) is a familiar space: what is it?

(b) Repeat part a for \( I_a = \{ (x, y) \mid x^2 + y^2 = a \} \subset \mathbb{R}^2 \) and \( \mathcal{I} = \{ I_a \mid a \in \mathbb{R}, a \geq 0 \} \subset \mathcal{P}(\mathbb{R}^2) \).